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A Combined Approach of a Sequential Life Testing and an Accelerated Life Testing Applied to a Low-Alloy High Strength Steel Component

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Abstract**—**Sometimes the amount of time available for testing could be considerably less than the expected lifetime of the component. To overcome such a problem, there is the accelerated life-testing alternative aimed at forcing components to fail by testing them at much higher-than-intended application conditions. These models are known as acceleration models. One possible way to translate test results obtained under accelerated conditions to normal using conditions could be through the application of the "Maxwell Distribution Law." In this paper we will apply a combined approach of a sequential life testing and an accelerated life testing to a low alloy high-strength steel component used in the construction of overpasses in Brazil. The underlying sampling distribution will be three-parameter Inverse Weibull model. To estimate the three parameters of the Inverse Weibull model we will use a maximum likelihood approach for censored failure data. We will be assuming a linear acceleration condition. To evaluate the accuracy (significance) of the parameter values obtained under normal conditions for the underlying Inverse Weibull model we will apply to the expected normal failure times a sequential life testing using a truncation mechanism. An example will illustrate the application of this procedure.

Keywords**—**Sequential Life Testing, Accelerated Life Testing, Underlying Three-Parameter Weibull Model, Maximum Likelihood Approach, Hypothesis Testing.

I. Introduction

The Inverse Weibull model was developed by [2]. It has a minimum life (or location), a scale and a shape parameter. This model has been used in reliability estimation of some industrial components where it seems to have a better answer to the accuracy problem presented by the Weibull model, as shown by [1]. It happens that when the shape parameter of the Weibull model is greater than 7, the Weibull curve becomes highly pointed, resulting in some computational difficulty (accuracy) in calculating the component's characteristics of interest values. The three-parameter Inverse Weibull model has a density function given by:

$$f(t) = \frac{\beta}{\theta}\left(\frac{\theta}{t-\varphi}\right)^{\beta+1} \exp\left[-\left(\frac{\theta}{t-\varphi}\right)^{\beta}\right] \quad (1)$$

$t \geq 0;\ \theta, \beta, \varphi > 0$

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II. Acceleration Mechanism

The "Maxwell Distribution Law" is given by:

$$M_{TE} = M_{tot} \times e^{-E/KT} \quad (2)$$

In (2) M_{TE} represents the number of molecules at a particular absolute Kelvin temperature T (Kelvin = 273.16 plus the temperature in Centigrade), that passes a kinetic energy greater than E among the total number of molecules present, M_{tot}; E is the energy of activation of the reaction and K represents the gas constant (1.986 calories per mole). Equation (1) expresses the probability of a molecule having energy in excess of E. The accelerating factor $AF_{2/1}$ (or the ratio of the number of molecules at two different stress Kelvin temperatures T_2 and T_1; $M_{TE}(2)/M_{TE}(1)$), will be given by:

$$AF_{2/1} = \frac{M_{TE}(2)}{M_{TE}(1)} = \frac{e^{-E/KT_2}}{e^{-E/KT_1}}$$

$$AF_{2/1} = \exp\left[\frac{E}{K}\left(\frac{1}{T_1} - \frac{1}{T_2}\right)\right] \quad (3)$$

Applying natural logarithm to both sides of (3) and after some algebraic manipulation, we will obtain:

$$\ln\left(AF_{2/1}\right) = \frac{\mathrm{E}}{\mathrm{K}}\left(\frac{1}{T_1} - \frac{1}{T_2}\right) \quad (4)$$

From (4) we can estimate the term E/K by testing at two different stress temperatures and computing the acceleration factor on the basis of the fitted distributions. Then:

$$\frac{E}{K} = \frac{\ln\left(AF_{2/1}\right)}{\left(\frac{1}{T_1} - \frac{1}{T_2}\right)} \quad (5)$$

The acceleration factor $AF_{2/1}$ will be given by the relationship θ_1/θ_2, with θ_i representing a scale parameter or a percentile at a stress level corresponding to T_i. Once the term E/K is determined, the acceleration factor $AF_{2/n}$ to be applied at the normal stress temperature is obtained from (3) by replacing the stress temperature T_1 with the temperature at normal condition of use T_n. Then:

$$AF_{2/n} = \exp\left[\frac{E}{K}\left(\frac{1}{T_n} - \frac{1}{T_2}\right)\right] \quad (6)$$

De Souza [3] has shown that under a linear acceleration assumption, if a three-parameter Inverse Weibull model represents the life distribution at one stress level, a three-parameter Inverse Weibull model also represents the life distribution at any other stress level. We will be assuming a linear acceleration condition. In general, the scale parameter and the minimum life can be estimated by using two different stress levels (temperature or cycles or miles, etc.), and their ratios will provide the desired value for the acceleration factors AF_θ and AF_φ. Then:

$$AF_\theta = \frac{\theta_n}{\theta_a} \quad (7)$$

$$AF_\varphi = \frac{\varphi_n}{\varphi_a} \quad (8)$$

Now, based on papers by [1] and [3], for the three-parameter Inverse Weibull model the cumulative distribution function at normal testing condition $F_n(t_n - \varphi_n)$ for a certain testing time $t = t_n$, will be given by:

$$F_n(t) = F_a\left(\frac{t}{AF}\right) = \exp\left[-\left(\frac{\frac{\theta_n}{AF}}{AF\left(t - \frac{\varphi_n}{AF}\right)}\right)^{\beta_n}\right] \quad (9)$$

Equation (9) tells us that, under a linear acceleration assumption, if a three-parameter Inverse Weibull model represents the life distribution at one stress level, a three-parameter Inverse Weibull model also represents the life distribution at any other stress level. The shape parameter remains the same while the accelerated scale parameter and the accelerated minimum life are multiplied by the acceleration factor. The equal shape parameter is a necessary mathematical consequence to the other two assumptions, that is; assuming a linear acceleration model and assuming a three-parameter Inverse Weibull sampling distribution. If different stress levels yield data with very different shape parameters, then either the Inverse Weibull sampling distribution is the wrong model for the data or we do not have a linear acceleration condition.

III. The Maximum Likelihood Approach

The likelihood function for the shape, scale and minimum life parameters of an Inverse Weibull sampling distribution for censored Type II data (failure censored) will be given by:

$$L(\beta;\theta;\varphi) = k!\left[\prod_{i=1}^{r} f(t_i)\right]\left[R(t_r)\right]^{n-r} \; ; \; t > 0 \quad (10)$$

The density function $f(t_i)$ will be given by:

$$f(t_i) = \frac{\beta}{\theta}\left(\frac{\theta}{t_i - \varphi}\right)^{\beta+1} \exp\left[-\left(\frac{\theta}{t_i - \varphi}\right)^{\beta}\right] \quad (11)$$

The reliability function $R(t_r)$ will be:

$$R(t_r) = \exp\left[-\left(\frac{\theta}{t_r - \varphi}\right)^{\beta}\right] \quad (12)$$

Then, we will have:

$$L(\beta;\theta;\varphi) = k!\; \beta^r \theta^{\beta r}\left[\prod_{i=1}^{r}\frac{1}{(t_i - \varphi)}\right]^{\beta+1} \times$$
$$\times\; e^{-\sum_{i=1}^{r}(\theta/t_i - \varphi)^{\beta}}\left[e^{-(\theta/t_r - \varphi)^{\beta}}\right]^{n-r} \quad (13)$$

The log likelihood function $L = \ln[L(\beta;\theta;\varphi)]$ will be given by:

$$L = \ln(k!) + r\ln(\beta) + r\beta\ln(\theta) - (\beta+1)\times\sum_{i=1}^{r}\ln(t_i - \varphi) -$$
$$-\sum_{i=1}^{r}\left(\frac{\theta}{t_i - \varphi}\right)^{\beta} - (n-r)\left(\frac{\theta}{t_r - \varphi}\right)^{\beta} \quad (14)$$

To find the value of θ and β that maximizes the log likelihood function, we take θ, β and φ derivatives and make them equal to zero. Then, we will have:

$$\frac{dL}{d\theta} = \frac{r\beta}{\theta} - \beta\theta^{\beta-1}\sum_{i=1}^{r}\left(\frac{1}{t_i - \varphi}\right)^{\beta} - (n-r)\beta\theta^{\beta-1}\left(\frac{1}{t_r - \varphi}\right)^{\beta} = 0 \quad (15)$$

$$\frac{dL}{d\beta} = \frac{r}{\beta} + r\ln(\theta) - \sum_{i=1}^{r}\ln(t_i - \varphi) - \sum_{i=1}^{r}\left(\frac{\theta}{t_i - \varphi}\right)^{\beta} \times$$
$$\times\ln\left(\frac{\theta}{t_i - \varphi}\right) - (n-r)\left(\frac{\theta}{t_r - \varphi}\right)^{\beta}\ln\left(\frac{\theta}{t_r - \varphi}\right) = 0 \quad (16)$$

$$\frac{dL}{d\varphi} = (\beta+1)\sum_{i=1}^{r}\frac{1}{(t_i - \varphi)} - \beta\theta^{\beta} \times$$
$$\times\left[\sum_{i=1}^{r}\left(\frac{1}{t_i - \varphi}\right)^{\beta+1} + (n-r)\left(\frac{1}{t_r - \varphi}\right)^{\beta+1}\right] = 0 \quad (17)$$

From (15) we obtain:

$$\theta = \left(\frac{r}{\sum_{i=i}^{r} \left(\frac{1}{t_i - \varphi} \right)^{\beta} + (n-r) \left(\frac{1}{t_r - \varphi} \right)^{\beta}} \right)^{1/\beta} \quad (18)$$

Notice that, when β = 1, (18) reduces to the maximum likelihood estimator for the inverse two-parameter exponential distribution. Using (18) for θ in (16) and (17) and applying some algebra, (16) and (17) reduce to:

$$\frac{r}{\beta} - \sum_{i=1}^{r} \ln(t_i - \varphi) + \frac{r \times \left[\sum_{i=1}^{r} \left(\frac{1}{t_i - \varphi} \right)^{\beta} \ln(t_i - \varphi) + (n-r) \left(\frac{1}{t_r - \varphi} \right)^{\beta} \ln(t_r - \varphi) \right]}{\sum_{i=1}^{r} \left(\frac{1}{t_i - \varphi} \right)^{\beta} + (n-r) \left(\frac{1}{t_r - \varphi} \right)^{\beta}} = 0 \quad (19)$$

$$(\beta + 1) \sum_{i=1}^{r} \frac{1}{(t_i - \varphi)} - \frac{\beta r \left[\sum_{i=1}^{r} \left(\frac{1}{t_i - \varphi} \right)^{\beta+1} + (n-r) \left(\frac{1}{t_r - \varphi} \right)^{\beta+1} \right]}{\sum_{i=1}^{r} \left(\frac{1}{t_i - \varphi} \right)^{\beta} + (n-r) \left(\frac{1}{t_r - \varphi} \right)^{\beta}} = 0 \quad (20)$$

Equations (19) and (20) must be solved iteratively. The problem was reduced to the simultaneous solution of the two iterative equations (19) and (20). The simultaneous solution of two iterative equations can be seen as relatively simple when compared to the arduous task of solving three simultaneous iterative equations (15)-(17) as outlined by [4]. Even though this is the present case, one possible simplification in solving for estimates when all three parameters are unknown could be the following approach proposed by [5]. For example, let us suppose that $\hat{\beta}$ and $\hat{\theta}$ represent the good linear unbiased estimators (GLUEs) of the shape parameter β and of the scale parameter θ for a fixed value of the minimum life φ. We could choose an initial value for φ to obtain the estimators $\hat{\beta}$ and $\hat{\theta}$, and then apply these two values in (20), that is, the maximum likelihood equation for the minimum life φ. An estimate $\breve{\varphi}$ can then be obtained from (20), then the GLUEs of β and of θ can be recalculated for the new estimate $\breve{\varphi}$, and a second estimate for the minimum life φ obtained from (20). Continuing this iteration would lead to approximate values of the maximum likelihood estimators. As we can notice, the advantage of using the GLUEs in this iteration is that only one equation must be solved implicitly. The existence of solutions to the above set of equations (19) and (20) has been frequently addressed by researchers as there can be more than one solution or none at all; see [6].

The standard maximum likelihood method for estimating the parameters of the three-parameter Wibull and Inverse Weibull models can have problems since the regularity conditions are not met, see [6], [7] and [8]. To overcome this regularity problem, one of the approaches proposed [9] is to replace (20) with the equation

$$n\theta \times \frac{g}{3} \times \left\{ \sum_{i=1}^{k+1} \left[\left(U_i^{-1/\beta} \right) \left(1 - e^{-U_i} \right)^{n} \times (1, 2 \; or \; 4) \right] \right\} + n\,\varphi_j \times \frac{g}{3} \times \left\{ \sum_{i=1}^{k+1} \left[\left(1 - e^{-U_i} \right)^{n} \times (1, 2 \; or \; 4) \right] \right\} = t_1 \quad (21)$$

Here, t_1 is the first order statistic in a sample of size *n*. In solving the maximum likelihood equations, we will use this approach proposed by [9]. The derivation of (21) can be found in the Appendix.

IV. Hypothesis Testing

The hypothesis testing situations were given by [1], [10] and [11]:

1. For the scale parameter θ: H_0: $\theta \geq \theta_0$; H_1: $\theta < \theta_0$

The probability of accepting the null hypothesis H_0 will be set at $(1-\alpha)$ if $\theta = \theta_0$. Now, if $\theta = \theta_1$ where $\theta_1 < \theta_0$, then the probability of accepting H_0 will be set at a low level γ. H_1 represents the alternative hypothesis.

2. For the shape parameter β: H_0: $\beta \geq \beta_0$; H_1: $\beta < \beta_0$

The probability of accepting H_0 will be set again at $(1-\alpha)$ in the case of $\beta = \beta_0$. Now, if $\beta = \beta_1$, where $\beta_1 < \beta_0$, then the probability of accepting H_0 will also be set at a low level γ.

3. For the minimum life parameter φ.

Again, the probability of accepting H_0 will be set at $(1-\alpha)$ if $\varphi = \varphi_0$. Now, if $\varphi = \varphi_1$ where $\varphi < \varphi_0$, then the probability of accepting H_0 will be once more set at a low level γ.

Again, the probability of accepting H_0 will be set at (1-α) if φ = φ_0. Now, if φ = φ_1 where φ < φ_0, then the probability of accepting H_0 will be once more set at a low level γ.

V. Sequential Testing

According to [1], [10] and [11], the development of a sequential test uses the likelihood ratio given by the following relationship:

$$L_{1;n} \big/ L_{0;n} \quad (22)$$

The sequential probability ratio (*SPR*) will be given by *SPR* = $L_{1,1,1,n} / L_{0,0,0,n}$, or yet, according to [1], for the Inverse Weibull model, the sequential probability ratio (*SPR*) will be:

$$SPR=\left(\frac{\beta_1}{\theta_0^{\beta_0}}\times\frac{\theta_1^{\beta_1}}{\beta_0}\right)^n\prod_{i=1}^{n}\left[\frac{\left(t_i-\varphi_0\right)^{\beta_0+1}}{\left(t_i-\varphi_1\right)^{\beta_1+1}}\right]\times$$

$$\times\exp\left[-\sum_{i=1}^{n}\left(\frac{\theta_1^{\beta_1}}{\left(t_i-\varphi_1\right)^{\beta_1}}-\frac{\theta_0^{\beta_0}}{\left(t_i-\varphi_0\right)^{\beta_0}}\right)\right] \quad (23)$$

So, the continue region becomes $A< SPR < B$, where $A = \gamma/(1-\alpha)$ and $B = (1-\gamma)/\alpha$.. We will accept the null hypothesis H_0 if $SPR \geq B$ and we will reject H_0if $SPR \leq A$. Now, if we have $A <SPR< B$, we will take one more observation. Then, by taking the natural logarithm of each term in the above inequality and rearranging them, we get:

$$n\ln\left(\frac{\beta_1}{\theta_0^{\beta_0}}\times\frac{\theta_1^{\beta_1}}{\beta_0}\right)-\ln\left[\frac{(1-\gamma)}{\alpha}\right]< X< n\ln\left(\frac{\beta_1}{\theta_0^{\beta_0}}\times\frac{\theta_1^{\beta_1}}{\beta_0}\right)+\ln\left[\frac{(1-\alpha)}{\gamma}\right] \quad (24)$$

$$X=\sum_{i=1}^{n}\left(\frac{\theta_1^{\beta_1}}{\left(t_i-\varphi_1\right)^{\beta_1}}-\frac{\theta_0^{\beta_0}}{\left(t_i-\varphi_0\right)^{\beta_0}}\right)-\left(\beta_0+1\right)\times$$

$$\times\sum_{i=1}^{n}\ln\left(t_i-\phi_0\right)+\left(\beta_1+1\right)\sum_{i=1}^{n}\ln\left(t_i-\phi_1\right) \quad (25)$$

VI. Expected Sample Size

According to [12] an approximate expression for the expected sample size E(n) of a sequential life testing will be given by:

$$E(n)=\frac{E\left(W_n^*\right)}{E(w)} \quad (26)$$

$$w=\ln\frac{f\left(t;\theta_1,\beta_1\right)}{f\left(t;\theta_o,\beta_o\right)} \quad (27)$$

$$E\left(W_n^*\right)\cong P(\theta,\beta)\ln A+\left[1-P(\theta,\beta)\right]\ln B \quad (28)$$

Again, according to [12], the variate W^*_n takes on only values in which W^*_n exceeds $ln(A)$ or falls short of $ln(B)$. When the true distribution is $f(t;\theta,\beta,\varphi)$, the probability that W^*_n takes the value $ln(A)$ is $P(\theta,\beta,\varphi)$, while the probability that it takes the value $ln(B)$ is equal to $1-P(\theta,\beta,\varphi)$. Hence, with Again, according to [12], the variate W^*_n takes on only values in which W^*_n exceeds $ln(A)$ or falls short of $ln(B)$. When the true distribution is $f(t;\theta,\beta,\varphi)$, the probability that W^*_n takes the value $ln(A)$ is $P(\theta,\beta,\varphi)$, while the probability that it takes the value $ln(B)$ is equal to $1-P(\theta,\beta,\varphi)$. Hence, with $A = \gamma/(1-\alpha)$ and also with $B = (1-\gamma)/\alpha$, (26) becomes:

$$E(n)\approx\frac{P(\theta,\beta,\phi)\ \ln(A)+\left[1-P(\theta,\beta,\phi)\right]\ \ln(B)}{E(w)} \quad (29)$$

Equation (27) enables one to compare sequential tests with fixed sample size tests. The proofs of the existence of (26) to (29) can be found in [12], pp. 391-392. For a three-parameter Inverse Weibull sampling distribution, the expected value of (27) will be given by:

$$E(w)=\ln\left(\frac{\beta_1}{\theta_0^{\beta_0}}\times\frac{\theta_1^{\beta_1}}{\beta_0}\right)+\left(\beta_0+1\right)\ E\left[\ln\left(t_i-\varphi_0\right)\right]$$

$$-\left(\beta_1+1\right)\ E\left[\ln\left(t_i-\varphi_1\right)\right]-\theta_1^{\beta_1}\ E\left[\frac{1}{\left(t_i-\varphi_1\right)}\right]^{\beta_1}+$$

$$+\theta_0^{\beta_0}\ E\left[\frac{1}{\left(t_i-\varphi_0\right)}\right]^{\beta_0} \quad (30)$$

The solution for the components of (30) can be found in [1].

VII. Example

We are trying to determine the values of the shape, scale and minimum life parameters of an underlying three-parameter Inverse Weibull model, representing the life cycle of a low alloy high-strength steel component used in the construction of overpasses in Brazil. Once a life curve for this steel component is determined, we will be able to verify using sequential life testing, if new units produced will have the necessary required characteristics. It happens that the amount of time available for testing is considerably less than the expected lifetime of the component. So, we will have to rely on an accelerated life testing procedure to obtain failure times used on the parameters estimation procedure. The steel component has a normal operating temperature of 296 K (about 23 degrees Centigrade). Under stress testing at 490 K, 16 steel component items were subjected to testing, with the testing being truncated at the moment of occurrence of the twelfth failure. Table I shows these failure time data (hours multiply by10^{-2}).

TABLE I
FAILURE TIMES (HOURS × 10^2) OF STEEL COMPONENT ITEMS TESTED UNDER ACCELERATED TEMPERATURE CONDITIONS (490 K)

661.1	687.6	683.1
697.3	719.8	740.5
753.6	785.2	792.8
801.2	833.0	1,002.4

Now, under stress testing at 530 K, 16 steel rail items were again subjected to testing, with the testing being truncated at the moment of occurrence of the twelfth failure. Table II shows these failure time data (hours x 10^{-2}).

TABLE II
FAILURE TIMES (HOURS × 10^2) OF STEEL COMPONENT ITEMS TESTED UNDER ACCELERATED TEMPERATURE CONDITIONS (530 K)

781.0	861.2	868.1
880.2	895.6	909.6
928.3	950.3	972.2
993.5	1,035.8	1,147.0

Using the maximum likelihood estimator approach for the shape parameter β, for the scale parameter θ and for the minimum life φ of the Inverse Weibull model for censored Type II data (failure censored), we obtain the following values for these three parameters under accelerated conditions of testing:

At 490 K.

$\beta_1 = \beta_n = \beta = 8.38;\ \theta_1 = 655.7 \text{ x } 10^2$ hours;

$\varphi_1 = 120.3 \text{ x } 10^2$ hours

At 530 K.

$\beta_2 = \beta_n = \beta = 8.41;\ \theta_2 = 559.4 \text{ x } 10^2$ hours;

$\varphi_2 = 102.3 \text{ x } 10^2$ hours

The shape parameter did not change with $\beta \approx 8.4$. The acceleration factor for the scale parameter $AF\theta_{2/1}$ will be given by:

$$AF\theta_{2/1} = \theta_1/\theta_2 = 655.7/559.4$$

Using (5), we can now estimate the term *E/K*.

$$\frac{E}{K} = \frac{\ln\left(AF_{2/1}\right)}{\left(\frac{1}{T_1} - \frac{1}{T_2}\right)} = \frac{\ln(655.7/559.4)}{\left(\frac{1}{490} - \frac{1}{530}\right)} = 1031.3$$

Using (6), the acceleration factor for the scale parameter, to be applied at the normal stress temperature $AF\theta_{2/n}$, will be:

$$AF_{2/n} = \exp\left[\frac{E}{K}\left(\frac{1}{T_n} - \frac{1}{T_2}\right)\right]$$

$$AF_{2/n} = \exp\left[1{,}031.3\left(\frac{1}{296} - \frac{1}{530}\right)\right] = 4.66$$

Therefore, the scale parameter of the component at normal operating temperatures is estimated to be:

$$\theta_n = AF_{2/n} \times \theta_2 = 4.66 \times 559.4 \text{ x } 10^2$$

$$\theta_n = 2{,}606.8 \text{ x } 10^2 \text{ hours}$$

The acceleration factor for the minimum life parameter $AF\varphi_{2/1}$ will be given by:

$$AF\varphi_{2/1} = \frac{\varphi_1}{\varphi_2} = \frac{120.3}{102.3}$$

Again applying (5), we can again estimate the term *E/K*. Then:

$$\frac{E}{K} = \frac{\ln\left(AF_{2/1}\right)}{\left(\frac{1}{T_1} - \frac{1}{T_2}\right)} = \frac{\ln(120.3/102.3)}{\left(\frac{1}{490} - \frac{1}{530}\right)} = 1{,}052.3$$

Using once more (6), the acceleration factor for the minimum life parameter, to be applied at the normal stress temperature $AF\varphi_{2/n}$, will be:

$$AF\varphi_{2/n} = \exp\left[1{,}052.3\left(\frac{1}{296} - \frac{1}{530}\right)\right] = 4.80$$

Then, as we expected, $AF_\theta = 4.66 \approx AF_\varphi = 4.80 \approx AF = 4.7$. Finally, the minimum life parameter of the component at normal operating temperatures is estimated to be:

$$\varphi_n = AF\varphi_{2/n} \times \varphi_2 = 4.7 \times 102.3 \text{ x } 10^2$$

$$\varphi_n = 480.8 \text{ x } 10^2 \text{ hours}$$

Then, the steel rail life when operating at normal use conditions could be represented by a three-parameter Inverse Weibull model having a shape parameter β of 8.4; a scale parameter θ of 2,606.8 x 10^2 hours and a minimum life φ of 480.8 x 10^2 hours. To evaluate the accuracy (significance) of the three-parameter values obtained under normal conditions for the underlying Inverse Weibull model we will apply, to the expected normal failure times, a sequential life testing using a truncation mechanism developed by [1]. These expected normal failure times will be acquired by multiplying the twelve failure times obtained under accelerated testing conditions at 530 K given by Table II by the accelerating factor *AF* of 4.7. It was decided that the value of α was 0.05 and γ was 0.10. In this example, the following values for the alternative and null parameters were chosen: alternative scale parameter θ_1= 2,400 x 10^2 hours, alternative shape parameter β_1 = 7.8 and alternative location parameter φ_1 = 420 x 10^2 hours; null scale parameter θ_0 = 2,607 x 10^2 hours, null shape parameter β_0 = 8.4 and null minimum life parameter φ_0 = 480 x 10^2 hours. Now electing $P(\theta,\beta,\varphi)$ to be 0.01, we can calculate the expected sample size *E(n)* of this sequential life testing under analysis. Using now (30), the expression for the expected sample size of the sequential life testing for truncation purpose *E(n)*, we will have:

$E(w) = -5.439 + 9.4 \times 7.702 - 8.8 \times 7.663 - 0.038 + 1 = 0.4874$

Now, with $P(\theta, \beta, \varphi) = 0.01$; also with

$$\ln(B) = \ln\left[\frac{(1-\gamma)}{\alpha}\right] = \ln\left[\frac{(1-0.10)}{0.05}\right] = 2.8904,$$

and having

$$\ln(A) = \ln\left(\frac{\gamma}{1-\alpha}\right) = \ln\left(\frac{0.10}{1-0.05}\right) = -2.2513,$$

we will obtain:

$$P(\theta,\beta)\ln(A) + [1 - P(\theta,\beta)]\ln(B) =$$

$$= -0.01 \times 2.2513 + 0.99 \times 2.8904 = 2.8390$$

$$\text{Finally: } E(n) = \frac{2.8390}{0.4874} = 5.825 \approx 6 \text{ items}$$

Therefore, we could make a decision about accepting or rejecting the null hypothesis H_0after the analysis of observation number 6. Using (24) and (25) and the twelve failure times obtained under accelerated conditions at 530 K given by Table II, multiplied by the accelerating factor *AF* of 4.7, we calculate the sequential life testing limits. Table III shows these estimated failure times (hours x 10^{-2}) under normal temperature conditions (296 K).

TABLE III
FAILURE TIMES (HOURS × 10^2) OF STEEL COMPONENT ITEMS TESTED UNDER NORMAL TEMPERATURE CONDITIONS (296 K)

3,130.7	3,231.7	3,210.6
3,277.3	3,383.1	3,480.4
3,541.9	3,690.4	3,726.2
3,765.6	3,915.1	4,711.3

VIII. EARLY TRUNCATION

Fig. 1 below shows the sequential life-testing for the three-parameter Inverse Weibull model. According to [10], when the truncation point is reached, a line partitioning the sequential graph can be drawn as shown in Fig. 1.

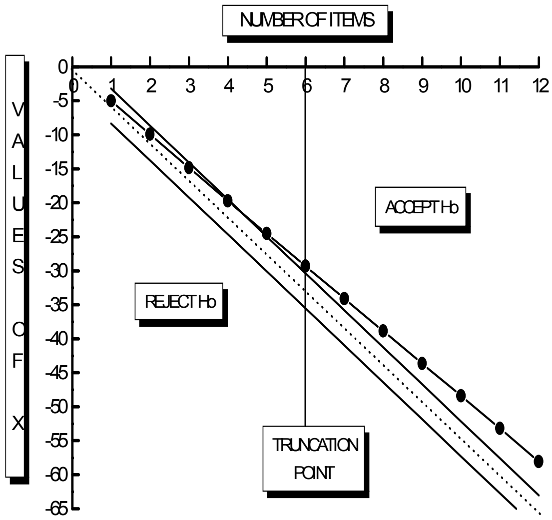

Fig. 1 Sequential test graph for the three-parameter Inverse Weibull model

This line is drawn through the origin of the graph parallel to the accept and reject lines. The decision to accept or reject H_0 simply depends on which side of the line the final outcome lies. Obviously this procedure changes the levels of α and γ of the original test; however, the change is slight if the truncation point is not too small (less than four). But since we were able to make a decision about accepting or rejecting the null hypothesis H_0 after the analysis of observation number 5, we do not have to analyze a number of observations corresponding to the truncation point (6 observations). As we can see in Fig. 1, the null hypothesis H_0 should be accepted since the final observation (observation number 5) lays on the region related to the acceptance of H_0.

IX. CONCLUSIONS

In this work we life-tested a new industrial component using an accelerated mechanism. We assumed a linear acceleration condition. To estimate the parameters of the three-parameter Inverse Weibull model we used a maximum likelihood approach for censored failure data, since the life-testing will be terminated at the moment the truncation point is reached. The shape parameter remained the same while the accelerated scale parameter and the accelerated minimum life parameter were multiplied by the acceleration factor. The equal shape parameter is a necessary mathematical consequence of the other two assumptions; that is, assuming a linear acceleration model and a three-parameter Inverse Weibull sampling distribution. If different stress levels yield data with very different shape parameters, then either the three-parameter Inverse Weibull sampling distribution is the wrong model for the data or we do not have a linear acceleration condition. In order to translate test results obtained under accelerated conditions to normal using conditions we applied some reasoning given by the "Maxwell Distribution Law." To evaluate the accuracy (significance) of the three-parameter values estimated under normal conditions for the underlying Inverse Weibull model we employed, to the expected normal failure times, a sequential life testing using a truncation mechanism developed by [1]. These expected normal failure times were acquired by multiplying the twelve failure times obtained under accelerated testing conditions at 530 K given by Table II, by the accelerating factor AF of 4.7. Since we were able to make a decision about accepting or rejecting the null hypothesis H_0 after the analysis of observation number 5, we did not have to analyze a number of observations corresponding to the truncation point (6 observations). As we saw in Fig. 1, the null hypothesis H_0 should be accepted since the final observation (observation number 5) lays on the region related to the acceptance of H_0. Therefore, we accept the hypothesis that the friction-resistant low alloy-high strength steel rails life when operating at normal use conditions could be represented by a three-parameter Inverse Weibull model having a shape parameter β of 8.4; a scale parameter θ of 2,606.8 x 10^2 hours and a minimum life φ of 480.8 x 10^2 hours.

APPENDIX

EQUATION FOR THE MINIMUM LIFE Φ

The probability density function (pdf) of t_1 will be given by:

$$f(t_1) = \mathrm{n}\left[R(t_1)\right]^{n-1} f(t_1)$$

For the three-parameter Inverse Weibull sampling distribution, we will obtain:

$$f(t_1) = \frac{n\beta}{\theta}\left(\frac{\theta}{t-\varphi}\right)^{\beta+1}\left\{1-\exp-\left[\left(\frac{\theta}{t-\varphi}\right)^{\beta}\right]\right\}^{n}$$

The expected value of t_1 is given by:

$$E(t_1) = \int_{\varphi}^{\infty}\frac{n\beta}{\theta}\,\mathrm{t}\left(\frac{\theta}{t-\varphi}\right)^{\beta+1}\left\{1-\exp-\left[\left(\frac{\theta}{t-\varphi}\right)^{\beta}\right]\right\}^{n}\mathrm{dt}$$

Letting $U=\left(\frac{\theta}{t-\varphi}\right)^{\beta}$, we will have:

$$du = \left(-\frac{\beta}{\theta}\right)\left(\frac{\theta}{t-\varphi}\right)^{\beta+1}dt\;;\; t = \frac{\theta}{U^{1/\beta}}+\varphi$$

when $t\to\infty, U\to 0$; When $t\to\varphi$, $U\to\infty$. Then, we will have:

$$E(t_1) = \int_0^{\infty} n\left(\theta U^{-1/\beta}+\varphi\right)\left[1-e^{-U}\right]^{n}du$$

$$E(t_1) = n\theta\int_0^{\infty}U^{-1/\beta}\left[1-e^{-U}\right]^{n}du + n\varphi\int_0^{\infty}\left[1-e^{-U}\right]^{n}du$$

The above integrals have to be solved by using a numerical integration procedure, such as Simpson's 1/3 rule. Remembering that Simpson's 1/3 rule is given by:

$$\int_a^b f(x)\,dx = \frac{g}{3}\left(f_1+4f_2+2f_3+\cdots+4f_k+f_{k+1}\right) - \text{error}$$

Then, making the error = 0; and with i= 1,2,...,k+1, we will get:

$$n\theta\int_0^{\infty}U^{-1/\beta}\left[1-e^{-U}\right]^{n}du = n\times\theta\times\frac{g}{3}\times \times\left\{\sum_{i=1}^{k+1}\left[\left(U_i^{-1/\beta}\right)\left(1-e^{-U_i}\right)^{n}\times(1,2\;or\;4)\right]\right\} \quad \text{(A)}$$

$$n\varphi\int_0^{\infty}\left[1-e^{-U}\right]^{n}du = n\times\varphi_j\times\frac{g}{3}\times \times\left\{\sum_{i=1}^{k+1}\left[\left(1-e^{-U_i}\right)^{n}\times(1,2\;or\;4)\right]\right\} \quad \text{(B)}$$

Using (A) and (B), we will have:

$$E(t_1) = n\theta\int_0^{\infty}U^{-1/\beta}\left[1-e^{-U}\right]^{n}du + n\varphi\int_0^{\infty}\left[1-e^{-U}\right]^{n}du$$

Finally:

$$n\theta\times\frac{g}{3}\times\left\{\sum_{i=1}^{k+1}\left[\left(U_i^{-1/\beta}\right)\left(1-e^{-U_i}\right)^{n}\times(1,2\;or\;4)\right]\right\}+ +n\varphi_j\times\frac{g}{3}\times\left\{\sum_{i=1}^{k+1}\left[\left(1-e^{-U_i}\right)^{n}\times(1,2\;or\;4)\right]\right\} = t_1$$

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