

Almost periodic solution for an impulsive neural networks with distributed delays

Lili Wang

Abstract—By using the estimation of the Cauchy matrix of linear impulsive differential equations and Banach fixed point theorem as well as Gronwall-Bellman's inequality, some sufficient conditions are obtained for the existence and exponential stability of almost periodic solution for an impulsive neural networks with distributed delays. An example is presented to illustrate the feasibility and effectiveness of the results.

Keywords—Almost periodic solution; Exponential stability; Neural networks; Impulses.

I. INTRODUCTION

IN the past few years, different types of recurrent neural networks have been extensively studied due to their promising potential for applications in the areas of signal and image processing, associative memories and pattern classification, parallel computation and optimization problems. One of the most popular models in the literature of recurrent neural network is the following shunting inhibitory cellular neural networks (SICNNs) with delays:

$$\begin{cases} \dot{x}_{ij}(t) = -a_{ij}(t)x_{ij}(t) \\ \quad - \sum_{C^{kl} \in N_r(i,j)} C_{ij}^{kl}(t)f_{ij}(x_{kl}(t))x_{ij}(t) \\ \quad + L_{ij}(t), t \geq 0, \\ x_{ij}(t) = \varphi_{ij}(t), t \in [-\tau, 0], \\ i = 1, 2, \dots, n, j = 1, 2, \dots, m, \end{cases}$$

where $C_{ij}(t)$ denotes the cell at the (i, j) position of the lattice at the t ; the r -neighborhood $N_r(i, j)$ of $C_{ij}(t)$ is

$$N_r(i, j) = \{C^{kl}(t) : \max(|k - i|, |l - j|) \leq r, 1 \leq k \leq n, 1 \leq l \leq m\},$$

$x_{ij}(t)$ is the activity of the cell $C_{ij}(t)$; $L_{ij}(t)$ is the external inputs to $C_{ij}(t)$; $a_{ij}(t) > 0$ represents the passive decay rate of the cell activity; $C_{ij}^{kl}(t) \geq 0$ is the connection or coupling strength of postsynaptic activity of the cell transmitted to the cell C_{ij} ; the activity functions $f_{ij}(\cdot)$ are continuous functions representing the output or firing rate of the cell $C^{kl}(t)$, $\varphi_{ij}(t)$ are the initial functions.

Since Bouzerdout and Pinter in [1-3] described SICNNs as a new cellular neural networks, SICNNs have been extensively applied in psychophysics, speech, perception, robotics, adaptive pattern recognition, vision, and image processing. It is well known that studies on neural dynamic systems not only involve a discussion of stability properties, but also involve many dynamic behaviors such as periodic oscillatory behavior,

almost periodic oscillatory properties, chaos and bifurcation. In applications, if the various constituent components of the temporally nonuniform environment is with incommensurable (nonintegral multiples) periods, then one has to consider the environment to be almost periodic since there is no a priori reason to expect the existence of periodic solutions. If we consider the effects of the environmental factors, the assumption of almost periodicity is more realistic, more important and more general. Recently, a lot of sufficient conditions have been given for almost periodic oscillation of SICNNs with constant time delays or time-varying delays in the literature, see [4-10] and the references cited therein.

On the other hand, impulsive effects widely exist in many dynamical systems involving such areas as population dynamics, automatic control, neural networks and so on. For example, in implementation of electronic networks in which state is subject to instantaneous perturbations and experiences abrupt change at certain moments, which may be caused by switching phenomenon, frequency change or other sudden noise, that is, does exhibit impulsive effects. For significance of impulsive effects, one can see [11-18].

Let $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}^+ = [0, \infty)$, $\Omega \subset \mathbb{R}$, $\Omega \neq \emptyset$ and $\mathbb{B} = \{\{\tau_k\} \in \mathbb{R} : \tau_k < \tau_{k+1}, k \in \mathbb{Z}, \lim_{k \rightarrow \pm\infty} \tau_k = \pm\infty\}$ denote the set of all sequences that are unbounded and strictly increasing.

To the best of our knowledge, the almost periodic oscillatory behavior is seldom considered for Cohen-Grossberg SICNNs with continuously distributed delays and impulses, which is described by the following integro-differential equations:

$$\begin{cases} x'_{ij}(t) = -a_{ij}(x_{ij}(t)) \left[b_{ij}(x_{ij}(t)) \right. \\ \quad \left. + \sum_{C^{gl} \in N_r(i,j)} C_{ij}^{gl}(t)w_{ij} \left(\int_0^{+\infty} k_{ij}(s) \times \right. \right. \\ \quad \left. \left. x_{gl}(t-s)ds \right) x_{ij}(t) - I_{ij}(t) \right], \\ t \neq \tau_k, k \in \mathbb{Z}, \\ \Delta x_{ij}(\tau_k) = \alpha_{ijk}x_{ij}(\tau_k) + \gamma_{ijk}, \\ i = 1, 2, \dots, n, j = 1, 2, \dots, m, k \in \mathbb{Z}, \end{cases} \quad (1)$$

where $a_{ij}(x_{ij}(t))$ and $b_{ij}(x_{ij}(t))$ represent an amplification function at time t and an appropriately behaved function at time t , respectively; $w_{ij} \in C(\mathbb{R}, \mathbb{R}^+)$ denote the normal and the delayed activation functions; $\{\tau_k\} \in \mathbb{B}$, with the constants $\alpha_{ijk} \in \mathbb{R}$, $\gamma_{ijk} \in \mathbb{R}$, $k \in \mathbb{Z}$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$.

Let $t_0 \in \mathbb{R}$. Introduce the following notation:

$PC(t_0)$ is the space of all functions $\phi : [-\infty, t_0] \rightarrow \Omega$ having points of discontinuity at $\theta_1, \theta_2, \dots \in (-\infty, t_0)$ of the first kind and left continuous at these points.

Lili Wang is with the School of Mathematics and Statistics, Anyang Normal University, Anyang, Henan 455000, China.

E-mail address: ay_wanglili@126.com.

For $J \subset \mathbb{R}$, $PC(J, \mathbb{R})$ is the space of all piecewise continuous functions from J to \mathbb{R} with points of discontinuity of the first kind τ_k , at which it is left continuous.

Let $x(t) = x(t, t_0, x_0)$, $x = (x_{11}, \dots, x_{ij}, \dots, x_{mn})^T$, $x_0 = (x_{011}, \dots, x_{0ij}, \dots, x_{0mn})^T$. The system (1) is supplemented with initial values problem given by

$$x(t_0 + 0, t_0, x_0) = x_0.$$

The rest of this paper is organized as follows: In Section 2, we will introduce some necessary notations, definitions and lemmas which will be used in the paper. In Section 3, some sufficient conditions are derived ensuring the existence and exponential stability of the almost periodic solution. An example is given to illustrate the effectiveness of our results in section 4.

II. PRELIMINARIES

In this section, we shall first recall some basic definitions, lemmas which will be used in what follows.

Definition 2.1([19]) Let $x(t) \in C(\mathbb{R}, \mathbb{R})$ be continuous in t . $x(t)$ is said to be almost periodic in the sense of Bohr on \mathbb{R} , if for any $\epsilon > 0$, the set $T(x, \epsilon) = \{\tau : |x(t + \tau) - x(t)| < \epsilon, \forall t \in \mathbb{R}\}$ is relatively dense, i.e., for any $\epsilon > 0$, it is possible to find a real number $l = l(\epsilon) > 0$, for any interval with length $l(\epsilon)$, there exists a number $\tau = \tau(\epsilon)$ in this interval such that $|x(t + \tau) - x(t)| < \epsilon, \forall t \in \mathbb{R}$.

Definition 2.2([20]) A sequence $x : \mathbb{Z} \rightarrow \mathbb{R}$ is called an almost periodic sequence if the ϵ - translation set of x :

$$T\{\epsilon, x\} := \{\tau \in \mathbb{Z} : |x(n + \tau) - x(n)| < \epsilon \text{ for all } n \in \mathbb{Z}\}$$

is a relatively dense set in \mathbb{Z} for all $\epsilon > 0$, that is, for any given $\epsilon > 0$, there exists an integer $l > 0$ such that each discrete interval of length l contains a integer $\tau = \tau(\epsilon) \in T\{\epsilon, x\}$ such that

$$|x(n + \tau) - x(n)| < \epsilon \text{ for all } n \in \mathbb{Z},$$

τ is called the ϵ - translation number of $x(n)$.

Definition 2.3([21]) The set of sequences $\{\tau_k^j\}, \tau_k^j = \tau_{k+j} - \tau_k, k, j \in \mathbb{Z}, \{\tau_k\} \in \mathbb{B}$ is said to be uniformly almost periodic if for arbitrary $\epsilon > 0$ there exists a relatively dense set of ϵ -almost periods common for any sequences.

Definition 2.4([21]) The function $x(t) \in PC(\mathbb{R}, \mathbb{R})$ is said to be almost periodic, if the following hold:

- (a) The set of sequences $\{\tau_k^j\}, \tau_k^j = \tau_{k+j} - \tau_k, k, j \in \mathbb{Z}, \{\tau_k\} \in \mathbb{B}$ is uniformly almost periodic.
- (b) For any $\epsilon > 0$ there exists a real number $\delta > 0$ such that if the points t' and t'' belong to one and the same interval of continuity of $x(t)$ and satisfy the inequality $|t' - t''| < \delta$, then $|x(t') - x(t'')| < \epsilon$.
- (c) For any $\epsilon > 0$ there exists a relatively dense set T such that if $\tau \in T$, then $|x(t + \tau) - x(t)| < \epsilon$ for all $t \in \mathbb{R}$ satisfying the condition $|t - \tau_k| > \epsilon, k \in \mathbb{Z}$.

The elements of T are called ϵ -almost periods.

Throughout this paper, we assume that

- (H₁) $a_{ij}(\cdot) \in C(\mathbb{R}, \mathbb{R}^+)$ and there exist positive constants \underline{a}_{ij} and \bar{a}_{ij} such that $0 < \underline{a}_{ij} \leq a_{ij}(\cdot) \leq \bar{a}_{ij}$ and $\bar{a}_{ij} \leq \underline{a}_{ij}e, i = 1, 2, \dots, n, j = 1, 2, \dots, m$.

- (H₂) The set of sequences $\{\tau_k^j\}, \tau_k^j = \tau_{k+j} - \tau_k, k \in \mathbb{Z}, j \in \mathbb{Z}, \{\tau_k\} \in \mathbb{B}$ is uniformly almost periodic and there exists $\theta > 0$ such that $\inf_{k \in \mathbb{Z}} \tau_k^1 = \theta > 0$.

- (H₃) The sequence $\{\alpha_{ijk}\}$ is almost periodic and $\frac{\bar{a}_{ij}}{\underline{a}_{ij}} - 1 \leq \alpha_{ijk} \leq \frac{\underline{a}_{ij}}{\bar{a}_{ij}}e^2 - 1, k \in \mathbb{Z}, i = 1, 2, \dots, n, j = 1, 2, \dots, m$.

- (H₄) The sequence $\{\gamma_{ijk}\}$ is almost periodic and $\gamma = \sup_{k \in \mathbb{Z}} |\gamma_{ijk}|, k \in \mathbb{Z}, i = 1, 2, \dots, n, j = 1, 2, \dots, m$.

- (H₅) $b_{ij}(\cdot) \in C^1(\mathbb{R}, \mathbb{R})$ and $b_{ij}(0) = 0$. There exist positive constants b'_{ij}, \bar{b}'_{ij} and L_b such that $0 < b'_{ij} \leq b'_{ij}(t) \leq \bar{b}'_{ij}, i = 1, 2, \dots, n, j = 1, 2, \dots, m$, and for $u, v \in \mathbb{R}, \max_{1 \leq i \leq n, 1 \leq j \leq m} |b_{ij}(u) - b_{ij}(v)| \leq L_b|u - v|$.

- (H₆) The functions $C_{ij}^{gl}(t), I_{ij}(t)$ are almost periodic in the sense of Bohr and $|I_{ij}(t)| < \infty, t \in \mathbb{R}, i = 1, 2, \dots, n, j = 1, 2, \dots, m$.

- (H₇) The functions w_{ij} are almost periodic in the sense of Bohr and $w_{ij}(0) = 0, i = 1, 2, \dots, n, j = 1, 2, \dots, m$. There exist positive constant L_w such that for $u, v \in \mathbb{R}, \max_{1 \leq i \leq n, 1 \leq j \leq m} |w_{ij}(u) - w_{ij}(v)| \leq L_w|u - v|$.

- (H₈) The delay kernels $k_{ij} \in C(\mathbb{R}, \mathbb{R})$ and there exist positive constants \bar{k}_{ij} such that

$$\int_0^{+\infty} |k_{ij}(s)| ds \leq \bar{k}_{ij}, \quad i = 1, 2, \dots, n, j = 1, 2, \dots, m.$$

Now, we shall transform system (1) and state some notations, which will be used in later sections.

From (H₁), the antiderivative of $1/a_{ij}(x_{ij})$ exists. We choose an antiderivative $h_{ij}(x_{ij})$ of $1/a_{ij}(x_{ij})$ that satisfies $h_{ij}(0) = 0$. Obviously, $(d/dx_{ij})h_{ij}(x_{ij}) = 1/a_{ij}(x_{ij})$. By $a_{ij}(x_{ij}) > 0$, we obtain that $h_{ij}(x_{ij})$ is strictly monotone increasing about x_{ij} . In view of derivative theorem for inverse function, the inverse function $h_{ij}^{-1}(x_{ij})$ of $h_{ij}(x_{ij})$ is differentiable and $(d/dx_{ij})h_{ij}^{-1}(x_{ij}) = a_{ij}(h_{ij}^{-1}(x_{ij}))$. By (H₅), composition function $b_{ij}(h_{ij}^{-1}(z))$ is differentiable. Denote $u_{ij}(t) = h_{ij}(x_{ij}(t))$. It is easy to see that $u'_{ij}(t) = x'_{ij}(t)/a_{ij}(x_{ij}(t))$ and $x_{ij}(t) = h_{ij}^{-1}(u_{ij}(t))$. Substituting these equalities into system (1), we get

$$\begin{cases} u'_{ij}(t) = -b_{ij}(h_{ij}^{-1}(u_{ij}(t))) \\ \quad - \sum_{C^{gl} \in \mathbb{N}_r(i,j)} C_{ij}^{gl}(t)w_{ij} \left(\int_0^{+\infty} k_{ij}(s) \times \right. \\ \quad \left. h_{ij}^{-1}(u_{gl}(t-s)) ds \right) h_{ij}^{-1}(u_{ij}(t)) \\ \quad + I_{ij}(t), \quad t \neq \tau_k, \\ \Delta u_{ij}(\tau_k) = h_{ij}((1 + \alpha_{ijk})h_{ij}^{-1}(u_{ij}(\tau_k)) + \gamma_{ijk}) - u_{ij}(\tau_k) \\ \quad := r_{ij}(u_{ij}(\tau_k)), \end{cases} \quad (2)$$

where $i = 1, 2, \dots, n, j = 1, 2, \dots, m, k \in \mathbb{Z}$.

If $u_{ij}(t) \neq 0$ for all $t \in \mathbb{R}, i = 1, 2, \dots, n, j = 1, 2, \dots, m$, from the definitions of $h_{ij}(z)$ and $h_{ij}^{-1}(z)$, using Lagrange mean-value theorem, we have

$$\begin{aligned} & r_{ij}(u_{ij}(\tau_k)) \\ &= h_{ij}((1 + \alpha_{ijk})h_{ij}^{-1}(u_{ij}(\tau_k)) + \gamma_{ijk}) - u_{ij}(\tau_k) \\ &= \frac{h_{ij}((1 + \alpha_{ijk})h_{ij}^{-1}(u_{ij}(\tau_k)) + \gamma_{ijk})}{(1 + \alpha_{ijk})h_{ij}^{-1}(u_{ij}(\tau_k)) + \gamma_{ijk}} \end{aligned}$$

$$\begin{aligned} & \times [(1 + \alpha_{ijk})h_{ij}^{-1}(u_{ij}(\tau_k)) + \gamma_{ijk}] - u_{ij}(\tau_k) \\ & = \frac{(1 + \alpha_{ijk})h_{ij}((1 + \alpha_{ijk})h_{ij}^{-1}(u_{ij}(\tau_k)) + \gamma_{ijk})}{(1 + \alpha_{ijk})h_{ij}^{-1}(u_{ij}(\tau_k)) + \gamma_{ijk}} \\ & \quad \times \frac{h_{ij}^{-1}(u_{ij}(\tau_k))}{u_{ij}(\tau_k)} u_{ij}(\tau_k) - u_{ij}(\tau_k) \\ & \quad + \frac{h_{ij}((1 + \alpha_{ijk})h_{ij}^{-1}(u_{ij}(\tau_k)) + \gamma_{ijk})}{(1 + \alpha_{ijk})h_{ij}^{-1}(u_{ij}(\tau_k)) + \gamma_{ijk}} \gamma_{ijk} \\ & = \left[\frac{(1 + \alpha_{ijk})a_{ij}(\eta_{ijk})}{a_{ij}(\xi_{ijk})} - 1 \right] u_{ij}(\tau_k) + \frac{\gamma_{ijk}}{a_{ij}(\xi_{ijk})} \\ & = \mu_{ijk} u_{ij}(\tau_k) + \nu_{ijk}, \end{aligned}$$

where

$$\begin{aligned} a_{ij}(\xi_{ijk}) &= \frac{1}{h_{ij}'(\xi_{ijk})} \\ &= \frac{(1 + \alpha_{ijk})h_{ij}^{-1}(u_{ij}(\tau_k)) + \gamma_{ijk}}{h_{ij}((1 + \alpha_{ijk})h_{ij}^{-1}(u_{ij}(\tau_k)) + \gamma_{ijk})}, \end{aligned} \quad (3)$$

$$\begin{aligned} a_{ij}(\eta_{ijk}) &= a_{ij}(h_{ij}^{-1}(\zeta_{ijk})) = (h_{ij}^{-1})'(\zeta_{ijk}) \\ &= \frac{h_{ij}^{-1}(u_{ij}(\tau_k))}{u_{ij}(\tau_k)}, \end{aligned} \quad (4)$$

in which $\eta_{ijk} = h_{ij}^{-1}(\zeta_{ijk})$, ξ_{ijk} is between 0 and $(1 + \alpha_{ijk})h_{ij}^{-1}(u_{ij}(\tau_k)) + \gamma_{ijk}$, η_{ijk} is between 0 and $h_{ij}^{-1}(u_{ij}(\tau_k))$, and

$$\mu_{ijk} = \frac{(1 + \alpha_{ijk})a_{ij}(\eta_{ijk})}{a_{ij}(\xi_{ijk})} - 1, \quad (5)$$

$$\nu_{ijk} = \frac{\gamma_{ijk}}{a_{ij}(\xi_{ijk})}, \quad (6)$$

where $i = 1, 2, \dots, n, j = 1, 2, \dots, m, k \in \mathbb{Z}$.

Then system (2) can be rewritten as

$$\begin{cases} u_{ij}'(t) = -e_{ij}(u_{ij}(t))u_{ij}(t) \\ \quad - \sum_{C^{gl} \in N_r(i,j)} C_{ij}^{gl}(t)w_{ij} \left(\int_0^{+\infty} k_{ij}(s) \times \right. \\ \quad \left. h_{gl}^{-1}(u_{gl}(t-s)) ds \right) h_{ij}^{-1}(u_{ij}(t)) \\ \quad + I_{ij}(t), \quad t \neq \tau_k, \\ \Delta u_{ij}(\tau_k) = \mu_{ijk}u_{ij}(\tau_k) + \nu_{ijk}, \\ \quad i = 1, 2, \dots, n, j = 1, 2, \dots, m, k \in \mathbb{Z}, \end{cases} \quad (7)$$

where $e_{ij}(u_{ij}(t)) := b_{ij}(h_{ij}^{-1}(u_{ij}(t)))/u_{ij}(t)$ for all $t \in \mathbb{R}$, $i = 1, 2, \dots, n, j = 1, 2, \dots, m$.

System (1) has an almost periodic solution which is globally exponentially stable if and only if system (7) has an almost periodic solution which is globally exponentially stable.

Let $E_{ij}(t) = e_{ij}(u_{ij}(t)) = \frac{b_{ij}(h_{ij}^{-1}(u_{ij}(t)))}{u_{ij}(t)}$, $i = 1, 2, \dots, n, j = 1, 2, \dots, m$.

Together with the system (7) we consider the linear system

$$\begin{cases} u_{ij}'(t) = -E_{ij}(t)u_{ij}(t), & t \neq \tau_k, \\ \Delta u_{ij}(\tau_k) = \mu_{ijk}u_{ij}(\tau_k), & k \in \mathbb{Z}, \end{cases} \quad (8)$$

where $t \in \mathbb{R}, i = 1, 2, \dots, n, j = 1, 2, \dots, m$.

Now let us consider the equations

$$u_{ij}'(t) = -E_{ij}(t)u_{ij}(t), \quad \tau_{k-1} < t \leq \tau_k, \quad \{\tau_k\} \in \mathbb{B}$$

and their solutions

$$u_{ij}(t) = u_{ij}(s) \exp \left\{ - \int_s^t E_{ij}(\sigma) d\sigma \right\}$$

for $\tau_{k-1} < s < t \leq \tau_k, i = 1, 2, \dots, n, j = 1, 2, \dots, m$.

Then, recall [22], the Cauchy matrix of the linear system (8) is

$$W_{ij}(t, s) = \begin{cases} \exp \left\{ - \int_s^t E_{ij}(\sigma) d\sigma \right\}, & \tau_{k-1} < s < t < \tau_k; \\ \prod_{l=m}^{k+1} (1 + \mu_{ijl}) \exp \left\{ - \int_s^t E_{ij}(\sigma) d\sigma \right\}, & \tau_{m-1} < s \leq \tau_m < \tau_k < t \leq \tau_{k+1}, \end{cases}$$

and the solutions of system (8) are in the form

$$\begin{aligned} u_{ij}(t; t_0; u_{ij}(t_0)) &= W_{ij}(t, t_0)u_{ij}(t_0), \quad t_0 \in \mathbb{R}, \\ & i = 1, 2, \dots, n, j = 1, 2, \dots, m. \end{aligned}$$

Lemma 2.1 If the conditions $(H_1) - (H_2), (H_5)$ and the following condition:

(H) $u_{ij}(t) \in PC(\mathbb{R}, \mathbb{R})$ is almost periodic function satisfying

$$0 < M \leq \inf_{t \in \mathbb{R}} |u_{ij}(t)| \leq \sup_{t \in \mathbb{R}} |u_{ij}(t)| \leq N,$$

where M, N are positive constants, $i = 1, 2, \dots, n, j = 1, 2, \dots, m$.

hold. Then $E_{ij}(t) = e_{ij}(u_{ij}(t))$ is almost periodic, $i = 1, 2, \dots, n, j = 1, 2, \dots, m$.

Proof: From the definition of $e_{ij}(u_{ij}(t))$, $E_{ij}(t) = e_{ij}(u_{ij}(t)) = \frac{b_{ij}(h_{ij}^{-1}(u_{ij}(t)))}{u_{ij}(t)}$, $i = 1, 2, \dots, n, j = 1, 2, \dots, m$.

Case I: $\frac{dE_{ij}(t)}{du_{ij}} = \frac{de_{ij}(u_{ij}(t))}{du_{ij}} = 0$, then $E_{ij}(t) = C_i$, where C_{ij} is a constant, $i = 1, 2, \dots, n, j = 1, 2, \dots, m$. Hence $E_{ij}(t) \in C(\mathbb{R}, \mathbb{R})$ is almost periodic, $i = 1, 2, \dots, n, j = 1, 2, \dots, m$.

Case II: $\frac{dE_{ij}(t)}{du_{ij}} = \frac{de_{ij}(u_{ij}(t))}{du_{ij}} \neq 0$, then $E_{ij}(t)$ depends on $u_{ij}(t)$, and $E_{ij}(t) \in PC(\mathbb{R}, \mathbb{R}), i = 1, 2, \dots, n, j = 1, 2, \dots, m$.

We now check the three conditions given in Definition 2.4.

- (a) With (H_2) , it is trivially satisfied.
- (b) For any $\epsilon > 0$, if t' and t'' belong to one and the same interval of continuity $E_{ij}(t)$ (t' and t'' also belong to one and the same interval of continuity $u_{ij}(t)$), by condition (b) of Definition 2.4, there exists a positive numbers δ_{ij} , such that $|t' - t''| < \delta_i$ implies $|u_{ij}(t') - u_{ij}(t'')| < \frac{M^2 \epsilon}{2L_b a_{ij} N}$, $i = 1, 2, \dots, n, j = 1, 2, \dots, m$. We take $\delta = \min_{1 \leq i \leq n, 1 \leq j \leq m} \{\delta_{ij}\}$, when $|t' - t''| < \delta$, for $1 \leq i \leq n, 1 \leq j \leq m$.

$j \leq m$,

$$\begin{aligned} & |E_{ij}(t') - E_{ij}(t'')| \\ &= \left| \frac{b_{ij}(h_{ij}^{-1}(u_{ij}(t'))) - b_{ij}(h_{ij}^{-1}(u_{ij}(t'')))}{u_{ij}(t')} - \frac{b_{ij}(h_{ij}^{-1}(u_{ij}(t''))) - b_{ij}(h_{ij}^{-1}(u_{ij}(t'')))}{u_{ij}(t'')} \right| \\ &= \left| \frac{u_{ij}(t'')b_{ij}(h_{ij}^{-1}(u_{ij}(t'))) - u_{ij}(t')b_{ij}(h_{ij}^{-1}(u_{ij}(t''))) - u_{ij}(t')b_{ij}(h_{ij}^{-1}(u_{ij}(t''))) + u_{ij}(t'')b_{ij}(h_{ij}^{-1}(u_{ij}(t''))) }{u_{ij}(t')u_{ij}(t'')} \right| \\ &\leq \frac{|u_{ij}(t'')b_{ij}(h_{ij}^{-1}(u_{ij}(t'))) - u_{ij}(t')b_{ij}(h_{ij}^{-1}(u_{ij}(t''))) + u_{ij}(t')b_{ij}(h_{ij}^{-1}(u_{ij}(t''))) - u_{ij}(t'')b_{ij}(h_{ij}^{-1}(u_{ij}(t''))) |}{M^2} \\ &\leq \frac{L_b|h_{ij}^{-1}(u_{ij}(t')) - u_{ij}(t') - u_{ij}(t'')|}{M^2} \\ &\quad + \frac{N|b_{ij}(h_{ij}^{-1}(u_{ij}(t'))) - b_{ij}(h_{ij}^{-1}(u_{ij}(t''))) |}{M^2} \\ &\leq \frac{L_b\bar{a}_{ij}N|u_{ij}(t') - u_{ij}(t'')|}{M^2} \\ &\quad + \frac{L_bN|h_{ij}^{-1}(u_{ij}(t')) - h_{ij}^{-1}(u_{ij}(t'')) |}{M^2} \\ &\leq \frac{L_b\bar{a}_{ij}N|u_{ij}(t') - u_{ij}(t'')|}{M^2} \\ &\quad + \frac{L_b\bar{a}_{ij}N|u_{ij}(t') - u_{ij}(t'')|}{M^2} \\ &= \frac{2L_b\bar{a}_{ij}N}{M^2}|u_{ij}(t') - u_{ij}(t'')| < \epsilon, \end{aligned}$$

so condition (b) of Definition 2.4 is satisfied.

(c) Let $P_{ij} = \frac{M^2}{2L_b\bar{a}_{ij}N} + 1, i = 1, 2, \dots, n, j = 1, 2, \dots, m$.

It is obvious that $\frac{M^2}{2L_b\bar{a}_{ij}NP_{ij}} < 1$. For any $\epsilon > 0$, with the almost periodicity of $u_{ij}(t)$, by Definition 2.4, there exists a relatively dense set T such that if $\tau \in T$, then $|u_{ij}(t + \tau) - u_{ij}(t)| < \frac{M^2\epsilon}{2L_b\bar{a}_{ij}NP_{ij}}$ for all $t \in \mathbb{R}$ satisfying $|t - \tau_k| > \frac{M^2\epsilon}{2L_b\bar{a}_{ij}NP_{ij}}, k \in \mathbb{Z}, i = 1, 2, \dots, n, j = 1, 2, \dots, m$.

Then for all $t \in \mathbb{R}$ satisfying the condition $|t - \tau_k| > \epsilon > \frac{M^2\epsilon}{2L_b\bar{a}_{ij}NP_{ij}}$, we obtain

$$\begin{aligned} & |E_{ij}(t + \tau) - E_{ij}(t)| \\ &= \left| \frac{b_{ij}(h_{ij}^{-1}(u_{ij}(t + \tau))) - b_{ij}(h_{ij}^{-1}(u_{ij}(t)))}{u_{ij}(t + \tau)} - \frac{b_{ij}(h_{ij}^{-1}(u_{ij}(t))) - b_{ij}(h_{ij}^{-1}(u_{ij}(t)))}{u_{ij}(t)} \right| \\ &= \left| \frac{u_{ij}(t)b_{ij}(h_{ij}^{-1}(u_{ij}(t + \tau))) - u_{ij}(t + \tau)b_{ij}(h_{ij}^{-1}(u_{ij}(t))) - u_{ij}(t)b_{ij}(h_{ij}^{-1}(u_{ij}(t))) + u_{ij}(t + \tau)b_{ij}(h_{ij}^{-1}(u_{ij}(t))) }{u_{ij}(t)u_{ij}(t + \tau)} \right| \\ &\leq \frac{1}{M^2}|u_{ij}(t)b_{ij}(h_{ij}^{-1}(u_{ij}(t + \tau))) - u_{ij}(t + \tau)b_{ij}(h_{ij}^{-1}(u_{ij}(t)))| \\ &\quad + \frac{1}{M^2}|u_{ij}(t + \tau)b_{ij}(h_{ij}^{-1}(u_{ij}(t))) - u_{ij}(t)b_{ij}(h_{ij}^{-1}(u_{ij}(t)))| \\ &\leq \frac{L_b\bar{a}_{ij}N|u_{ij}(t + \tau) - u_{ij}(t)|}{M^2} \\ &\quad + \frac{N|b_{ij}(h_{ij}^{-1}(u_{ij}(t + \tau))) - b_{ij}(h_{ij}^{-1}(u_{ij}(t))) |}{M^2} \end{aligned}$$

$$\begin{aligned} & \leq \frac{L_b\bar{a}_{ij}N|u_{ij}(t + \tau) - u_{ij}(t)|}{M^2} \\ & \quad + \frac{L_bN|h_{ij}^{-1}(u_{ij}(t + \tau)) - h_{ij}^{-1}(u_{ij}(t))|}{M^2} \\ & \leq \frac{L_b\bar{a}_{ij}N|u_{ij}(t + \tau) - u_{ij}(t)|}{M^2} \\ & \quad + \frac{L_b\bar{a}_{ij}N|u_{ij}(t + \tau) - u_{ij}(t)|}{M^2} \\ & = \frac{2L_b\bar{a}_{ij}N}{M^2}|u_{ij}(t + \tau) - u_{ij}(t)| \\ & < \frac{\epsilon}{P_{ij}} < \epsilon, \end{aligned}$$

where $i = 1, 2, \dots, n, j = 1, 2, \dots, m$, hence the condition (c) of Definition 2.4 is satisfied.

From Definition 2.4, $E_{ij}(t) \in PC(\mathbb{R}, \mathbb{R})$ is almost periodic, $i = 1, 2, \dots, n, j = 1, 2, \dots, m$. This completes the proof. ■

Now from [21], we have

Lemma 2.2 If the conditions $(H_1) - (H_7)$ and **(H)** hold, then for each $\epsilon > 0$, there exist $\epsilon_1, 0 < \epsilon_1 < \epsilon$, relatively dense sets T of real numbers and Q of whole numbers, such that the following relations are fulfilled:

- (a) $|E_{ij}(t + \tau) - E_{ij}(t)| < \epsilon, t \in \mathbb{R}, \tau \in T, |t - \tau_k| > \epsilon, k \in \mathbb{Z}, i = 1, 2, \dots, n, j = 1, 2, \dots, m;$
- (b) $|C_{ij}^g(t + \tau) - C_{ij}^g(t)| < \epsilon, t \in \mathbb{R}, \tau \in T, |t - \tau_k| > \epsilon, k \in \mathbb{Z}, i = 1, 2, \dots, n, j = 1, 2, \dots, m;$
- (c) $|I_{ij}(t + \tau) - I_{ij}(t)| < \epsilon, t \in \mathbb{R}, \tau \in T, |t - \tau_k| > \epsilon, k \in \mathbb{Z}, i = 1, 2, \dots, n, j = 1, 2, \dots, m;$
- (d) $|w_{ij}(t + \tau) - w_{ij}(t)| < \epsilon, t \in \mathbb{R}, \tau \in T, |t - \tau_k| > \epsilon, k \in \mathbb{Z}, i = 1, 2, \dots, n, j = 1, 2, \dots, m;$
- (e) $|\alpha_{ij(k+q)} - \alpha_{ijk}| < \epsilon, q \in Q, k \in \mathbb{Z}, i = 1, 2, \dots, n, j = 1, 2, \dots, m;$
- (f) $|\gamma_{ij(k+q)} - \gamma_{ijk}| < \epsilon, q \in Q, k \in \mathbb{Z}, i = 1, 2, \dots, n, j = 1, 2, \dots, m;$
- (g) $|\tau_k^q - \tau| < \epsilon_1, q \in Q, \tau \in T, k \in \mathbb{Z}.$

Lemma 2.3 If the conditions $(H_1) - (H_4)$ and **(H)** hold, then the sequences $\{\mu_{ijk}\}$ and $\{\nu_{ijk}\}$ is almost periodic, $k \in \mathbb{Z}, i = 1, 2, \dots, n, j = 1, 2, \dots, m$.

Proof: For convenience, let

$$\begin{aligned} F_{ij} &= \max \left\{ \frac{2\bar{a}_{ij}N(\bar{a}_{ij} + 2a_{ij})}{a_{ij}^2M^2}, \frac{(\bar{a}_{ij} + a_{ij})N}{a_{ij}^2M^2} \right\}, \\ & \quad \frac{\bar{a}_{ij}(\bar{a}_{ij}N^2 + a_{ij}N^2 + \bar{a}_{ij}M^2)}{a_{ij}^2M^2}, \\ L_{ij} &= \max \left\{ \frac{2\bar{a}_{ij}N\gamma(\bar{a}_{ij} + a_{ij})}{a_{ij}^3M_{ij}^2}, \frac{\bar{a}_{ij}N^2\gamma(\bar{a}_{ij} + a_{ij})}{a_{ij}^3M^2}, \right. \\ & \quad \left. \frac{\bar{a}_{ij}N\gamma + a_{ij}N\gamma + \bar{a}_{ij}a_{ij}M^2}{a_{ij}^3M^2} \right\}, \end{aligned}$$

where $i = 1, 2, \dots, n, j = 1, 2, \dots, m, F = \max_{1 \leq i \leq n, 1 \leq j \leq m} \{F_{ij}\}, L = \max_{1 \leq i \leq n, 1 \leq j \leq m} \{L_{ij}\}$ and $H = \max\{F, L\}$.

For any $\epsilon > 0$, since $u_{ij}(t) \in PC(\mathbb{R}, \mathbb{R})$ is almost periodic function, by Definition 2.4, there exists $\delta > 0$ such that if the points t' and t'' belong to one and the same interval of continuity of $u_{ij}(t)$ and satisfy the inequality $|t' - t''| < \delta$, then $|u_{ij}(t') - u_{ij}(t'')| < \frac{\epsilon}{4H}, i = 1, 2, \dots, n, j = 1, 2, \dots, m$.

With the left continuousness of τ_k , take numbers $\epsilon_0, \tau'_k < \tau_k$ such that τ'_k and τ_k belong to one and the same interval of continuity of $u_{ij}(t)$, and $0 < \epsilon_0 < \tau_k - \tau'_k < \min\{\delta, \frac{\theta}{2}, \frac{\epsilon}{12H}\}$, $i = 1, 2, \dots, n, j = 1, 2, \dots, m, k \in \mathbb{Z}$.

From Lemma 2.2, for $\epsilon_0 < \frac{\epsilon}{12H}$, because the sequences $\{\tau_k^j\}, k \in \mathbb{Z}, j \in \mathbb{Z}, \{\tau_k\} \in \mathbb{B}$ is uniformly almost periodic, for $q \in Q$ (without loss of generality, assuming $q \geq 0$), let $\tau = \inf_{k \in \mathbb{Z}} \{\tau_k^q\}$, implying $\tau \in T$ and $\tau_k + \tau \leq \tau_{k+q}$, where the sets T and Q are determined in Lemma 2.2 such that $0 \leq \tau_k^q - \tau = \tau_{k+q} - \tau_k - \tau < \epsilon_0 < \min\{\frac{\epsilon}{12H}, \delta, \frac{\theta}{2}\}$. Then $\tau_k + \tau$ and $\tau_{k+q}, \tau'_k + \tau$ and $\tau_k + \tau$ belong to one and the same interval of continuity of $u_{ij}(t)$, respectively, $i = 1, 2, \dots, n, j = 1, 2, \dots, m, k \in \mathbb{Z}$.

By (H) and Lemma 2.2, we have

$$\begin{aligned} & |u_{ij}(\tau_{k+q}) - u_{ij}(\tau_k)| \\ & \leq |u_{ij}(\tau_k) - u_{ij}(\tau_k + \tau)| + |u_{ij}(\tau_k + \tau) - u_{ij}(\tau_{k+q})| \\ & \leq |u_{ij}(\tau'_k) - u_{ij}(\tau_k)| + |u_{ij}(\tau'_k) - u_{ij}(\tau_k + \tau)| \\ & \quad + |u_{ij}(\tau_k + \tau) - u_{ij}(\tau_{k+q})| \\ & \leq |u_{ij}(\tau'_k) - u_{ij}(\tau_k)| + |u_{ij}(\tau'_k + \tau) - u_{ij}(\tau'_k)| \\ & \quad + |u_{ij}(\tau'_k + \tau) - u_{ij}(\tau_k + \tau)| \\ & \quad + |u_{ij}(\tau_k + \tau) - u_{ij}(\tau_{k+q})| \\ & < \frac{\epsilon}{4H} + \frac{\epsilon}{12H} + \frac{\epsilon}{4H} + \frac{\epsilon}{4H} = \frac{5\epsilon}{6H}, \end{aligned} \tag{9}$$

where $i = 1, 2, \dots, n, j = 1, 2, \dots, m, k \in \mathbb{Z}$.

Let $G_{ijk} = (1 + \alpha_{ijk})h_{ij}^{-1}(u_{ij}(\tau_k)) + \gamma_{ijk}$, then $G_{ijk} = x_{ij}(\tau_k^+) = h_{ij}^{-1}(u_{ij}(\tau_k^+))$, $M \leq |h_{ij}(G_{ijk})| = |u_{ij}(\tau_k^+)| \leq N$, $|G_{ijk}| = |h_{ij}^{-1}(u_{ij}(\tau_k^+))| \leq \bar{a}_{ij}|u_{ij}(\tau_k^+)| \leq \bar{a}_{ij}N$, $i = 1, 2, \dots, n, j = 1, 2, \dots, m, k \in \mathbb{Z}$. Then

$$\begin{aligned} & |G_{ij(k+q)} - G_{ijk}| \\ & = |[(1 + \alpha_{ij(k+q)})h_{ij}^{-1}(u_{ij}(\tau_{k+q})) + \gamma_{ij(k+q)}] \\ & \quad - [(1 + \alpha_{ijk})h_{ij}^{-1}(u_{ij}(\tau_k)) + \gamma_{ijk}]| \\ & \leq |h_{ij}^{-1}(u_{ij}(\tau_{k+q})) - h_{ij}^{-1}(u_{ij}(\tau_k))| \\ & \quad + |\alpha_{ij(k+q)}h_{ij}^{-1}(u_{ij}(\tau_{k+q})) - \alpha_{ijk}h_{ij}^{-1}(u_{ij}(\tau_k))| \\ & \quad + |\gamma_{ij(k+q)} - \gamma_{ijk}| \\ & \leq \bar{a}_{ij}|u_{ij}(\tau_{k+q}) - u_{ij}(\tau_k)| + |\alpha_{ij(k+q)}h_{ij}^{-1}(u_{ij}(\tau_{k+q})) \\ & \quad - \alpha_{ijk}h_{ij}^{-1}(u_{ij}(\tau_k))| \\ & \quad + |\alpha_{ij(k+q)}h_{ij}^{-1}(u_{ij}(\tau_k)) - \alpha_{ijk}h_{ij}^{-1}(u_{ij}(\tau_k))| \\ & \quad + |\gamma_{ij(k+q)} - \gamma_{ijk}| \\ & \leq 2\bar{a}_{ij}|u_{ij}(\tau_{k+q}) - u_{ij}(\tau_k)| \\ & \quad + |h_{ij}^{-1}(u_{ij}(\tau_k))| |\alpha_{ij(k+q)} - \alpha_{ijk}| + |\gamma_{ij(k+q)} - \gamma_{ijk}| \\ & \leq 2\bar{a}_{ij}|u_{ij}(\tau_{k+q}) - u_{ij}(\tau_k)| \\ & \quad + \bar{a}_{ij}|u_{ij}(\tau_k)| |\alpha_{ij(k+q)} - \alpha_{ijk}| + |\gamma_{ij(k+q)} - \gamma_{ijk}| \\ & \leq 2\bar{a}_{ij}|u_{ij}(\tau_{k+q}) - u_{ij}(\tau_k)| + \bar{a}_{ij}N|\alpha_{ij(k+q)} - \alpha_{ijk}| \\ & \quad + |\gamma_{ij(k+q)} - \gamma_{ijk}|, \end{aligned} \tag{10}$$

where $i = 1, 2, \dots, n, j = 1, 2, \dots, m, k \in \mathbb{Z}$.

By (3), (4) and (10), we obtain

$$\begin{aligned} & |a_{ij}(\xi_{ij(k+q)}) - a_{ij}(\xi_{ijk})| \\ & = \left| \frac{G_{ij(k+q)}}{h_{ij}(G_{ij(k+q)})} - \frac{G_{ijk}}{h_{ij}(G_{ijk})} \right| \end{aligned}$$

$$\begin{aligned} & = \frac{|h_{ij}(G_{ijk})G_{ij(k+q)} - G_{ijk}h_{ij}(G_{ij(k+q)})|}{|h_{ij}(G_{ij(k+q)})h_{ij}(G_{ijk})|} \\ & \leq \frac{|h_{ij}(G_{ijk})G_{ij(k+q)} - h_{ij}(G_{ij(k+q)})G_{ij(k+q)}|}{M^2} \\ & \quad + \frac{|h_{ij}(G_{ij(k+q)})G_{ij(k+q)} - G_{ijk}h_{ij}(G_{ij(k+q)})|}{M^2} \\ & \leq \frac{1}{M^2}|G_{ij(k+q)}||h_{ij}(G_{ij(k+q)}) - h_{ij}(G_{ijk})| \\ & \quad + \frac{1}{M^2}|h_{ij}(G_{ij(k+q)})||G_{ij(k+q)} - G_{ijk}| \\ & \leq \frac{\bar{a}_{ij}N|G_{ij(k+q)} - G_{ijk}| + N|G_{ij(k+q)} - G_{ijk}|}{M^2} \\ & \leq \frac{(\bar{a}_{ij} + \underline{a}_{ij})N}{\underline{a}_{ij}M^2}|G_{ij(k+q)} - G_{ijk}| \\ & \leq \frac{2\bar{a}_{ij}(\bar{a}_{ij} + \underline{a}_{ij})N}{\underline{a}_{ij}M^2}|u_{ij}(\tau_{k+q}) - u_{ij}(\tau_k)| \\ & \quad + \frac{\bar{a}_{ij}(\bar{a}_{ij} + \underline{a}_{ij})N^2}{\underline{a}_{ij}M^2}|\alpha_{ij(k+q)} - \alpha_{ijk}| \\ & \quad + \frac{(\bar{a}_{ij} + \underline{a}_{ij})N}{\underline{a}_{ij}M^2}|\gamma_{ij(k+q)} - \gamma_{ijk}| \end{aligned} \tag{11}$$

and

$$\begin{aligned} & |a_{ij}(\eta_{ij(k+q)}) - a_{ij}(\eta_{ijk})| \\ & = \left| \frac{h_{ij}^{-1}(u_{ij}(\tau_{k+q}))}{u_{ij}(\tau_{k+q})} - \frac{h_{ij}^{-1}(u_{ij}(\tau_k))}{u_{ij}(\tau_k)} \right| \\ & = \frac{|h_{ij}^{-1}(u_{ij}(\tau_{k+q}))u_{ij}(\tau_k) - u_{ij}(\tau_{k+q})h_{ij}^{-1}(u_{ij}(\tau_k))|}{|u_{ij}(\tau_{k+q})u_{ij}(\tau_k)|} \\ & \leq \frac{|h_{ij}^{-1}(u_{ij}(\tau_{k+q}))u_{ij}(\tau_k) - h_{ij}^{-1}(u_{ij}(\tau_{k+q}))u_{ij}(\tau_{k+q})|}{M^2} \\ & \quad + \frac{|h_{ij}^{-1}(u_{ij}(\tau_{k+q}))u_{ij}(\tau_{k+q}) - u_{ij}(\tau_{k+q})h_{ij}^{-1}(u_{ij}(\tau_k))|}{M^2} \\ & \leq \frac{|h_{ij}^{-1}(u_{ij}(\tau_{k+q}))||u_{ij}(\tau_{k+q}) - u_{ij}(\tau_k)|}{M^2} \\ & \quad + \frac{|u_{ij}(\tau_{k+q})||h_{ij}^{-1}(u_{ij}(\tau_{k+q})) - h_{ij}^{-1}(u_{ij}(\tau_k))|}{M^2} \\ & \leq \frac{2\bar{a}_{ij}N}{M^2}|u_{ij}(\tau_{k+q}) - u_{ij}(\tau_k)|, \end{aligned} \tag{12}$$

where $i = 1, 2, \dots, n, j = 1, 2, \dots, m, k \in \mathbb{Z}$.

Finally, with (5), (6) and (9) – (12), for each $q \in Q$,

$$\begin{aligned} & |\mu_{ij(k+q)} - \mu_{ijk}| \\ & = \left| \left(\frac{(1 + \alpha_{ij(k+q)})a_{ij}(\eta_{ij(k+q)})}{a_{ij}(\xi_{ij(k+q)})} - 1 \right) \right. \\ & \quad \left. - \left(\frac{(1 + \alpha_{ijk})a_{ij}(\eta_{ijk})}{a_{ij}(\xi_{ijk})} - 1 \right) \right| \\ & \leq \frac{1}{\underline{a}_{ij}^2} |1 + \alpha_{ij(k+q)}| |a_{ij}(\eta_{ij(k+q)})a_{ij}(\xi_{ijk}) \\ & \quad - a_{ij}(\xi_{ij(k+q)})a_{ij}(\eta_{ijk})| \\ & \quad + \frac{|a_{ij}(\xi_{ij(k+q)})a_{ij}(\eta_{ijk})| |\alpha_{ij(k+q)} - \alpha_{ijk}|}{\underline{a}_{ij}^2} \\ & \leq \frac{|a_{ij}(\eta_{ij(k+q)})a_{ij}(\xi_{ijk}) - a_{ij}(\eta_{ij(k+q)})a_{ij}(\xi_{ij(k+q)})|}{\bar{a}_{ij}\underline{a}_{ij}} \end{aligned}$$

$$\begin{aligned}
 & + \frac{|a_{ij}(\eta_{ij(k+q)})a_{ij}(\xi_{ij(k+q)}) - a_{ij}(\xi_{ij(k+q)})a_{ij}(\eta_{ijk})|}{\bar{a}_{ij}a_{ij}} \\
 & + \frac{\bar{a}_{ij}^2|\alpha_{ij(k+q)} - \alpha_{ijk}|}{a_{ij}^2} \\
 \leq & \frac{|a_{ij}(\xi_{ij(k+q)}) - a_{ij}(\xi_{ijk})| + |a_{ij}(\eta_{ij(k+q)}) - a_{ij}(\eta_{ijk})|}{a_{ij}} \\
 & + \frac{\bar{a}_{ij}^2|\alpha_{ij(k+q)} - \alpha_{ijk}|}{a_{ij}^2} \\
 \leq & \frac{2\bar{a}_{ij}N(\bar{a}_{ij} + 2a_{ij})}{a_{ij}^2M^2}|u_{ij}(\tau_{k+q}) - u_{ij}(\tau_k)| \\
 & + \frac{\bar{a}_{ij}(\bar{a}_{ij}N^2 + a_{ij}N^2 + \bar{a}_{ij}M^2)}{a_{ij}^2M^2}|\alpha_{ij(k+q)} - \alpha_{ijk}| \\
 & + \frac{(\bar{a}_{ij} + a_{ij})N}{a_{ij}^2M^2}|\gamma_{ij(k+q)} - \gamma_{ijk}| \\
 < & H\frac{5\epsilon}{6H} + H\frac{\epsilon}{12H} + H\frac{\epsilon}{12H} = \epsilon,
 \end{aligned}$$

and

$$\begin{aligned}
 |\nu_{ij(k+q)} - \nu_{ijk}| & = \left| \frac{\gamma_{ij(k+q)}}{a_{ij}(\xi_{ij(k+q)})} - \frac{\gamma_{ijk}}{a_{ij}(\xi_{ijk})} \right| \\
 & = \frac{|\gamma_{ij(k+q)}a_{ij}(\xi_{ijk}) - \gamma_{ijk}a_{ij}(\xi_{ij(k+q)})|}{|a_{ij}(\xi_{ij(k+q)})a_{ij}(\xi_{ijk})|} \\
 \leq & \frac{1}{a_{ij}^2}|\gamma_{ij(k+q)}a_{ij}(\xi_{ijk}) - \gamma_{ij(k+q)}a_{ij}(\xi_{ij(k+q)})| \\
 & + \frac{1}{a_{ij}^2}|\gamma_{ij(k+q)}a_{ij}(\xi_{ij(k+q)}) - \gamma_{ijk}a_{ij}(\xi_{ij(k+q)})| \\
 \leq & \frac{\gamma|a_{ij}(\xi_{ij(k+q)}) - a_{ij}(\xi_{ijk})| + \bar{a}_{ij}|\gamma_{ij(k+q)} - \gamma_{ijk}|}{a_{ij}^2} \\
 \leq & \frac{2\bar{a}_{ij}N\gamma(\bar{a}_{ij} + a_{ij})}{a_{ij}^3M^2}|u_{ij}(\tau_{k+q}) - u_{ij}(\tau_k)| \\
 & + \frac{\bar{a}_{ij}N^2\gamma(\bar{a}_{ij} + a_{ij})}{a_{ij}^3M^2}|\alpha_{ij(k+q)} - \alpha_{ijk}| \\
 & + \frac{(\bar{a}_{ij}N\gamma + a_{ij}N\gamma + \bar{a}_{ij}a_{ij}M^2)}{a_{ij}^3M^2}|\gamma_{ij(k+q)} - \gamma_{ijk}| \\
 < & H\frac{5\epsilon}{6H} + H\frac{\epsilon}{12H} + H\frac{\epsilon}{12H} = \epsilon,
 \end{aligned}$$

where $i = 1, 2, \dots, n, j = 1, 2, \dots, m, k \in \mathbb{Z}$. Since Q is relatively dense set of whole numbers, hence the sequences $\{\mu_{ijk}\}$ and $\{\nu_{ijk}\}$ is almost periodic, $k \in \mathbb{Z}, i = 1, 2, \dots, n, j = 1, 2, \dots, m$. This completes the proof. ■

Lemma 2.4([21]) Let $\{\tau_k\} \in \mathbb{B}$ and the condition (H_2) hold. Then for $l > 0$ there exists a positive integer A such that on each interval of length l , we have no more than A elements of the sequence $\{\tau_k\}$, i.e.,

$$i(s, t) \leq A(t - s) + A,$$

where $i(s, t)$ is the number of the points τ_k in the interval (s, t) .

Lemma 2.5 If the conditions $(H_1) - (H_3), (H_5), (\mathbf{H})$ and the following condition:

$$(H_9) \alpha_{ij} = \underline{a}_{ij}b'_{ij} - 2A > 0, \text{ where constant } A \text{ is determined in Lemma 2.4, } i = 1, 2, \dots, n, j = 1, 2, \dots, m.$$

hold, then:

(i) For the Cauchy matrix $W_{ij}(t, s)$ of system (8), there exist positive numbers α_{ij} and β_{ij} such that

$$e^{-\beta_{ij}(t-s)} \leq W_{ij}(t, s) \leq e^{2A}e^{-\alpha_{ij}(t-s)},$$

where $\beta_{ij} = \bar{a}_{ij}b'_{ij}, t \geq s, t, s \in \mathbb{R}, i = 1, 2, \dots, n, j = 1, 2, \dots, m$.

(ii) For any $\epsilon > 0, t \geq s, t, s \in \mathbb{R}, |t - \tau_k| > \epsilon, |s - \tau_k| > \epsilon, k \in \mathbb{Z}$ there exists a relatively dense set T of the function $E_{ij}(t)$ and a positive constant Γ such that for $\tau \in T$ it follows that

$$|W_{ij}(t + \tau, s + \tau) - W_{ij}(t, s)| \leq \epsilon \Gamma e^{-\frac{\alpha_{ij}}{2}(t-s)}, t \geq s, t, s \in \mathbb{R}, i = 1, 2, \dots, n, j = 1, 2, \dots, m.$$

Proof: Because the proof of the second part of this lemma is similar to Lemma 3 in [23], hence we will only prove the first part of the lemma.

From the definition of $E_{ij}(t)$, using Lagrange mean-value theorem, one gets

$$\begin{aligned}
 E_{ij}(t) & = \frac{b_{ij}(h_{ij}^{-1}(u_{ij}(t)))}{u_{ij}(t)} = \frac{b'_{ij}(o_i)h_{ij}^{-1}(u_{ij}(t))}{u_{ij}(t)} \\
 & = b'_{ij}(o_{ij})a_{ij}(\rho_{ij}),
 \end{aligned}$$

where o_i is between 0 and $h_{ij}^{-1}(u_{ij}(t)), \rho_{ij}$ is between 0 and $u_{ij}(t), i = 1, 2, \dots, n, j = 1, 2, \dots, m$.

Thus

$$\underline{a}_{ij}b'_{ij} \leq E_{ij}(t) = b'_{ij}(o_{ij})a_{ij}(\rho_{ij}) \leq \bar{a}_{ij}b'_{ij} = \beta_{ij}, i = 1, 2, \dots, n, j = 1, 2, \dots, m. \quad (13)$$

Since the sequence $\{\mu_{ijk}\}$ is almost periodic, then it is bounded. From (H_3) and (5) it follows that $1 \leq 1 + \mu_{ijk} \leq e^2$ for $i = 1, 2, \dots, n, j = 1, 2, \dots, m, k \in \mathbb{Z}$.

With the presentation of $W_{ij}(t, s)$, the last inequality and (13) it follows that

$$\begin{aligned}
 e^{-\beta_{ij}(t-s)} & \leq W_{ij}(t, s) \leq (1 + \mu_{ijk})^{i(s,t)}e^{-\alpha_{ij}b'_{ij}(t-s)} \\
 & \leq (1 + \mu_{ijk})^{A(t-s)+A}e^{-\alpha_{ij}b'_{ij}(t-s)} \\
 & = e^{2A}e^{(2A - \alpha_{ij}b'_{ij})(t-s)} \\
 & = e^{2A}e^{-\alpha_{ij}(t-s)},
 \end{aligned}$$

where $t \geq s, t, s \in \mathbb{R}, k \in \mathbb{Z}, i = 1, 2, \dots, n, j = 1, 2, \dots, m$. This completes the proof. ■

For convenience, we introduce the notation:

$$\bar{f} = \sup_{t \in \mathbb{R}} |f(t)|, \quad \underline{f} = \inf_{t \in \mathbb{R}} |f(t)|.$$

III. MAIN RESULTS

Let

$$M = \min_{1 \leq i \leq n, 1 \leq j \leq m} \left\{ \frac{\Theta_{ij}}{\beta_{ij}} - \frac{\gamma e^{2A}}{1 - e^{-\alpha_{ij}}} \right\},$$

$$K = \max_{1 \leq i \leq n, 1 \leq j \leq m} \left\{ \frac{\bar{I}_{ij} e^{2A}}{\alpha_{ij}} + \frac{\gamma e^{2A}}{1 - e^{-\alpha_{ij}}} \right\},$$

$$r = \max_{1 \leq i \leq n, 1 \leq j \leq m} \left\{ \frac{e^{2A}}{\alpha_{ij}} \left[\sum_{C^{gl} \in N_r(i,j)} \bar{C}_{ij}^{gl} L_w \bar{k}_{ij} \bar{a}_{ij}^2 \right] \right\},$$

$$\lambda = \inf_{1 \leq i \leq n, 1 \leq j \leq m} \alpha_{ij},$$

where $\Theta_{ij} = \inf_{t \in \mathbb{R}} \left\{ - \sum_{C^{gl} \in N_r(i,j)} C_{ij}^{gl}(t) w_{ij} \left(\int_0^{+\infty} k_{ij}(\sigma) \times h_{ij}^{-1}(\varphi_{gl}(t - \sigma)) d\sigma \right) h_{ij}^{-1}(\varphi_{ij}(t)) + I_{ij}(t) \right\}$.

Theorem 3.1 Assume that $(H_1) - (H_9)$ and **(H)** hold. If $M > 0$, $r < 1$ and $\frac{K}{1-r} < 1$, then there exists a unique nonzero almost periodic solution of (1).

Proof: Set $\mathbb{X} = \{\varphi(t) \in PC(\mathbb{R}, \mathbb{R}^n) : \varphi(t) = (\varphi_{11}(t), \dots, \varphi_{ij}(t), \dots, \varphi_{nm}(t))^T, \text{ where } \varphi_{ij}(t) \text{ is a almost periodic function satisfying } 0 < M \leq \inf_{t \in \mathbb{R}} |\varphi_{ij}(t)| \leq \sup_{t \in \mathbb{R}} |\varphi_{ij}(t)| \leq N = \frac{K}{1-r}, i = 1, 2, \dots, n, j = 1, 2, \dots, m\}$ with the norm

$$\|\varphi\| = \max_{1 \leq i \leq n, 1 \leq j \leq m} \left\{ \sup_{t \in \mathbb{R}} |\varphi_{ij}(t)| \right\},$$

then $(\mathbb{X}, \|\cdot\|)$ is a Banach space.

Define an operator Φ on \mathbb{X} by

$$(\Phi\varphi)(t) = ((\Phi_{11}\varphi)(t), \dots, (\Phi_{ij}\varphi)(t), \dots, (\Phi_{nm}\varphi)(t))^T, t \in \mathbb{R},$$

where

$$(\Phi_{ij}\varphi)(t) = \int_{-\infty}^t W_{ij}(t, s) \left[- \sum_{C^{gl} \in N_r(i,j)} C_{ij}^{gl}(s) \times w_{ij} \left(\int_0^{+\infty} k_{ij}(\sigma) h_{ij}^{-1}(\varphi_{ij}(s - \sigma)) d\sigma \right) h_{ij}^{-1}(\varphi_{gl}(s)) + I_{ij}(s) \right] ds + \sum_{\tau_k < t} W_{ij}(t, \tau_k) \nu_{ijk},$$

$$i = 1, 2, \dots, n, j = 1, 2, \dots, m. \tag{14}$$

Set \mathbb{X}^* be a subset of \mathbb{X} defined by

$$\mathbb{X}^* = \left\{ \varphi \in \mathbb{X} : \|\varphi - \varphi_0\| \leq \frac{rK}{1-r} \right\},$$

where

$$\varphi_0 = (\varphi_{011}, \dots, \varphi_{0ij}, \dots, \varphi_{0nm})^T$$

and

$$\varphi_{0ij} = \int_{-\infty}^t W_{ij}(t, s) I_{ij}(s) ds + \sum_{\tau_k < t} W_{ij}(t, \tau_k) \nu_{ijk},$$

$$i = 1, 2, \dots, n, j = 1, 2, \dots, m.$$

Then, it follows Lemma 2.5 that

$$\begin{aligned} \|\varphi_0\| &= \max_{1 \leq i \leq n, 1 \leq j \leq m} \left\{ \sup_{t \in \mathbb{R}} \left| \int_{-\infty}^t W_{ij}(t, s) I_{ij}(s) ds + \sum_{\tau_k < t} W_{ij}(t, \tau_k) \nu_{ijk} \right| \right\} \\ &\leq \max_{1 \leq i \leq n, 1 \leq j \leq m} \left\{ \sup_{t \in \mathbb{R}} \left[\int_{-\infty}^t |W_{ij}(t, s)| |I_{ij}(s)| ds + \sum_{\tau_k < t} |W_{ij}(t, \tau_k)| \nu_{ijk} \right] \right\} \\ &\leq \max_{1 \leq i \leq n, 1 \leq j \leq m} \left\{ \sup_{t \in \mathbb{R}} \left[\int_{-\infty}^t e^{2A} e^{-\alpha_{ij}(t-s)} \bar{I}_{ij} ds + \sum_{\tau_k < t} e^{2A} e^{-\alpha_{ij}(t-\tau_k)} \nu_{ijk} \right] \right\} \\ &\leq \max_{1 \leq i \leq n, 1 \leq j \leq m} \left\{ \frac{\bar{I}_{ij} e^{2A}}{\alpha_{ij}} + \frac{\gamma e^{2A}}{1 - e^{-\alpha_{ij}}} \right\} = K. \end{aligned} \tag{15}$$

Then for arbitrary $\varphi \in \mathbb{X}^*$, from (14) and (15) we have

$$\|\varphi\| \leq \|\varphi - \varphi_0\| + \|\varphi_0\| \leq \frac{rK}{1-r} + K = \frac{K}{1-r}.$$

Now we prove that Φ is self-mapping from \mathbb{X}^* to \mathbb{X}^* .

Firstly, we shall show that for arbitrary $\varphi \in \mathbb{X}^*$, then $\Phi\varphi \in \mathbb{X}^*$. In fact

$$\begin{aligned} &\|\Phi\varphi - \varphi_0\| \\ &= \max_{1 \leq i \leq n, 1 \leq j \leq m} \left\{ \sup_{t \in \mathbb{R}} \left| \int_{-\infty}^t W_{ij}(t, s) \left[- \sum_{C^{gl} \in N_r(i,j)} C_{ij}^{gl}(s) \times w_{ij} \left(\int_0^{+\infty} k_{ij}(\sigma) h_{ij}^{-1}(\varphi_{gl}(s - \sigma)) d\sigma \right) \times h_{ij}^{-1}(\varphi_{ij}(s)) \right] ds \right| \right\} \\ &\leq \max_{1 \leq i \leq n, 1 \leq j \leq m} \left\{ \sup_{t \in \mathbb{R}} \left[\int_{-\infty}^t e^{2A} e^{-\alpha_{ij}(t-s)} \times \left[\sum_{C^{gl} \in N_r(i,j)} \bar{C}_{ij}^{gl} L_w \bar{k}_{ij} \bar{a}_{ij} \|\varphi\| \right] \bar{a}_{ij} \|\varphi\| ds \right] \right\} \\ &\leq \max_{1 \leq i \leq n, 1 \leq j \leq m} \left\{ \frac{e^{2A}}{\alpha_{ij}} \left[\sum_{C^{gl} \in N_r(i,j)} \bar{C}_{ij}^{gl} L_w \bar{k}_{ij} \bar{a}_{ij}^2 \right] \right\} \|\varphi\| \\ &= r \|\varphi\| \leq \frac{rK}{1-r}. \end{aligned} \tag{16}$$

Moreover, we get

$$\begin{aligned} &\sup_{t \in \mathbb{R}} |(\Phi_{ij}\varphi)(t)| \\ &= \sup_{t \in \mathbb{R}} \left\{ \left| \int_{-\infty}^t W_{ij}(t, s) \left[- \sum_{C^{gl} \in N_r(i,j)} C_{ij}^{gl}(s) \times w_{ij} \left(\int_0^{+\infty} k_{ij}(\sigma) h_{ij}^{-1}(\varphi_{gl}(s - \sigma)) d\sigma \right) h_{ij}^{-1}(\varphi_{ij}(s)) + I_{ij}(s) \right] ds + \sum_{\tau_k < t} W_{ij}(t, \tau_k) \nu_{ijk} \right| \right\} \end{aligned}$$

$$\leq \max_{1 \leq i \leq n, 1 \leq j \leq m} \left\{ \frac{e^{2A}}{\alpha_{ij}} \left[\sum_{C^{gl} \in N_r(i,j)} \overline{C}_{ij}^{gl} L_w \bar{k}_{ij} \bar{a}_{ij} \|\varphi\| \right] \times \bar{a}_{ij} \|\varphi\| \right\} + K$$

$$\leq \frac{rK}{1-r} + K = \frac{K}{1-r} = N \tag{17}$$

and

$$\inf_{t \in \mathbb{R}} |(\Phi_{ij}\varphi)(t)|$$

$$= \inf_{t \in \mathbb{R}} \left\{ \left| \int_{-\infty}^t W_{ij}(t,s) \left[- \sum_{C^{gl} \in N_r(i,j)} C_{ij}^{gl}(s) \times w_{ij} \left(\int_0^{+\infty} k_{ij}(\sigma) h_{ij}^{-1}(\varphi_{gl}(s-\sigma)) d\sigma \right) \times h_{ij}^{-1}(\varphi_{ij}(s)) + I_{ij}(s) \right] ds + \sum_{\tau_k < t} W_{ij}(t, \tau_k) \nu_{ijk} \right| \right\}$$

$$\geq \inf_{t \in \mathbb{R}} \left\{ \left| \int_{-\infty}^t W_{ij}(t,s) \left[- \sum_{C^{gl} \in N_r(i,j)} C_{ij}^{gl}(s) \times w_{ij} \left(\int_0^{+\infty} k_{ij}(\sigma) h_{ij}^{-1}(\varphi_{gl}(s-\sigma)) d\sigma \right) \times h_{ij}^{-1}(\varphi_{ij}(s)) + I_{ij}(s) \right] ds - \left| \sum_{\tau_k < t} W_{ij}(t, \tau_k) \nu_{ijk} \right| \right| \right\}$$

$$\geq \inf_{t \in \mathbb{R}} \left\{ \int_{-\infty}^t W_{ij}(t,s) \Theta_{ij} ds \right\} - \sup_{t \in \mathbb{R}, k \in \mathbb{Z}} \left\{ \left| \sum_{\tau_k < t} W_{ij}(t, \tau_k) \nu_{ijk} \right| \right\}$$

$$\geq \inf_{t \in \mathbb{R}} \left\{ \int_{-\infty}^t e^{-\beta_{ij}(t-s)} \Theta_{ij} ds \right\} - \sup_{k \in \mathbb{Z}} |\gamma_{ijk}| \frac{e^{2A}}{1-e^{-\alpha_{ij}}}$$

$$\geq \min_{1 \leq i, j \leq n} \left\{ \frac{\Theta_{ij}}{\beta_{ij}} - \frac{\gamma e^{2A}}{1-e^{-\alpha_{ij}}} \right\} = M, \tag{18}$$

$$i = 1, 2, \dots, n, j = 1, 2, \dots, m.$$

Now, we shall prove that $\Phi\varphi$ is almost periodic. In fact, let $\tau \in T, q \in Q$, where the sets T and Q are determined in Lemma 2.3. By Lemma 2.5, we have

$$|(\Phi_{ij}\varphi)(t+\tau) - (\Phi_{ij}\varphi)(t)|$$

$$= \left| \int_{-\infty}^{t+\tau} W_{ij}(t+\tau, s) \left[- \sum_{C^{gl} \in N_r(i,j)} C_{ij}^{gl}(s) \times w_{ij} \left(\int_0^{+\infty} k_{ij}(\sigma) h_{ij}^{-1}(\varphi_{gl}(s-\sigma)) d\sigma \right) \times h_{ij}^{-1}(\varphi_{ij}(s)) + I_{ij}(s) \right] ds + \sum_{\tau_k < t+\tau} W_{ij}(t+\tau, \tau_k) \nu_{ijk} - \int_{-\infty}^t W_{ij}(t, s) \left[- \sum_{C^{gl} \in N_r(i,j)} C_{ij}^{gl}(s) \times w_{ij} \left(\int_0^{+\infty} k_{ij}(\sigma) h_{ij}^{-1}(\varphi_{gl}(s-\sigma)) d\sigma \right) \times h_{ij}^{-1}(\varphi_{ij}(s)) + I_{ij}(s) \right] ds - \sum_{\tau_k < t} W_{ij}(t, \tau_k) \nu_{ijk} \right|$$

$$\leq \int_{-\infty}^t |W_{ij}(t+\tau, s+\tau) - W_{ij}(t, s)| \left| - \sum_{C^{gl} \in N_r(i,j)} C_{ij}^{gl}(s+\tau) w_{ij} \left(\int_0^{+\infty} k_{ij}(\sigma) \times h_{ij}^{-1}(\varphi_{gl}(s+\tau-\sigma)) d\sigma \right) h_{ij}^{-1}(\varphi_{ij}(s+\tau)) + I_{ij}(s+\tau) \right| ds + \int_{-\infty}^t |W_{ij}(t, s)| \left| - \sum_{C^{gl} \in N_r(i,j)} C_{ij}^{gl}(s+\tau) w_{ij} \left(\int_0^{+\infty} k_{ij}(\sigma) \times h_{ij}^{-1}(\varphi_{gl}(s+\tau-\sigma)) d\sigma \right) h_{ij}^{-1}(\varphi_{ij}(s+\tau)) + \sum_{C^{gl} \in N_r(i,j)} C_{ij}^{gl}(s) w_{ij} \left(\int_0^{+\infty} k_{ij}(\sigma) \times h_{ij}^{-1}(\varphi_{gl}(s-\sigma)) d\sigma \right) h_{ij}^{-1}(\varphi_{ij}(s)) + I_{ij}(s+\tau) - I_{ij}(s) \right| ds + \sum_{\tau_k < t} |W_{ij}(t+\tau, \tau_{k+q}) - W_{ij}(t, \tau_k)| |\nu_{ij(k+q)}| + \sum_{\tau_k < t} |W_{ij}(t, \tau_k)| |\nu_{ij(k+q)} - \nu_{ijk}|$$

$$\leq C\epsilon, \quad i = 1, 2, \dots, n, j = 1, 2, \dots, m, \tag{19}$$

where

$$C = \max_{1 \leq i, j \leq n} \left\{ \frac{1}{\alpha_{ij}} \sum_{C^{gl} \in N_r(i,j)} (2\Gamma \overline{C}_{ij}^{gl} L_w \bar{k}_{ij} + L_w \bar{k}_{ij} e^{2A}) \bar{a}_{ij}^2 N + \frac{e^{2A} + 2\Gamma \bar{I}_{ij}}{\alpha_{ij}} + \frac{e^{2A}}{\alpha_{ij}} \sum_{C^{gl} \in N_r(i,j)} \overline{C}_{ij}^{gl} L_w \bar{k}_{ij} \bar{a}_{ij}^2 + \frac{\gamma \Gamma}{1 - e^{-\frac{\alpha_{ij}}{2}}} + \frac{e^{2A}}{1 - e^{-\alpha_{ij}}} \right\}.$$

It follows from (16) – (19) that $\Phi\varphi \in \mathbb{X}^*$.

For arbitrary $\varphi \in \mathbb{X}^*, \psi \in \mathbb{X}^*$, we can get

$$\|\Phi\varphi - \Phi\psi\|$$

$$= \max_{1 \leq i \leq n, 1 \leq j \leq m} \left\{ \sup_{t \in \mathbb{R}} \left| \int_{-\infty}^t W_{ij}(t, s) \left[- \sum_{C^{gl} \in N_r(i,j)} C_{ij}^{gl}(s) \times w_{ij} \left(\int_0^{+\infty} k_{ij}(\sigma) h_{ij}^{-1}(\varphi_{gl}(s-\sigma)) d\sigma \right) h_{ij}^{-1}(\varphi_{ij}(s)) ds - \int_{-\infty}^t W_{ij}(t, s) \left[- \sum_{C^{gl} \in N_r(i,j)} C_{ij}^{gl}(s) \times w_{ij} \left(\int_0^{+\infty} k_{ij}(\sigma) h_{ij}^{-1}(\psi_{gl}(s-\sigma)) d\sigma \right) \times h_{ij}^{-1}(\psi_{ij}(s)) ds \right] \right| \right| \right\}$$

$$\leq \max_{1 \leq i \leq n, 1 \leq j \leq m} \left\{ \frac{e^{2A}}{\alpha_{ij}} \left[\sum_{C^{gl} \in N_r(i,j)} \overline{C}_{ij}^{gl} L_w \bar{k}_{ij} \bar{a}_{ij}^2 \right] \right\}$$

$$\begin{aligned} & \times \max_{1 \leq i \leq n, 1 \leq j \leq m} \left\{ \sup_{t \in \mathbb{R}} |\varphi_{ij}(t) - \psi_{ij}(t)| \right\} \\ & = r \|\varphi - \psi\| < \|\varphi - \psi\|. \end{aligned} \tag{20}$$

Then from (20), it follows that Φ is a contraction operator in \mathbb{X}^* . So, Φ has exactly a unique nonzero fixed point φ^* in \mathbb{X}^* such that $\Phi\varphi^* = \varphi^*$. It is easy to verify that φ^* satisfies (7). Thus, system (1) has exactly one nonzero almost periodic solution. This completes the proof. ■

Theorem 3.2 Assume that the conditions in Theorem 3.1 hold. If $r < \lambda$, then the unique nonzero almost periodic solution of (1) is exponentially stable.

Proof: Let $x(t)$ be arbitrary solution of (7) with the initial condition $x(t_0 + 0, t_0, x_0) = x_0$, and $y(t) = (y_{11}(t), \dots, y_{ij}(t), \dots, y_{mn}(t))^T$ be the unique almost periodic solution of (7) with the initial condition $y(t_0 + 0, t_0, y_0) = y_0$. Then from (14), we have

$$\begin{aligned} & x(t) - y(t) \\ & = W(t, t_0)(x_0 - y_0) \\ & + \int_{t_0}^t W(t, s) \left[- \sum_{C^{gl} \in N_r(i, j)} C_{ij}^{gl}(s) \times \right. \\ & w_{ij} \left(\int_0^{+\infty} k_{ij}(\sigma) h_{ij}^{-1}(x_{gl}(s - \sigma)) d\sigma \right) h_{ij}^{-1}(x_{ij}(s)) ds \\ & + \sum_{C^{gl} \in N_r(i, j)} C_{ij}^{gl}(s) w_{ij} \left(\int_0^{+\infty} k_{ij}(\sigma) \times \right. \\ & \left. \left. h_{ij}^{-1}(y_{gl}(s - \sigma)) d\sigma \right) h_{ij}^{-1}(y_{ij}(s)) ds \right] ds. \end{aligned} \tag{21}$$

It follows from Lemma 2.5, (20) and (21) that

$$\begin{aligned} & \|x - y\| \\ & \leq e^{2A} e^{-\lambda(t-t_0)} \|x_0 - y_0\| \\ & + \max_{1 \leq i \leq n, 1 \leq j \leq m} \left\{ \frac{e^{2A}}{\alpha_{ij}} \left[\sum_{C^{gl} \in N_r(i, j)} \bar{C}_{ij}^{gl} \bar{L}_w \bar{k}_{ij} \bar{a}_{ij}^2 \right] \right\} \\ & \int_{t_0}^t e^{-\lambda(t-t_0)} \|x(s) - y(s)\| ds \\ & \leq e^{2A} e^{-\lambda(t-t_0)} \|x_0 - y_0\| \\ & + r \int_{t_0}^t e^{-\lambda(t-s)} \|x(s) - y(s)\| ds, \end{aligned}$$

that is

$$\begin{aligned} \|x - y\| e^{\lambda t} & \leq e^{2A} e^{\lambda t_0} \|x_0 - y_0\| \\ & + r \int_{t_0}^t e^{\lambda s} \|x(s) - y(s)\| ds. \end{aligned}$$

By Gronwall-Bellman's Lemma, we have

$$\|x - y\| \leq e^{2A} \|x_0 - y_0\| e^{(r-\lambda)(t-t_0)}.$$

So, the almost periodic solution $y(t)$ is exponentially stable since $r - \lambda < 0$. Thus the unique almost periodic solution of (1) is exponentially stable. This completes the proof. ■

IV. AN EXAMPLE

Consider the following CGSICNNs:

$$\begin{cases} x'_{ij}(t) = -a_{ij}(x_{ij}(t)) \left[b_{ij}(x_{ij}(t)) + \sum_{C^{gl} \in N_r(i, j)} C_{ij}^{gl}(t) \right. \\ \quad \times w_{ij} \left(\int_0^{+\infty} k_{ij}(s) x_{gl}(t-s) ds \right) x_{ij}(t) \\ \quad \left. - I_{ij}(t) \right], \quad t \neq \tau_k, \\ \Delta x_{ij}(\tau_k) = \alpha_{ijk} x_{ij}(\tau_k) + \frac{1-e^{4-4e^4}}{30e^8}, \\ \quad i = 1, 2, \dots, n, \quad j = 1, 2, \dots, m, \quad k \in \mathbb{Z}, \end{cases} \tag{22}$$

where

$$\begin{aligned} (a_{ij})_{2 \times 2} & = \begin{bmatrix} 3 + \sin u & 3 + \cos u \\ 3 - \sin u & 3 - \cos u \end{bmatrix}, \\ (b_{ij})_{2 \times 2} & = \begin{bmatrix} 0.5e^4 u & e^4 u \\ 1.5e^4 u & 2e^4 u \end{bmatrix}, \\ (C_{ij})_{2 \times 2} & = \begin{bmatrix} 0.5 & 1.5 \\ 1.5 & 0.5 \end{bmatrix}, \\ (I_{ij})_{2 \times 2} & = \begin{bmatrix} 0.45 + 0.05 \sin t & 0.45 - 0.05 \sin t \\ 0.45 + 0.05 \cos t & 0.45 - 0.05 \cos t \end{bmatrix}, \\ \omega_{ij} & = \frac{1}{32} |u|, \quad K_{ij}(t) = e^{-e^4 t}, \quad i, j = 1, 2, \end{aligned}$$

and (H_2) holds with $A = 2$. Obviously,

$$\begin{aligned} \bar{a}_{ij} & = 4, \quad \underline{a}_{ij} = 2, \quad i, j = 1, 2; \quad \bar{b}'_{11} = \underline{b}'_{11} = 0.5e^4, \\ \bar{b}'_{12} & = \underline{b}'_{12} = e^4, \quad \bar{b}'_{21} = \underline{b}'_{21} = 1.5e^4, \quad \bar{b}'_{22} = \underline{b}'_{22} = 2e^4, \\ \Sigma_{C^{gl} \in N_1(i, j)} \bar{C}_{11}^{gl} & = \Sigma_{C^{gl} \in N_1(i, j)} \bar{C}_{12}^{gl} = \Sigma_{C^{gl} \in N_1(i, j)} \bar{C}_{21}^{gl} \\ & = \Sigma_{C^{gl} \in N_1(i, j)} \bar{C}_{22}^{gl} = 4, \quad L_b = 2e^4, \quad L_\omega = \frac{1}{32}, \\ \bar{I}_{11} & = \bar{I}_{12} = \bar{I}_{21} = \bar{I}_{22} = 0.5, \quad \underline{I}_{11} = \underline{I}_{12} = \underline{I}_{21} = \underline{I}_{22} \\ & = 0.4, \quad 1 \leq \alpha_{ijk} \leq \frac{e^2}{2} - 1, \quad k \in \mathbb{Z}, \quad \alpha_{11} = \underline{a}_{11} \underline{b}'_{11} - 2A \\ & = e^4 - 4, \quad \alpha_{12} = \underline{a}_{12} \underline{b}'_{12} - 2A = 2e^4 - 4, \quad \alpha_{21} = \underline{a}_{21} \underline{b}'_{21} \\ & - 2A = 3e^4 - 4, \quad \alpha_{22} = \underline{a}_{22} \underline{b}'_{22} - 2A = 4e^4 - 4, \\ \beta_{11} & = \bar{a}_{11} \bar{b}'_{11} = 2e^4, \quad \beta_{12} = \bar{a}_{12} \bar{b}'_{12} = 4e^4, \\ \beta_{21} & = \bar{a}_{21} \bar{b}'_{21} = 6e^4, \quad \beta_{22} = \bar{a}_{22} \bar{b}'_{22} = 8e^4. \end{aligned}$$

By a direct calculation, we get

$$\begin{aligned} \min_{1 \leq i, j \leq 2} \Theta_{ij} & = 0.4 - \frac{2}{e^4}, \\ M & = \frac{0.4 - \frac{2}{e^4}}{8e^4} - \frac{e^4}{1 - e^{4-4e^4}} \frac{1 - e^{4-4e^4}}{30e^8} > 0, \\ K & = \frac{0.5e^4}{e^4 - 4} + \frac{1}{30e^4}, \quad r = \frac{2}{e^4 - 4} < 1, \end{aligned}$$

then $\frac{K}{1-r} < 1$ and $r < \alpha_{ij}, i, j = 1, 2$.

Now, we can see that all conditions are hold, according to Theorem 3.1 and Theorem 3.2, system (22) has one unique nonzero almost periodic solution which is exponentially stable.

ACKNOWLEDGMENT

This work is supported by the National Natural Sciences Foundation of People's Republic of China under Grant 61073065.

REFERENCES

- [1] A. Bouzerdoum, R.B. Pinter, Analysis and analog implementation of directionally sensitive shunting inhibitory cellular neural networks, *Visual Information Processing: From Neurons to Chips*, vol. 1473, SPIE, 1991, pp. 29-38.
- [2] A. Bouzerdoum, R.B. Pinter, Nonlinear lateral inhibition applied to motion detection in the fly visual system, in: R.B. Pinter, B. Nabet (Eds.), *Nonlinear Vision*, CRC Press, Boca Raton, FL, 1992, pp. 423-450.
- [3] A. Bouzerdoum, R.B. Pinter, Shunting inhibitory cellular neural networks: derivation and stability analysis, *IEEE Trans. Circuits and Systems 1-Fundamental Theory and Applications* 40 (1993) 215-221.
- [4] M. Cai, W. Xiong, Almost periodic solutions for shunting inhibitory cellular neural networks without global Lipschitz and bounded activation functions, *Phys. Lett. A* 362 (2007) 417-423.
- [5] M. Cai, H. Zhang, Z. Yuan, Positive almost periodic solutions for shunting inhibitory cellular neural networks with time-varying delays, *Math. Comput. Simul.* 78 (2008) 548-558.
- [6] A. Chen, J. Cao, L. Huang, Almost periodic solution of shunting inhibitory CNNs with delays, *Phys. Lett. A* 298 (2002) 161-170.
- [7] X. Huang, J. Cao, Almost periodic solutions of inhibitory cellular neural networks with time-vary delays, *Phys. Lett. A* 314 (2003) 222-231.
- [8] Y.K. Li, C. Liu, L. Zhu, Global exponential stability of periodic solution of shunting inhibitory CNNs with delays, *Phys. Lett. A* 337 (2005) 46-54.
- [9] B. Liu, L. Huang, Existence and stability of almost periodic solutions for shunting inhibitory cellular neural networks with continuously distributed delays, *Phys. Lett. A* 349 (2006) 177-186.
- [10] B. Liu, L. Huang, Existence and stability of almost periodic solutions for shunting inhibitory cellular neural networks with continuously distributed delays, *Phys. Lett. A* 349 (2006) 177-186.
- [11] S. Mohamad, K. Gopalsamy, H. Akca, Exponential stability of artificial neural networks with distributed delays and large impulses, *Nonlinear Anal.* 9 (2008) 872-888.
- [12] L. Zhou, W. Li, Y. Chen, Suppressing the periodic impulse in partial discharge detection based on chaotic control, *Automat. Electric Power Syst.* 28 (2004) 90-94.
- [13] V. Nagesh, Automatic detection and elimination of periodic pulse shaped interference in partial discharge measurements, *IEE Proc. Meas. Technol.* 141 (1994) 335-339.
- [14] C. Bai, Stability analysis of Cohen-Crossberg BAM neural networks with delays and impulses, *Chaos, Solitons & Fractals* 35 (2008) 263-267.
- [15] Z. Yang, D. Xu, Existence and exponential stability of periodic solution for impulsive delay differential equations and applications, *Nonlinear Anal.* 64 (2006) 130-145.
- [16] D. Bainov, P. Simeonov, *Impulsive Differential Equations: Periodic Solutions and Applications*, Longman Scientific and Technical Group Limited, New York, 1993.
- [17] Y. Li, L. Lu, Global exponential stability and existence of periodic solution of Hopfield-type neural networks with impulses *Phys. Lett. A* 333 (2004) 62-71.
- [18] Y. Li, Global exponential stability of BAM neural networks with delays and impulses *Chaos, Solitons & Fractals*, 24 (2005) 279-285.
- [19] A.M. Fink, *Almost Periodic Differential Equations*, Springer, Berlin, 1974.
- [20] D. Cheban, C. Mamma, Invariant manifolds, global attractors and almost periodic solutions of nonautonomous difference equations, *Nonlinear Anal.* 56(4) (2004) 465-484.
- [21] A.M. Samoilenko, N.A. Perestyuk, *Differential Equations with Impulse Effect*, World Scientific, Singapore, 1995.
- [22] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, New Jersey, London, 1989.
- [23] G.T. Stamov, On the existence of almost periodic solutions for the impulsive Lasota-Ważewska model, *Appl. Math. Lett.* 22 (2009) 516-520.