

Properties of a stochastic predator-prey system with Holling II functional response

Xianqing Liu, Shouming Zhong, Fuli Zhong and Zijian Liu

Abstract—In this paper, a stochastic predator-prey system with Holling II functional response is studied. First, we show that there is a unique positive solution to the system for any given positive initial value. Then, stochastically bounded of the positive solution to the stochastic system is derived. Moreover, sufficient conditions for global asymptotic stability are also established. In the end, some simulation figures are carried out to support the analytical findings.

Keywords—stochastically bounded, global stability, Holling II functional response, white noise, Markovian switching.

I. INTRODUCTION

RECENTLY, more and more scholars pay attention to the predator-prey system because of its theoretical and practical significance. The earliest predator-prey system is the Lotka-Volterra model^[1,2], governed by the following differential equations

$$\begin{cases} \dot{x}(t) = x(t)(a - by(t)), \\ \dot{y}(t) = y(t)(-c + fx(t)). \end{cases} \quad (1.1)$$

As a matter of fact, there are many significant elements of the predator-prey relationship, such as the predator's functional response which is the rate of prey consumption by an average predator. Much works which considering different functional response types of the original model have been done. And Holling II functional response is one of the most important functional responses. In [3], Li and Gao introduced the special predator-prey system with Holling II functional response, which also considered the density-dependence of predator population:

$$\begin{cases} \dot{x}(t) = b_1x \left[1 - a_{11}x - \frac{a_{12}y}{1+x} \right], \\ \dot{y}(t) = b_2y \left[-1 + \frac{a_{21}x}{1+x} - a_{22}y \right]. \end{cases} \quad (1.2)$$

However, we have no choice but to admit that all population systems are often subject to environmental noises. So, considering the corresponding stochastic population is necessary and important. Therefore, many people start to be concerned about stochastic population systems (see e.g.[4-8]). Lv and Wang^[6] proposed and investigated the stochastic predator-prey system with Holling II functional response, which taking

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into account the effect of randomly fluctuating environment in the growth rates. There are two noise sources in [6], but their coupled mode is very simple. We know one noise source not only has influence on the growth rate of predator but also on the prey's. In many cases, the growth rates are often subject to environmental noise as well. For example, the growth rates of many species will be different between the rainy season and the dry season. Moreover, the changes in nutrition and food resources also have effect on the carrying capacities. Generally, we regard the growth rates as Markov jump process which can modelled by Markov chains. Also, [6] only considered white noise and ignored Markovian switching. Therefore, from the argument above, we study the following form in this paper:

$$\begin{cases} dx = x \left[1 - a_{11}(\xi(t))x - \frac{a_{12}(\xi(t))y}{1+x} \right] \left[b_1(\xi(t))dt + \sigma_1(\xi(t))dB_1(t) + \mu_1(\xi(t))dB_2(t) \right], \\ dy = y \left[-1 + \frac{a_{21}(\xi(t))x}{1+x} - a_{22}(\xi(t))y \right] \left[b_2(\xi(t))dt + \mu_2(\xi(t))dB_1(t) + \sigma_2(\xi(t))dB_2(t) \right], \end{cases} \quad (1.3)$$

where $x(t)$ and $y(t)$ stand for the population densities of prey and predator at time t , respectively; $b_i(\cdot)$ is called the intrinsic growth rate; $a_{ij}(\cdot)$ is positive parameter function, $i, j = 1, 2$. For system (1.3), $\xi(t)$ is a right-continuous Markov chain taking values in a finite state space $S = \{1, 2, \dots, N\}$ and $B_i(t)$ is the 1-dimensional standard Brownian motion. We always suppose $\xi(t)$ is F_t -adapted but independent of the Brownian motion $B_i(t)$, $i = 1, 2$. $\sigma_i^2(\cdot)$ and $\mu_i^2(\cdot)$ represent the intensities of the white noises, $i = 1, 2$. The model (1.3) together with the initial conditions $x(0) = x_0 > 0, y(0) = y_0 > 0$ will be referred to as model (SBD).

Throughout this paper, we denote by R_+^2 the positive cone in R^2 , and also denote by $X(t) = (x(t), y(t))$ and $|X(t)| = (x(t)^2 + y(t)^2)^{\frac{1}{2}}$. Let $(\Omega, F, \{F_t\}_{t \geq 0}, P)$ be a complete probability space with a filtration $\{F_t\}_{t \geq 0}$ satisfying the usual conditions, i.e. it is right continuous and increasing while F_0 contains all P -null sets. We also denote $f^* = \sup_{1 \leq i \leq N} f(n)$ and $f_* = \inf_{1 \leq i \leq N} f(n)$ for any bounded sequence $\{f(n)\}$. What's more, let $\{\xi(t), t \geq 0\}$ be a right-continuous Markov chain on the probability space (Ω, F, Υ) and take values in a finite space $S = \{1, 2, \dots, N\}$ with the generator $\Gamma = (\pi_{ij})_{N \times N}$ given by

$$\Gamma[\xi(t + \Delta) = j | \xi(t) = i] = \begin{cases} \pi_{ij}\Delta + o(\Delta), & \text{if } i \neq j, \\ 1 + \pi_{ii}\Delta + o(\Delta), & \text{if } i = j, \end{cases}$$

where $\Delta > 0$, $\lim_{\Delta \rightarrow 0} \frac{o(\Delta)}{\Delta} = 0$. Here $\pi_{ij} \geq 0 (\forall i, j \in S, i \neq j)$ is the transition rate from mode i to mode j , represents the expectation of the species which migrate from patch i to patch j , while

$$\pi_{ii} = - \sum_{j=1, j \neq i}^N \pi_{ij}.$$

The rest of this paper is organized as follows. In the next section, we show that the stochastic system (1.3) has a unique global (no explosion in a finite time) solution for any positive initial value. The stochastic boundedness of system (1.3) is studied in Section 3, and some sufficient conditions which guarantee the global asymptotic stability of positive solutions are obtained in Section 4. In the end, numerical simulations are carried out to support our analytical findings.

II. POSITIVE AND GLOBAL SOLUTION

For $x(t)$ and $y(t)$, we are only interested in the positive solutions because they are population densities of the prey and the predator respectively in system (1.3). Furthermore, the coefficients of the equation satisfying the linear growth condition and local Lipschitz condition is a guarantee of the stochastic differential equation to have a unique global (i.e., no explosion in a finite time) solution for any given initial value, (see e.g. [9]).

Lemma 2.1 There is a unique positive local solution $(x(t), y(t))$ for $t \in [0, \tau_e)$ of system (1.3) almost surely (a.s.) for any initial value $x_0 > 0, y_0 > 0$.

Proof: We consider the equation

$$\begin{cases} du = \left\{ b_1(\xi(t)) \left[1 - a_{11}(\xi(t))e^u - \frac{a_{12}(\xi(t))e^v}{1+e^u} \right] - \frac{\sigma_1^2(\xi(t)) + \mu_1^2(\xi(t))}{2} \left[1 - a_{11}(\xi(t))e^u - \frac{a_{12}(\xi(t))e^v}{1+e^u} \right] \times e^v \right\} dt + \left[1 - a_{11}(\xi(t))e^u - \frac{a_{12}(\xi(t))e^v}{1+e^u} \right] \times [\sigma_1(\xi(t))dB_1(t) + \mu_1(\xi(t))dB_2(t)], \\ dv = \left\{ b_2(\xi(t)) \left[-1 + \frac{a_{21}(\xi(t))e^u}{1+e^u} - a_{22}(\xi(t))e^v \right] - \frac{\sigma_2^2(\xi(t)) + \mu_2^2(\xi(t))}{2} \left[-1 + \frac{a_{21}(\xi(t))e^u}{1+e^u} - a_{22}(\xi(t))e^v \right] \times e^v \right\} dt + \left[-1 + \frac{a_{21}(\xi(t))e^u}{1+e^u} - a_{22}(\xi(t))e^v \right] \times [\mu_2(\xi(t))dB_1(t) + \sigma_2(\xi(t))dB_2(t)], \end{cases} \quad (2.1)$$

on $t \geq 0$ with initial value $u(0) = \ln x_0, v(0) = \ln y_0$. Obviously, the coefficients of system (1.3) satisfy the local Lipschitz condition, then there is a unique local solution $u(t), v(t)$ on $[0, \tau_e)$. Therefore, by Itô's formula, $x(t) = e^{u(t)}, y(t) = e^{v(t)}$ is the unique positive local solution to (1.3) with initial value $x_0 > 0, y_0 > 0$.

Lemma 2.1 only shows that there is a unique positive local solution of system (1.3). Next, we show this solution is global, i.e., $\tau_e = \infty$. Before giving Theorem 2.1, we assume that

$$(A_1) \quad b_{1*} - \frac{(\sigma_1^*)^2 + (\mu_1^*)^2}{2} a_{11}^* > 0;$$

$$(A_2) \quad b_{2*} - \frac{1}{2} \left[((\sigma_2^*)^2 + (\mu_2^*)^2) a_{22}^* + ((\sigma_1^*)^2 + (\mu_1^*)^2) \frac{(a_{12}^*)^2}{a_{22}^*} \right] > 0.$$

Theorem 2.1 Assume (A_1) and (A_2) hold. For any given initial value $X_0 = (x_0, y_0) \in R_+^2$, there is a unique solution $X(t) = (x(t), y(t))$ to stochastic differential equation (1.3) on $t \geq 0$ and the solution will remain in R_+^2 with probability 1, where $R_+^2 = \{x \in R^2 | x_i > 0, i = 1, 2\}$.

Proof: The following proof is motivated by the work Luo and Mao^[10]. Let $m_0 > 0$ be sufficiently large for x_0 and y_0 lying within the interval $[\frac{1}{m_0}, m_0]$. For each integer $m > m_0$, define the stopping times

$$\tau_m = \inf \left\{ t \in [0, \tau_e) : x(t) \notin \left(\frac{1}{m}, m \right) \text{ or } y(t) \notin \left(\frac{1}{m}, m \right) \right\}$$

where throughout this paper we set $\inf \emptyset = \infty$ ($\emptyset$ stands for the empty set). Clearly, τ_m is increasing as $m \rightarrow \infty$. Let $\tau_\infty = \lim_{m \rightarrow +\infty} \tau_m$, whence $\tau_\infty \leq \tau_e$ a.s. Now, we only need to show $\tau_\infty = \infty$. If this statement is false, there is a pair of constants $T > 0$ and $\varepsilon \in (0, 1)$ such that $P\{\tau_\infty \leq T\} > \varepsilon$. Consequently, there exists an integer $m_1 \geq m_0$ such that

$$P\{\tau_m \leq T\} \geq \varepsilon, m \geq m_1. \quad (2.2)$$

Define a C_2 -function $V : R_+^2 \rightarrow R^+$ by $V(x, y) = (x-1) - \ln x + (y-1) - \ln y$. The non-negativity of this function can be observed from $m-1-\ln m \geq 0$ on $m > 0$. If $(x(t), y(t)) \in R_+^2$, we obtain that

$$\begin{aligned} dV(x, y) = & (x-1) \left[1 - a_{11}(\xi(t))x - \frac{a_{12}(\xi(t))y}{1+x} \right] \times \\ & [b_1(\xi(t))dt + \sigma_1(\xi(t))dB_1(t) + \mu_1(\xi(t))dB_2(t)] \\ & + (y-1) \left[-1 + \frac{a_{21}(\xi(t))x}{1+x} - a_{22}(\xi(t))y \right] \times \\ & [b_2(\xi(t))dt + \mu_2(\xi(t))dB_1(t) + \sigma_2(\xi(t))dB_2(t)] \\ & + \frac{1}{2} \left[1 - a_{11}(\xi(t))x - \frac{a_{12}(\xi(t))y}{1+x} \right]^2 \times (\sigma_1^2(\xi(t)) + \mu_1^2(\xi(t)))dt \\ & + \frac{1}{2} \left[-1 + \frac{a_{21}(\xi(t))x}{1+x} - a_{22}(\xi(t))y \right]^2 \times (\sigma_2^2(\xi(t)) + \mu_2^2(\xi(t)))dt. \end{aligned} \quad (2.3)$$

Hence we have

$$\begin{aligned} LV(x, y) \leq & b_1^* |x-1| (1 - a_{11*}x) + \frac{(\sigma_1^*)^2 + (\mu_1^*)^2}{2} 2 \left[(1 - a_{11*}x)^2 + (a_{12*}^2 y^2) \right] + b_2^* (a_{21*}^2 - a_{22*} y) |y-1| \\ & + \frac{(\sigma_2^*)^2 + (\mu_2^*)^2}{2} \left[(1 + a_{22*} y)^2 + (a_{21*}^2)^2 \right] \\ = & b_1^* (1 - a_{11*}x) |x-1| + [(a_{11*}^2 + (\mu_1^*)^2) (1 - 2a_{11*}x + a_{11*}^2 x^2 + (a_{12*}^2)^2 y^2) + (b_2^* a_{21*}^2 - a_{22*} y) |y-1|] \\ & + \frac{(\sigma_2^*)^2 + (\mu_2^*)^2}{2} (1 + 2a_{22*} y + (a_{22*}^2)^2 y^2 + (a_{21*}^2)^2) \\ \leq & K, \end{aligned} \quad (2.4)$$

where K is positive numbers. That is

$$\begin{aligned} dV(x, y) \leq & K dt + (x+1) \left[1 - a_{11*}x - \frac{a_{12*}y}{1+x} \right] [\sigma_1^* dB_1(t) \\ & + \mu_1^* dB_2(t)] + (y-1) \left[-1 + \frac{a_{21*}x}{1+x} - a_{22*}y \right] \\ & \times [\mu_2^* dB_1(t) + \sigma_2^* dB_2(t)]. \end{aligned} \quad (2.5)$$

Integrating both sides of the above inequality from 0 to $\tau_n \wedge T$ and then taking the expectations leads to

$$EV(x(\tau_n \wedge T), y(\tau_n \wedge T)) \leq V(x_0, y_0) + KT. \quad (2.6)$$

Set $\Omega = \{\tau_m \leq T\}$, then by inequality (2.2) we obtain $P(\Omega) \geq \varepsilon$. Note that for every $\omega \in \Omega_m$, there is some i such that $x_i(\tau_m, \omega)$ equals either m or $\frac{1}{m}$ for $i = 1, 2$, hence $V(x(\tau_m, \omega), y(\tau_m, \omega))$ is no less than

$$\min \left\{ (m-1 - \ln m), \frac{1}{m} - 1 - \ln \frac{1}{m} \right\}.$$

From (2.6), we have

$$V(x_0, y_0) + KT \geq E[l_{\Omega_m}(\omega)V(x(\tau_m), y(\tau_m))] \geq \varepsilon \min \left\{ (m-1 - \ln m), \left(\frac{1}{m} - 1 - \ln \frac{1}{m} \right) \right\},$$

where 1_{Ω_m} is the indicator function of Ω_m . Letting $m \rightarrow \infty$ leads to the contradiction

$$\infty > V(x_0, y_0) + KT \geq \infty.$$

Theorem 2.1 is proved.

III. STOCHASTIC BOUNDEDNESS

Theorem 2.1 tells us the solutions to system (1.3) will remain in R_+^2 . Before discussing how the solution varies in R_+^2 in more detail, we first present the definition of stochastic ultimate boundedness.

Definition 3.1 (see [11]) The solution $X(t) = (x(t), y(t))$ of Eq. (1.3) is said to be stochastically ultimately bounded, if for any $\varepsilon \in (0, 1)$, there is a positive constant $\delta = \delta(\varepsilon)$, such that for any given initial value $X_0 \in R_+^2$, the solution $X(t)$ to (1.3) has the property that

$$\limsup_{t \rightarrow \infty} P\{|X(t)| > \delta\} < \varepsilon.$$

Theorem 3.1 The solutions of system (1.3) are stochastically ultimately bounded for any initial value $X_0 = (x_0, y_0) \in R_+^2$.

Proof: From theorem 2.1, we know that the solution $X(t)$ will remain in R_+^2 for all $t \geq 0$ with probability 1. Define the functions $V_1 = e^t x^\theta$ and $V_2 = e^t y^\theta$, for $(x, y) \in R_+^2$ and $\theta > 0$. By the Itô's formula, we compute

$$\begin{aligned} LV_1 &\leq e^t x^\theta \left\{ 1 + \theta b_1^* \left(1 - a_{11}x - \frac{a_{12}y}{1+x} \right) + \frac{\theta(\theta-1)}{2} \right. \\ &\quad \times \left(1 - a_{11}x - \frac{a_{12}y}{1+x} \right)^2 [(\sigma_1^*)^2 + (\mu_1^*)^2] \Big\} \\ &\leq e^t x^\theta \left\{ 1 + \theta b_1^* (1 - a_{11}x - a_{12}y) + \theta(\theta-1) \right. \\ &\quad \times [(1 - a_{11}x)^2 + a_{12}^2 y^2] [(\sigma_1^*)^2 + (\mu_1^*)^2] \Big\} \\ &\leq K_1 e^t \end{aligned}$$

and

$$\begin{aligned} LV_2 &\leq e^t y^\theta \left\{ 1 + \theta b_2^* \left(-1 - \frac{a_{21}x}{1+x} - a_{22}y \right) + \frac{\theta(\theta-1)}{2} \right. \\ &\quad \times \left(-1 - \frac{a_{21}x}{1+x} - a_{22}y \right)^2 [(\mu_2^*)^2 + (\sigma_2^*)^2] \Big\} \\ &\leq e^t y^\theta \left\{ 1 + \theta b_2^* (-1 + a_{21}^* - a_{22}y) + \theta(\theta-1) \right. \\ &\quad \times [(a_{21}^* - 1)^2 + a_{22}^2 y^2] [(\mu_2^*)^2 + (\sigma_2^*)^2] \Big\} \\ &\leq K_2 e^t. \end{aligned}$$

Therefore,

$$\begin{aligned} d(e^t x^\theta) &= LV_1 dt + \theta e^t x^\theta \left[1 - a_{11}(\xi(t))x - \frac{a_{12}(\xi(t))y}{1+x} \right] \\ &\quad \times [\sigma_1(\xi(t))dB_1(t) + \mu_1(\xi(t))dB_2(t)] \\ &\leq K_1 e^t dt + \theta e^t x^\theta \left[1 - a_{11}x - \frac{a_{12}y}{1+x} \right] \\ &\quad \times [\sigma_1^* dB_1(t) + \mu_1^* dB_2(t)] \end{aligned}$$

and

$$\begin{aligned} d(e^t y^\theta) &= LV_2 dt + \theta e^t y^\theta \left[-1 + \frac{a_{21}(\xi(t))x}{1+x} - a_{22}(\xi(t))y \right] \\ &\quad \times [\mu_2(\xi(t))dB_1(t) + \sigma_2(\xi(t))dB_2(t)] \\ &\leq K_2 e^t dt + \theta e^t y^\theta \left[-1 - \frac{a_{21}x}{1+x} - a_{22}y \right] \\ &\quad \times [\mu_2^* dB_1(t) + \sigma_2^* dB_2(t)]. \end{aligned}$$

Thus, we have $e^t Ex^\theta - Ex_0^\theta \leq K_1 e^t$ and $e^t Ey^\theta - Ey_0^\theta \leq K_2 e^t$.

Hence obtain that

$$\limsup_{t \rightarrow \infty} Ex^\theta \leq K_1 < +\infty, \quad \limsup_{t \rightarrow \infty} Ey^\theta \leq K_2 < +\infty.$$

For $X(t) = (x(t), y(t)) \in R_+^2$, we have $|X(t)|^\theta \leq 2^{\frac{\theta}{2}}(x^\theta + y^\theta)$. Consequently,

$$\limsup_{t \rightarrow \infty} E|X(t)|^\theta \leq K < +\infty.$$

Applying the Chebyshev inequality, we can obtain the desired assertion.

IV. GLOBAL ASYMPTOTIC STABILITY

In this section, we show that the equilibrium positive $(\tilde{x}, \tilde{y})$ of system(1.3) is stochastically asymptotically stable in the large. If $w = w(t)$ is the solution of the n -dimensional stochastic differential equation

$$d(w(t)) = f(t, w(t))dt + g(t, w(t))dB(t), \quad (4.1)$$

where $\tilde{w}$ is the equilibrium positive of (4.1), and

$$LV(w) = V_w(w)f(t, w)dt + \frac{1}{2} \text{trance}[g^T(t, w)V_{ww}g(t, w)].$$

From the theory of stability of stochastic differential equations, we only need to find a Lyapunov function $V(w)$ satisfying $LV(w) \leq 0$ and the identity holds if and only if $w = \tilde{w}$ (see, e.g.[9]).

Now, we give our main result.

Theorem 4.1 Let

$$c_1 = [b_1^* - (\sigma_1^{*2} + \mu_1^{*2})a_{11}^*\tilde{x}]a_{21}^*(1 + \tilde{x}), \quad c_2 = b_2^*a_{21}^*,$$

$$\begin{aligned} P &= -a_{11}b_1^* + \frac{b_1^*a_{12}^*\tilde{y}}{(1 + \tilde{x})} + \frac{((\sigma_1^*)^2 + (\mu_1^*)^2)(a_{11}^*)^2\tilde{x}}{2} \\ &\quad + \frac{((\sigma_1^*)^2 + (\mu_1^*)^2)(a_{12}^*)^2\tilde{x}(\tilde{y})^2}{2(1 + \tilde{x})^2} + \frac{c_1((\sigma_2^*)^2 + (\mu_2^*)^2)(a_{21}^*)^2\tilde{y}}{2c_2(1 + \tilde{x})^2}, \end{aligned}$$

$$\begin{aligned} Q &= \frac{c_1}{c_2} \left[-b_2^*a_{22}^* + \frac{((\sigma_2^*)^2 + (\mu_2^*)^2)(a_{22}^*)^2\tilde{y}}{2} \right] \\ &\quad + \frac{((\sigma_1^*)^2 + (\mu_1^*)^2)a_{12}^{*2}}{2}, \end{aligned}$$

$$H = \frac{((\sigma_1^*)^2 + (\mu_1^*)^2)(a_{12}^*)^2 \tilde{x} \tilde{y}}{(1 + \tilde{x})} + \frac{c_1((\sigma_2^*)^2 + (\mu_2^*)^2)a_{21}^* a_{22}^* \tilde{y}}{c_2(1 + \tilde{x})}.$$

It is easy to see that $1 + x \geq 1$, hence we obtain

If

$$b_1 - (\sigma_1^2 + \mu_1^2)a_{11}\tilde{x} > 0 \quad (4.2)$$

and

$$P < 0, \quad (4.3)$$

as well as

$$4PQ - H^2 > 0. \quad (4.4)$$

Then the equilibrium position $(\tilde{x}, \tilde{y})$ of system(1.3) is stochastically asymptotically stable in the large, i.e., for any initial data $(x(0), y(0))$, the solution of system(1.3) has the property that

$$\lim_{t \rightarrow +\infty} x(t) = \tilde{x}, \quad \lim_{t \rightarrow +\infty} y(t) = \tilde{y} \quad (4.5)$$

almost surely.

Proof: Now, we define Lyapunov functions

$$V_1(x) = x - \tilde{x} - \tilde{x} \ln\left(\frac{x}{\tilde{x}}\right), \quad V_2(y) = y - \tilde{y} - \tilde{y} \ln\left(\frac{y}{\tilde{y}}\right).$$

The nonnegativity of this function can be observed from $m - 1 - \ln m \geq 0$ on $m > 0$. If $(x(t), y(t)) \in R^2$, by applying Itô's formula we compute

$$\begin{aligned} LV_1(x) &= b_1(\xi(t))(x - \tilde{x}) \left[1 - a_{11}(\xi(t))x - \frac{a_{12}(\xi(t))y}{1 + x} \right] \\ &\quad + \frac{((\sigma_1(\xi(t)))^2 + (\mu_1(\xi(t)))^2) \tilde{x}}{2} \left[1 - a_{11}(\xi(t))x - \frac{a_{12}(\xi(t))y}{1 + x} \right]^2 \\ &= b_1(\xi(t))(x - \tilde{x}) \left[a_{11}(\xi(t))\tilde{x} - a_{11}(\xi(t))x + \frac{a_{12}(\xi(t))\tilde{y}}{1 + \tilde{x}} - \frac{a_{12}(\xi(t))y}{1 + x} \right] \\ &\quad + \frac{((\sigma_1(\xi(t)))^2 + (\mu_1(\xi(t)))^2) \tilde{x}}{2} \left[a_{11}(\xi(t))\tilde{x} - a_{11}(\xi(t))x + \frac{a_{12}(\xi(t))\tilde{y}}{1 + \tilde{x}} - \frac{a_{12}(\xi(t))y}{1 + x} \right]^2 \\ &= \left\{ -a_{11}(\xi(t))b_1(\xi(t)) + \frac{b_1(\xi(t))a_{12}(\xi(t))\tilde{y}}{(1+x)(1+\tilde{x})} + \frac{((\sigma_1(\xi(t)))^2 + (\mu_1(\xi(t)))^2)(a_{11}(\xi(t)))^2 \tilde{x}}{2} \right. \\ &\quad \left. - \frac{((\sigma_1(\xi(t)))^2 + (\mu_1(\xi(t)))^2)a_{11}(\xi(t))a_{12}(\xi(t))\tilde{x}\tilde{y}}{(1+x)(1+\tilde{x})} + \frac{((\sigma_1(\xi(t)))^2 + (\mu_1(\xi(t)))^2)(a_{12}(\xi(t)))^2 \tilde{x}(\tilde{y})^2}{2(1+x)^2(1+\tilde{x})^2} \right\} \\ &\quad \times (x - \tilde{x})^2 + (y - \tilde{y})^2 \\ &\quad \times \frac{((\sigma_1(\xi(t)))^2 + (\mu_1(\xi(t)))^2)(a_{12}(\xi(t)))^2(1 + \tilde{x})^2}{2(1+x)^2(1+\tilde{x})^2} \\ &\quad - \frac{[b_1(\xi(t)) - ((\sigma_1(\xi(t)))^2 + (\mu_1(\xi(t)))^2)a_{11}(\xi(t))\tilde{x}]}{(1+x)(1+\tilde{x})} \\ &\quad \times a_{12}(\xi(t))(1 + \tilde{x})(x - \tilde{x})(y - \tilde{y}) - (x - \tilde{x})(y - \tilde{y}) \\ &\quad \times \frac{((\sigma_1(\xi(t)))^2 + (\mu_1(\xi(t)))^2)(a_{12}(\xi(t)))^2 \tilde{x}\tilde{y}(1 + \tilde{x})}{(1+x)^2(1+\tilde{x})^2}. \end{aligned} \quad (4.6)$$

$$\begin{aligned} LV_1(x) &\leq \left\{ -a_{11*}b_{1*} + \frac{b_{1*}a_{12}^*\tilde{y}}{(1+\tilde{x})} + \frac{((\sigma_1^*)^2 + (\mu_1^*)^2)a_{11}^2\tilde{x}}{2} \right. \\ &\quad \left. + \frac{((\sigma_1^*)^2 + (\mu_1^*)^2)a_{12}^2\tilde{x}(\tilde{y})^2}{2(1+\tilde{x})^2} \right\} (x - \tilde{x})^2 \\ &\quad + \frac{((\sigma_1^*)^2 + (\mu_1^*)^2)(a_{12}^*)^2}{2} (y - \tilde{y})^2 \\ &\quad + \frac{((\sigma_1^*)^2 + (\mu_1^*)^2)(a_{12}^*)^2 \tilde{x} \tilde{y}}{(1 + \tilde{x})} |x - \tilde{x}| |y - \tilde{y}| \\ &\quad - \frac{[b_{1*} - ((\sigma_1^*)^2 + (\mu_1^*)^2)a_{11}^*\tilde{x}] a_{12*}(1 + \tilde{x})}{(1+x)(1+\tilde{x})} \\ &\quad \times (x - \tilde{x})(y - \tilde{y}). \end{aligned} \quad (4.7)$$

Similarly, we can derive that

$$\begin{aligned} LV_2(y) &\leq \left[-b_{2*}a_{22*} + \frac{((\sigma_1^*)^2 + (\mu_1^*)^2)(a_{22}^*)^2 \tilde{y}}{2} \right] (y - \tilde{y})^2 \\ &\quad + \frac{((\sigma_1^*)^2 + (\mu_1^*)^2)(a_{21}^*)^2 \tilde{y}}{2(1+\tilde{x})^2} \times (x - \tilde{x})^2 \\ &\quad + \frac{((\sigma_1^*)^2 + (\mu_1^*)^2)a_{21}^* a_{22}^* \tilde{y}}{(1+\tilde{x})} |x - \tilde{x}| |y - \tilde{y}| \\ &\quad + \frac{b_{2*}a_{21}^*}{(1+\tilde{x})(1+x)} (x - \tilde{x})(y - \tilde{y}). \end{aligned} \quad (4.8)$$

From (4.2), we have $\frac{c_1}{c_2} > 0$. Define

$$\begin{aligned} V(x, y) &= V_1(x) + \frac{c_1}{c_2} V_2(y) \\ &= V_1(x) + \left[\frac{b_1(\xi(t)) - ((\sigma_1(\xi(t)))^2)}{b_2(\xi(t))a_{21}(\xi(t))} \right. \\ &\quad \left. + \frac{(\mu_1(\xi(t)))^2 a_{11}(\xi(t)) \tilde{x}}{b_2(\xi(t))a_{21}(\xi(t))} \right] a_{21}(\xi(t))(1 + \tilde{x}) V_2(y). \end{aligned}$$

Then, we compute

$$\begin{aligned} LV(x, y) &= LV_1(x) + \frac{c_1}{c_2} LV_2(y) \\ &\leq \left\{ -a_{11*}b_{1*} + \frac{b_{1*}a_{12}^*\tilde{y}}{(1+\tilde{x})} + \frac{((\sigma_1^*)^2 + (\mu_1^*)^2)}{2} \right. \\ &\quad \times (a_{11}^*)^2 \tilde{x} + \frac{((\sigma_1^*)^2 + (\mu_1^*)^2)(a_{12}^*)^2 \tilde{x}(\tilde{y})^2}{2(1+\tilde{x})^2} \left. \right\} \\ &\quad \times (x - \tilde{x})^2 + \frac{((\sigma_1^*)^2 + (\mu_1^*)^2)(a_{12}^*)^2}{2} (y - \tilde{y})^2 \\ &\quad + \frac{((\sigma_1^*)^2 + (\mu_1^*)^2)(a_{12}^*)^2 \tilde{x} \tilde{y}}{(1 + \tilde{x})} |x - \tilde{x}| |y - \tilde{y}| \\ &\quad + \frac{c_1}{c_2} \frac{b_{2*}a_{21}^*}{(1 + \tilde{x})(1 + x)} (x - \tilde{x})(y - \tilde{y}) - (y - \tilde{y}) \\ &\quad \times \frac{[b_{1*} - ((\sigma_1^*)^2 + (\mu_1^*)^2)a_{11}^*(1 + \tilde{x})]}{(1 + x)(1 + \tilde{x})} (x - \tilde{x}) \\ &\quad + \frac{c_1}{c_2} \left[-b_{2*}a_{22*} + \frac{((\sigma_2^*)^2 + (\mu_2^*)^2)(a_{22}^*)^2 \tilde{y}}{2} \right] \\ &\quad \times (y - \tilde{y})^2 + \frac{c_1((\sigma_2^*)^2 + (\mu_2^*)^2)(a_{21}^*)^2 \tilde{y}}{2c_2(1 + \tilde{x})^2} (x - \tilde{x})^2 \\ &\quad + \frac{c_1((\sigma_2^*)^2 + (\mu_2^*)^2)a_{21}^* a_{22}^* \tilde{y}}{c_2(1 + \tilde{x})} |x - \tilde{x}| |y - \tilde{y}| \end{aligned}$$

$$\begin{aligned}
&= \left\{ -a_{11*}b_{1*} + \frac{b_1^*a_{12}\tilde{y}}{(1+\tilde{x})} + \frac{((\sigma_1^*)^2 + (\mu_1^*)^2)}{2} \right. \\
&\quad \times (a_{11}^*)^2\tilde{x} + \frac{((\sigma_1^*)^2 + (\mu_1^*)^2)(a_{12}^*)^2\tilde{x}(\tilde{y})^2}{2(1+\tilde{x})^2} \Big\} \\
&\quad + \frac{c_1((\sigma_2^*)^2 + (\mu_2^*)^2)(a_{21}^*)^2\tilde{y}}{2c_2(1+\tilde{x})^2} \Big\} (x - \tilde{x})^2 \\
&\quad + \left\{ \frac{c_1}{c_2} \left[-b_{2*}a_{22*} + \frac{((\sigma_2^*)^2 + (\mu_2^*)^2)(a_{22}^*)^2\tilde{y}}{2} \right] \right. \\
&\quad + \frac{((\sigma_1^*)^2 + (\mu_1^*)^2)(a_{12}^*)^2}{2} \Big\} (y - \tilde{y})^2 \\
&\quad + \left\{ \frac{((\sigma_1^*)^2 + (\mu_1^*)^2)(a_{12}^*)^2\tilde{x}\tilde{y}}{(1+\tilde{x})} \right. \\
&\quad + \frac{c_1((\sigma_2^*)^2 + (\mu_2^*)^2)a_{21}^*a_{22}^*\tilde{y}}{c_2(1+\tilde{x})} \Big\} |x - \tilde{x}||y - \tilde{y}| \\
&= P(x - \tilde{x})^2 + Q(y - \tilde{y})^2 + H|x - \tilde{x}||y - \tilde{y}|. \quad (4.9)
\end{aligned}$$

Let $|w - \tilde{w}| = (|x - \tilde{x}|, |y - \tilde{y}|)^T$, thus we have

$$LV(x, y) \leq \frac{1}{2}|w - \tilde{w}|^T \begin{pmatrix} P & \frac{1}{2}H \\ \frac{1}{2}H & Q \end{pmatrix} |w - \tilde{w}|.$$

Obviously, if (4.3) and (4.4) hold then the above inequality implies $LV(x, y) < 0$ along all trajectories in the first quadrant except $(\tilde{x}, \tilde{y})$. This completes the proof.

V. NUMERICAL EXPERIMENTS

To substantiate the analytical findings, we let $\xi(t)$ which is a right-continuous Markov chain take value in a two-state space $S = \{1, 2\}$. We will use the Mistein method mentioned in Higham^[12]. Consider the discretization equations:

$$\begin{cases}
x_{k+1} = x_k + b_1(i)x_k \left[1 - a_{11}(i)x_k - \frac{a_{12}(i)y_k}{1+x_k} \right] \Delta t \\
\quad + x_k \left[1 - a_{11}(i)x_k - \frac{a_{12}(i)y_k}{1+x_k} \right] \left[\sigma_1(i)\zeta_k + \mu_1(i)\eta_k \right] \\
\quad \times \sqrt{\Delta t} + \frac{1}{2}x_k^2 \left[1 - a_{11}(i)x_k - \frac{a_{12}(i)y_k}{1+x_k} \right]^2 \\
\quad \times [\sigma_1^2(i)(\zeta_k^2 - 1) + \mu_1^2(i)(\eta_k^2 - 1)]\Delta t, \\
y_{k+1} = y_k + b_2(i)x_k \left[-1 + \frac{a_{21}(i)x_k}{1+x_k} - a_{22}(i)y_k \right] \Delta t \\
\quad + y_k \left[-1 + \frac{a_{21}(i)x_k}{1+x_k} - a_{22}(i)y_k \right] \left[\mu_2(i)\zeta_k + \sigma_2(i)\eta_k \right] \\
\quad \times \sqrt{\Delta t} + \frac{1}{2}y_k^2 \left[-1 + \frac{a_{21}(i)x_k}{1+x_k} - a_{22}(i)y_k \right]^2 \\
\quad \times [\mu_2^2(i)(\zeta_k^2 - 1) + \sigma_2^2(i)(\eta_k^2 - 1)]\Delta t.
\end{cases}$$

where ζ_k and η_k , $k = 1, 2, \dots, n$ are the Gaussian random variables $N(0, 1)$, and $i = 1, 2$.

In Theorem 4.1, we know if $\frac{c_1}{c_2} > 0, P < 0$ and $4PQ - H^2 > 0$, then the positive equilibrium position $(\tilde{x}, \tilde{y})$ is stochastically asymptotically stable in the large. In Fig.1, Fig.2 and Fig.3, we choose $a_{11}(1) = a_{12}(1) = a_{22}(1) = 1, a_{21}(1) = 3, b_1(1) = 0.8, b_2(1) = 0.7$, then $\frac{a_{21}(1)/a_{11}(1)}{1 + 1/a_{11}(1)} > 1$ holds and $\tilde{x} = 0.8105, \tilde{y} = 0.3430$. The only difference between conditions of Fig.1, Fig.2, and Fig.3 is that the values of $\sigma_1^2(1), \sigma_2^2(1)$ and $\mu_1^2(1), \mu_2^2(1)$ are different. We choose $\sigma_1^2(1) = \sigma_2^2(1) = \mu_1^2(1) = \mu_2^2(1) = 0$ in Fig.1,

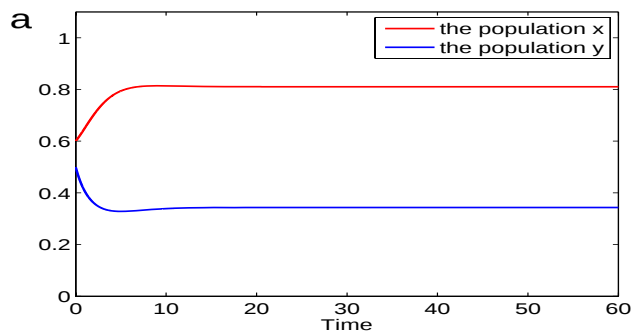


Fig. 1. is with $\sigma_1^2(1) = \sigma_2^2(1) = 0, \mu_1^2(1) = \mu_2^2(1) = 0$.

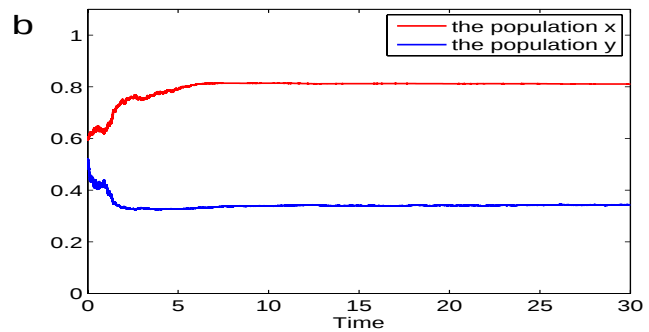


Fig. 2. is with $\sigma_1^2(1) = \sigma_2^2(1) = 0.1, \mu_1^2(1) = 0.13, \mu_2^2(1) = 0.15$.

then the conditions of (4.2), (4.3) and (4.4) are satisfied. That is to say that the equilibrium position $(\tilde{x}, \tilde{y})$ is asymptotically stable. In Fig.2, we choose $\sigma_1^2(1) = \sigma_2^2(1) = 0.1$ and $\mu_1^2(1) = 0.13, \mu_2^2(1) = 0.15$ then the conditions of (4.2), (4.3) and (4.4) are fulfilled. In other words, the equilibrium position $(\tilde{x}, \tilde{y})$ is stochastically asymptotically stable in the large. In Fig.3, we choose $\sigma_1^2(1) = 0.1, \sigma_2^2(1) = 30$ and $\mu_1^2(1) = 0.2, \mu_2^2(1) = 30$ which violates conditions (4.2). We can find that the population y will die out while $\lim_{t \rightarrow +\infty} x(t) = 1$. That is to say, when condition (4.2) is not satisfied the positive equilibrium $(\tilde{x}, \tilde{y})$ is no longer globally stable. By comparing Fig.1 with Fig.2, when the noise is not very large, we can see the positive equilibrium of the stochastic model is globally stable if the deterministic model has this property.

Obviously, we observe Fig.4, Fig.5 and Fig.6, when the continuous time Markov chain $\xi(t)$ takes values in another state space, i.e., $i = 2$. In Fig.4, Fig.5 and Fig.6, we choose $a_{11}(2) = 1.02, a_{12}(2) = 0.95, a_{22}(2) = 1.01, a_{21}(2) = 2.97, b_1(2) = 0.82, b_2(2) = 0.73$. We can obtain the same conclusions as in Fig.1, Fig.2 and Fig.3. Here it is omitted.

VI. CONCLUSION

In recent years, the predator and prey model has increasingly won attention as an important and fundamental model in biomathematics. However, population systems are often affected by environmental noise in reality. Although, Lv and Wang^[6] took into account two noise sources, the coupled mode of the two noise sources is simple. Our work is the first attempt

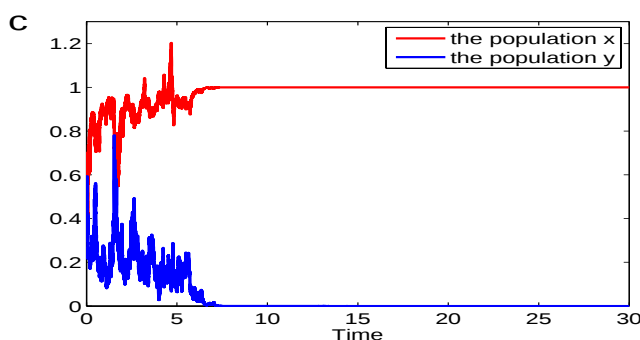


Fig. 3. is with $\sigma_1^2(1) = 0.1, \sigma_2^2(1) = 30, \mu_1^2(1) = 0.2, \mu_2^2(1) = 30$.

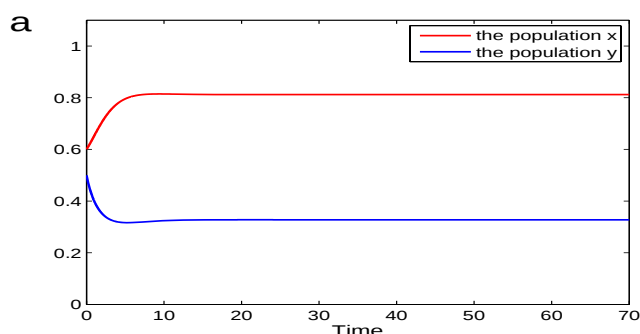


Fig. 4. is with $\sigma_1^2(1) = \sigma_2^2(1) = \mu_1^2(1) = \mu_2^2(1) = 0$.

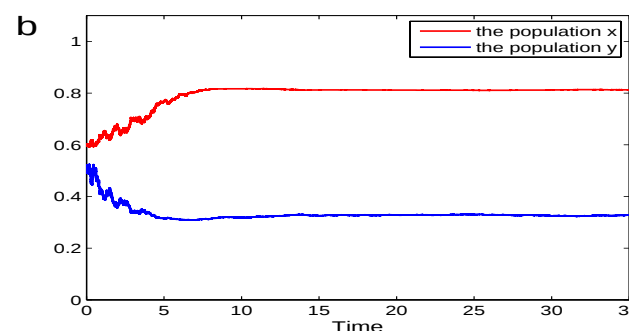


Fig. 5. is with $\sigma_1^2(1) = \sigma_2^2(1) = 0.1, \mu_1^2(1) = 0.13, \mu_2^2(1) = 0.15$.

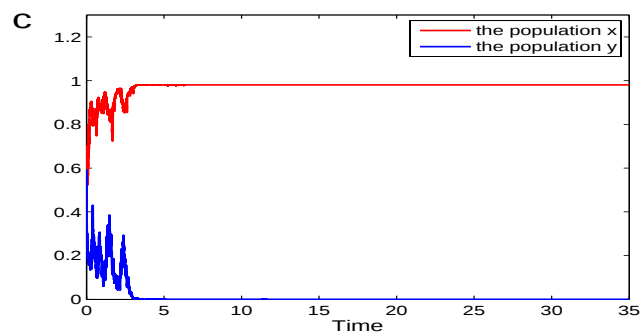


Fig. 6. is with $\sigma_1^2(1) = 0.1, \sigma_2^2(1) = 30, \mu_1^2(1) = 0.2, \mu_2^2(1) = 30$.

to consider the more complex coupled mode of the two noise sources. Moreover, Lv and Wang did not consider Markovian switching and they did not discuss the global stability which is one of the important properties.

In this paper, for any given positive initial value, we show that there is a unique positive solution to the system (1.3). What's more, we point out that the positive solution is stochastically bounded. At last, sufficient conditions for global stability are obtained. Some interesting questions deserve further investigation. One may consider other parameters such as $a_{ij}, i, j = 1, 2$ are made stochastic. Another is to study the permanence and extinction of system (1.3).

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