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Vibration Analysis of Functionally Graded Engesser-Timoshenko Beams Subjected to Axial Load Located on a Continuous Elastic Foundation

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Abstract**—**This paper studies free vibration of functionally graded beams Subjected to Axial Load that is simply supported at both ends lies on a continuous elastic foundation. The displacement field of beam is assumed based on Engesser-Timoshenko beam theory. The Young's modulus of beam is assumed to be graded continuously across the beam thickness. Applying the Hamilton's principle, the governing equation is established. Resulting equation is solved using the Euler's Equation. The effects of the constituent volume fractions and foundation coefficient on the vibration frequency are presented. To investigate the accuracy of the present analysis, a compression study is carried out with a known data.

Keywords**—**Functionally Graded Beam, Free Vibration, Elastic Foundation, Engesser-Timoshenko Beam Theory.

I. INTRODUCTION

PIEZOELECTRIC materials have been widely used in electromechanical systems such as piezoelectric transducers [1], micromechanical flying insects [2] and optical beam deflectors [3]. However, there are some weaknesses in the traditional laminated smart devices such as they may crack at low temperature at high temperature, which will reduce the life-time and reliability of these piezoelectric devices. Therefore, increasingly more sensors and actuators are being designed with graded properties [4]. Based on the theory of elasticity and piezoelectricity, Airy stress function method has been extended to solve some static piezoelectric problems such as piezoelectric cantilevers or curved beams with graded profiles under different loading cases [5]–[8]. Based on the layerwise theory, Saravanos [9] gave a numerical investigation of the dynamic behavior for piezoelectric laminates. Kapuria and Alam [10] presented a finite element model for dynamic analysis of hybrid piezoelectric beams based on the zigzag theory. By introducing displacement functions and using the Frobenius power-series method, Chen [11] recently considered the vibration problem of spherically isotropic piezoelastic spheres with a functionally graded property that the material constants vary with the radial co-ordinate in a power law. Also, some of the studies in piezoelectric materials and FGM structures can be found in [12], [13]. However, with the increasing number of involved layers, conventional methods used by many authors usually lead to lower numerical exigency. The state-space method has shown to be very effective in the analysis of laminated structures because of the associated lower order solving matrix. Its recent applications in piezoelectric materials and structures can be found in [14]-[17].

In the present work, the free vibration of a functionally graded beam Subjected to axial load that is simply supported at both ends lies on a continuous elastic foundation is studied. The displacement field of beam is assumed based on Engesser-Timoshenko beam theory. The elasticity modulus of functionally graded layer is assumed to vary as a power form of the thickness coordinate variable. Appling the Hamilton's principle, the governing equation of beam is derived and solved. The effects of foundation coefficient and functionally graded index on the vibration frequency of beam are also discussed.

II. FORMULATIONS

Consider a functionally graded beam with rectangular cross-section as shown in Fig. 1. The thickness, length, and width of the beam are denoted, respectively, by $h, L,$ and b. The $x-y$ plane coincides with the midplane of the beam and the $z-$axis located along the thickness direction.

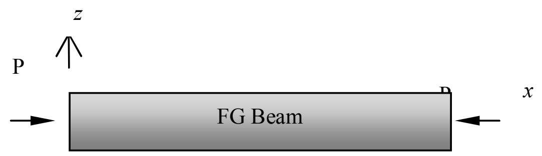

Fig. 1 Schematic view of the problem studied.

The Young's modulus E is assumed to vary as a power form of the thickness coordinate variable z $(-h/2 \le z \le h/2)$ as follow [18]

$$E(z) = (E_c - E_m)V + E_m$$
$$V = \left(\frac{2z+h}{2h}\right)^k \qquad (1)$$

where k is the power law index and the subscripts m and c refer to the metal and ceramic constituents, respectively. The Poisson's ratio ν is assumed to be constant. The displacement

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field of beam is assumed based on Engesser-Timoshenko beam theory. The constitutive relations for functionally graded beam are given by:

$$\sigma_{xx} = Q_{11}(z)\varepsilon_{xx}$$
$$\sigma_{xz} = Q_{55}(z)\gamma_{xz} \tag{2}$$

where

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = z\frac{d\phi}{dx}$$
$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \phi + \frac{dw}{dx} \tag{3}$$

where σ_{xx}, σ_{xz}, Q_{11} and Q_{55} are the normal ,shear stresses and plane stress-reduced stiffnesses respectively. Also, u and w are the displacement components in the $x-$ and $z-$ directions, respectively.

The potential energy can be expressed as [18]:

$$U = \frac{1}{2}\int_v (\sigma_{xx}\varepsilon_{xx} + \sigma_{xz}\gamma_{xz})\,\mathrm{d}v \tag{4}$$

Substituting (2) and (3) into (4) and neglecting the higher-order terms, we obtain

$$U = \frac{1}{2}\int_v [(Q_{11}\left(z\frac{d\varphi}{dx}\right))(z\frac{d\varphi}{dx}) + (Q_{55}(\varphi + \frac{dw}{dx}))(\varphi + \frac{dw}{dx})]dv \tag{5}$$

The width of beam is assumed to be constant, which is obtained by integrating along y over v. Then (5) becomes

$$U = \frac{b}{2}\int_0^L [D\left(\frac{d\varphi}{dx}\right)^2 + \frac{A}{2(1+\nu)}(\varphi^2 + (\frac{dw}{dx})^2 + 2\varphi\frac{dw}{dx})]dx \tag{6}$$

where

$$A = \int_{-\frac{h}{2}}^{+\frac{h}{2}} Q_{55}(\mathrm{z})\mathrm{d}z \tag{7}$$

and

$$D = \int_{-\frac{h}{2}}^{+\frac{h}{2}} z^2 Q_{11}(\mathrm{z})\mathrm{d}z \tag{8}$$

where A and D are the shear rigidity and flexural rigidity respectively. Note that, the extensional displacement is neglected. Thus, the potential energy can be written as

$$U = \frac{b}{2}\int_0^L [D\,(\frac{d\phi}{dx})^2 + A(\phi^2 + (\frac{dw}{dx})^2 + 2\phi\frac{dw}{dx}]dx \tag{9}$$

The work done by the axial compressive load can be expressed

$$W = \frac{1}{2}\int_0^L P\left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x \tag{10}$$

The kinetic energy can be expressed as [18]:

$$T = \frac{1}{2}\int_0^L \{I_0\left(\frac{\partial w}{\partial t}\right)^2 + I_2\left(\frac{\partial \phi_x}{\partial t}\right)^2\}dx \tag{11}$$

where

$$I_i = \int_{-h_B-\frac{h}{2}}^{h_T+\frac{h}{2}} \rho(z)^i dz$$
$$I_0 = \int_{-h_B-\frac{h}{2}}^{h_T+\frac{h}{2}} \rho(z)^0 dz, I_2 = \int_{-h_B-\frac{h}{2}}^{h_T+\frac{h}{2}} \rho(z)^2 dz \tag{12}$$

We apply the Hamilton's principle to derive the dynamic equation of beam, that is

$$\delta\int_0^t (T - U + W)\,\mathrm{d}t = 0 \tag{13}$$

Substitution from (9), (10) and (11) into (13) leads to the following dynamic equations

$$I_0\frac{\partial^2 w}{\partial t^2} - P\frac{\partial^2 w}{\partial x^2} - bA(\frac{\partial^2 w}{\partial x^2} + \frac{\partial \phi}{\partial x}) = 0$$
$$I_2\frac{\partial^2 \phi}{\partial t^2} - bD_{11}\frac{\partial^2 \phi}{\partial x^2} + bA(\phi + \frac{\partial w}{\partial x}) = 0 \tag{14}$$

Assume that a functionally graded beam that is simply supported at both ends lies on a continuous elastic foundation, whose reaction at every point is proportional to the deflection (Winkler foundation). The dynamic equation of the functionally graded beams located on a continuous elastic foundation subjected to a periodic axial compressive load is obtained from (14) by the addition of ηw for the foundation reaction as

$$I_0\frac{\partial^2 w}{\partial t^2} - P\frac{\partial^2 w}{\partial x^2} - bA(\frac{\partial^2 w}{\partial x^2} + \frac{\partial \phi}{\partial x}) + \eta w = 0$$
$$I_2\frac{\partial^2 \phi}{\partial t^2} - bD_{11}\frac{\partial^2 \phi}{\partial x^2} + bA(\phi + \frac{\partial w}{\partial x}) = 0 \tag{15}$$

where η is the foundation coefficient.

III. Vibration Analysis

For the simply supported boundary condition, the solution of the dynamic equation is assumed to be in the following form:

$$w(x,t) = Ce^{i\omega_n}\sin\frac{n\pi}{L}x \qquad n = 1,2,3,... \tag{16}$$

Substituting expression (16) into (15) and by using finite element method, the free vibration frequency of a functionally graded beam Subjected to axial load that is simply supported at both ends lies on a continuous elastic foundation will be derived. When the beam is assumed to be slender by setting the power law index equal to zero $(k = 0)$ and foundation coefficient the nth free vibration frequency of homogeneous beams can be written as

$$\Omega_a = \overline{\omega_k} \cdot \sqrt{1 - \frac{p}{p^*_k}} \tag{17}$$

where

$$\overline{\omega_n} = \left(\frac{n\pi}{L}\right)^2 \sqrt{\frac{bD}{m}} \tag{18}$$

$$P_n^* = \left(\frac{n\pi}{L}\right)^2 bD \tag{19}$$

and

$$D = \frac{1}{1-\nu^2} \int_{-\frac{h}{2}}^{\frac{h}{2}} z^2 E \mathrm{d}z \tag{20}$$

Equation (17) has been reported by Bolotin [19].

IV. Numerical Results

The free vibrations of functionally graded beams subjected to axial compressive loads located on a continuous elastic foundation are studied in this paper. The material properties of the beam are listed in Table I.

TABLE I
MATERIAL PROPERTIES

Property	FGM layer	
	Aluminium	Alumina
Young's modulus E (GPa)	70	380
Poisson's ratio ν	0.3	0.3
Length L (m)	0.4	0.4
Thickness h (m)	0.01	0.01
Density ρ ($\mathrm{Kgm^{-3}}$)	2707	3800

Fig. 2 illustrates the effect of the axial compressive load and the foundation coefficient on the free vibration frequency for $K = 1$. As the axial compressive load increases, the free vibration frequencies decrease. Also, the effect of the foundation coefficient on the free vibration frequency is shown in Fig. 2. It is evident that the free vibration frequencies increase when the foundation coefficient increase.

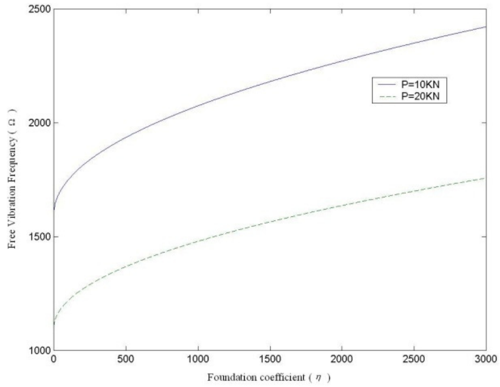

Fig. 2 Effect of axial compressive load on the free vibration frequency

V. Conclusions

The free vibration analysis of a functionally graded beam subjected to the axial compressive loads that is simply supported at both ends lies on a continuous elastic foundation has been presented. The displacement field of beam is assumed based on Engesser-Timoshenko beam theory. The free vibration frequencies increase when the foundation coefficient increase .The functionally graded beam with a smaller axial compressive load is more efficient in reducing the chance of resonance. The comparison of the stability for the functionally graded beam and isotropic beam shows that the functionally graded beam is more stable.

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