Permanence and global attractivity of a delayed predator-prey model with mutual interference

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Abstract—By utilizing the comparison theorem and Lyapunov second method, some sufficient conditions for the permanence and global attractivity of positive periodic solution for a predator-prey model with mutual interference $m \in (0,1)$ and delays τ_i are obtained. It is the first time that such a model is considered with delays. The significant is that the results presented are related to the delays and the mutual interference constant m. Several examples are illustrated to verify the feasibility of the results by simulation in the last part.

Keywords—Predator-prey model, Mutual interference, Delays, Permanence, Global attractivity

I. INTRODUCTION

IN this paper, we consider the permanence and global attractivity of a delayed predator-prey population model with mutual interference in the form:

$$\begin{cases} \dot{x}(t) = x(t)(a_1(t) - b_1(t)x(t-\tau_1) - c_1(t)y^m(t-\tau_2)), \\ \dot{y}(t) = y(t)(-a_2(t) - b_2(t)y(t-\tau_3) + c_2(t)x(t)y^{m-1}(t)), \\ x(\theta) = \varphi(\theta) \geqslant 0, \ \theta \in [-\tau, 0], \ \varphi(0) = \varphi_0 > 0, \\ y(\theta) = \psi(\theta) \geqslant 0, \ \psi \in [-\tau, 0], \ \psi(0) = \psi_0 > 0, \end{cases}$$
(PP)

where $\tau_i (i = 1,2,3)$ are nonnegative constants, $\tau = \max_{1 \leqslant i \leqslant 3} \{\tau_i\}$, φ and ψ are continuous on $[-\tau, 0]$; $x(t)$ and $y(t)$ denote the size of prey and predator at time t, respectively; a_i, b_i and $c_i (i = 1,2)$ are continuous and bounded above and below by positive constants; $m \in (0,1)$ is mutual interference constant, which was introduced by Hassell in 1971, see [1]–[3] for more details.

Recently, there are some literatures on studying the species dynamics, such as the permanence, positive periodic solutions, positive almost periodic solution, global attractivity etc., of the population model with mutual interference, see [4]–[15] for more details. It was pointed by Kuang(1993) [16] that any model of species dynamics without delays is an approximation at best, more detailed arguments on the importance and use-fulness of time-delays in realistic models may also be found in the classical books of Macdonald (1989) [17] and Gopalsamy (1992) [18]. But there are few literatures on considering the delays in population model with mutual interference in the form of model (PP).

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The structure of this paper is as follows. In section 2, some useful lemmas and definitions are presented. In section 3, by using the comparison theorem we estimate the eventually upper and lower bounds of all positive solutions for system (PP) and present some sufficient conditions for the permanence of the (PP) model, and by applying the Brouwer fixed point theorem we prove the existence of positive periodic solutions for system (PP) under some sufficient conditions. In section 4, sufficient conditions are presented for the global attractivity of the system (PP). Lastly, several examples are given to verify the feasibility of the results by simulation.

For the sake of convenience, we set

$$f^L = \inf_{t \in E} \{f(t)\}, \ f^U = \inf_{t \in E} \{f(t)\},$$

where f is a continuously bounded function defined on interval $E = [0, +\infty)$, and denote

$$M_1 = \frac{a_1^U}{b_1^L} \exp\{a_1^U \tau_1\}, \ M_2 = \left[\frac{c_2^U M_1}{a_2^L}\right]^{\frac{1}{1-m}},$$

$$M_3 = \min\left\{\frac{K_1}{b_1^U} \exp\left\{(K_1 - b_1^U M_1)\tau_1\right\}, \frac{K_1}{b_1^U}\right\},$$

$$M_4 = \left[\frac{K_2}{a_2^U}\right]^{\frac{1}{1-m}}, \ K_1 = \liminf_{t \to +\infty}(a_1(t) - M_2^m c_1(t)),$$

$$K_2 = \liminf_{t \to +\infty}\left(M_3 c_2(t) - M_2^{2-m} b_2(t)\right),$$

$$g_1(t) = \beta\left[b_1(t) - (a_1(t) + M_1 b_1(t) + M_2^m c_1(t)) \times \right.$$
$$\int_t^{t+\tau_1} b_1(l)dl - M_1 b_1(t+\tau_2) \int_{t+\tau_2}^{t+\tau_1+\tau_2} b_1(l)dl\right]$$
$$\left. - c_2(t)M_4^{m-1}\left[1 + M_2 \int_t^{t+\tau_3} b_2(l)dl\right]\right],$$

$$g_2(t) = \left[b_2(t) - \left(a_2(t) + M_2 b_2(t) + M_1 M_4^{m-1} c_1(t)\right) \times \right.$$
$$\int_t^{t+\tau_3} b_2(l)dl - M_2 b_2(t+\tau_3) \int_{t+\tau_3}^{t+2\tau_3} b_2(l)dl\right],$$

$$g_3(t) = c_2(t)\left[M_3 - M_1 M_2 \int_t^{t+\tau_3} b_2(l)dl\right],$$

$$g_4(t) = \beta c_1(t+\tau_2)\left[1 + M_1 b_1(t+\tau_2) \int_{t+\tau_2}^{t+\tau_1+\tau_2} b_1(l)dl\right],$$

$$G_1(t) = g_2(t) + (1-m)M_2^{m-2} g_3(t) - m M_4^{m-1} g_4(t),$$

$$G_2(t) = g_2(t) + (1-m)M_4^{m-2} g_3(t) - m M_4^{m-1} g_4(t),$$

$$G_3(t) = g_2(t) + (1-m)M_4^{m-2} g_3^L - m M_4^{m-1} g_4(t),$$

where β is a given positive constant.

II. Lemmas and definitions

In this section, we give some important lemmas and definitions, which will be used in next sections.

Lemma 2.1. (See [19]) If $a > 0, b > 0$ and $\dot{z}(t) \geqslant (\leqslant) b - a\, z(t)$ for $t \geqslant 0$, $z(0) > 0$, then the following inequality holds

$$z(t) \geqslant (\leqslant)\frac{b}{a} + \left(z(0) - \frac{b}{a}\right)\exp\{-a\,t\}$$

or

$$\liminf_{t\to+\infty} y(t) \geqslant \frac{b}{a}, \quad \limsup_{t\to+\infty} y(t) \leqslant \frac{b}{a}.$$

Lemma 2.2. (See [20], [21]) If $a > 0$, $b > 0$, $\tau \geqslant 0$, for $t \in R$, and $\dot{y}(t) \leqslant y(t)(b - a\, y(t-\tau))$, then there exists a constant $T > 0$ such that

$$y(t) \leqslant \frac{b}{a}\,\exp\{b\,\tau\}\ \text{ for } t \geqslant T.$$

Lemma 2.3. If $a > 0$, $b > 0$, $\tau \geqslant 0$, for $t \in R$, and $\dot{y}(t) \geqslant y(t)(b - a\, y(t-\tau))$, and $\limsup_{t\to+\infty} y(t) \leqslant M$, then

$$\liminf_{t\to+\infty} y(t) \geqslant \min\left\{\frac{b}{a}\,\exp\{(b - aM)\tau\}, \frac{b}{a}\right\}.$$

The proof is similar to that of [20] and [21], so we omit it here.

Definition 2.1. System (PP) is said to be permanent if there exist positive numbers $M \geqslant m$ such that any positive solution of (PP) satisfying

$$m \leq \liminf_{t\to+\infty}\{x(t), y(t)\} \leqslant \limsup_{t\to+\infty}\{x(t), y(t)\} \leqslant M.$$

Definition 2.2. System (PP) is called globally attractive, if

$$\lim_{t\to+\infty}(|x(t) - x_0(t)| + |y(t) - y_0(t)|) = 0,$$

for any two positive solutions $(x(t), y(t))$ and $(x_0(t), y_0(t))$ of system (PP).

III. Permanence and positive periodic solutions

In this section, the permanence of the (PP) model is discussed. Some sufficient conditions are presented for it.

Theorem 3.1. If $K_1 > 0$ and $K_2 > 0$, then system (PP) is permanent.

Proof. Suppose that $(x(t), y(t))$ is any positive solution of system (PP). We first estimate the eventually upper bounds of the positive solution $(x(t), y(t))$ of system (PP). From the first equation of model (PP), we get

$$\dot{x}(t) \leqslant x(t)\left(a_1^U - b_1^L\, x(t - \tau_1)\right),$$

it follows from Lemma 2.2 that there exists constant $T_1 > 0$ such that

$$x(t) \leqslant M_1\ \text{ for }\ t > T_1.$$

Similarly, from the second equation of system (PP) we can get

$$\dot{y}(t) \leqslant y^m(t)\left(c_2^U M_1 - a_2^L\, y^{1-m}(t)\right),$$

i.e.,

$$\frac{d(y^{1-m}(t))}{d\,t} \leqslant (1 - m)\left(c_2^U M_1 - a_2^L\, y^{1-m}(t)\right)\ \text{ for } t > T_1.$$

The above inequality and Lemma 2.1 yield

$$y^{1-m}(t) \leqslant \frac{c_2^U M_1}{a_2^L} + \left[\psi_0^{1-m} - \frac{c_2^U M_1}{a_2^L}\right]\exp\left\{(m-1)c_2^U M_1 t\right\},$$

which yields

$$\limsup_{t\to+\infty} y(t) \leqslant M_2. \tag{1}$$

We now estimate the eventually lower bounds of the positive solution $(x(t), y(t))$ of the system. From (1) and the first equation of model (PP) we get

$$\dot{x}(t) \geqslant x(t)\left(a_1(t) - c_1(t)M_2^m - b_1^U x(t - \tau_1)\right),\ t \to +\infty.$$

It follows from $K_1 > 0$ and Lemma 2.3 that

$$\liminf_{t\to+\infty} x(t) \geqslant M_3.$$

On the other hand, the second equation of system (PP) yields

$$\dot{y}(t) \geqslant y^m(t)\left(c_2(t)M_3 - b_2(t)M_2^{2-m} - a_2^U y^{1-m}(t)\right),$$

as $t \to +\infty$. Thus Lemma 2.1 and $K_2 > 0$ yield

$$\liminf_{t\to+\infty} y(t) \geqslant M_4.$$

The proof is now finished. ∎

If all coefficients in model (PP) are continuously periodic functions, i.e., it is a periodic system, then Theorem 3.1 and the Brouwer fixed point theorem yield the following result.

Theorem 3.2. If model (PP) is an ω-periodic system, then Theorem 3.1 yields that the ω-periodic model (PP) has at least one positive ω-periodic solution.

IV. Global attractivity

In this section, we will present some sufficient conditions for the global attractivity of model (PP). Assume that all conditions in Theorem 3.1 hold, and β can be chosen freely in R^+.

Theorem 4.1. If $g_3^L \geqslant 0$ and $\liminf_{t\to+\infty}\{g_1(t),\, G_1(t)\} > 0$, then system (PP) is globally attractive.

Theorem 4.2. If $g_3^U \leqslant 0$ and $\liminf_{t\to+\infty}\{g_1(t),\, G_2(t)\} > 0$, then system (PP) is globally attractive.

Theorem 4.3. If $g_3^L < 0 < g_3^U$ and $\liminf_{t\to+\infty}\{g_1(t),\, G_3(t)\} > 0$, then system (PP) is globally attractive.

The proofs of the above theorems are similar, thus we only present the complete proof of Theorem 4.1.

Proof. Suppose that $(x_0(t), y_0(t))$ and $(x(t), y(t))$ are any two

positive solutions of system (PP), by Theorem 3.1 one know that there exists positive constant T such that

$$M_3 \leqslant x_0(t), x(t) \leqslant M_1; \quad M_4 \leqslant y_0(t), y(t) \leqslant M_2 \text{ for } t > T.$$

Define the Lyapunov functional by

$$V(t) = V_1(t) + V_2(t) + V_3(t) \text{ for } t > T,$$

where

$$V_1(t) = \beta \left| \ln x(t) - \ln x_0(t) \right| + \left| \ln y(t) - \ln y_0(t) \right|,$$

$$
\begin{aligned}
V_2(t) &= \beta \int_{t-\tau_2}^{t} c_1(s+\tau_2)|y^m(s) - y_0^m(s)|ds \\
&+ \beta \int_{t}^{t+\tau_1} \int_{l-\tau_1}^{t} b_1(l)(a_1(s) + M_1 b_1(s) + M_2^m c_1(s)) \\
&\times |x(s) - x_0(s)|dsdl + \beta M_1 \int_{t}^{t+\tau_1} \int_{l-\tau_1}^{t} b_1(l)b_1(s) \\
&\times |x(s-\tau_2) - x_0(s-\tau_2)|dsdl + \beta M_1 \int_{t}^{t+\tau_1} \int_{l-\tau_1}^{t} \\
&\times b_1(l)b_1(s)c_1(s)|y^m(s-\tau_2) - y_0^m(s-\tau_2)|dsdl \\
&+ \int_{t}^{t+\tau_3} \int_{l-\tau_3}^{t} b_2(l)|y(s) - y_0(s)| \\
&\times (a_2(s) + M_2 b_2(s) + M_1 M_4^{m-1} c_1(s))dsdl \\
&+ M_2 \int_{t}^{t+\tau_3} \int_{l-\tau_3}^{t} b_2(l)b_2(s)|y(s-\tau_3) - y_0(s-\tau_3)|dsdl \\
&+ M_1 M_2 \int_{t}^{t+\tau_3} \int_{l-\tau_3}^{t} b_2(l)c_2(s)|y^{m-1}(s) - y_0^{m-1}(s)|dsdl \\
&+ M_2 M_4^{m-1} \int_{t}^{t+\tau_3} \int_{l-\tau_3}^{t} b_2(l)c_2(s)|x(s) - x_0(s)|dsdl
\end{aligned}
$$

and

$$
\begin{aligned}
V_3(t) &= \beta M_1 \int_{t-\tau_2}^{t} \int_{s+\tau_2}^{s+\tau_1+\tau_2} b_1(l)b_1(s+\tau_2)|x(s) - x_0(s)|dlds \\
&+ \beta M_1 \int_{t-\tau_2}^{t} \int_{s+\tau_2}^{s+\tau_1+\tau_2} b_1(l)b_1(s+\tau_2)c_1(s+\tau_2) \\
&\times |y^m(s) - y_0^m(s)|dlds \\
&+ M_2 \int_{t-\tau_3}^{t} \int_{s+\tau_3}^{s+2\tau_3} b_2(l)b_2(s+\tau_3)|y(s) - y_0(s)|dlds.
\end{aligned}
$$

Computing its Dini derivative along system (PP), we have

$$
\begin{aligned}
D^+ V_1(t)|_{(PP)} &\leqslant -\beta b_1(t)|x(t) - x_0(t)| - b_2(t)|y(t) - y_0(t)| \\
&+ b_2(t) \left| \int_{t-\tau_3}^{t} (\dot{y}(s) - \dot{y}_0(s))ds \right| \\
&+ \beta c_1(t)|y^m(t-\tau_2) - y_0^m(t-\tau_2)| \\
&+ \beta b_1(t) \left| \int_{t-\tau_1}^{t} (\dot{x}(s) - \dot{x}_0(s))ds \right| \\
&+ c_2(t)\mathrm{sgn}(y(t) - y_0(t))(x(t)y^{m-1}(t) - x_0(t)y_0^{m-1}(t)).
\end{aligned}
\tag{2}
$$

Meanwhile, we have

$$
\begin{aligned}
&\mathrm{sgn}(y(t) - y_0(t))(x(t)y^{m-1}(t) - x_0(t)y_0^{m-1}(t)) \\
&\leqslant -x(t)|y^{m-1}(t) - y_0^{m-1}(t)| + y_0^{m-1}(t)|x(t) - x_0(t)|
\end{aligned}
\tag{3}
$$

and

$$
\begin{aligned}
&\int_{t-\tau_1}^{t} (\dot{x}(s) - \dot{x}_0(s))ds \\
&= \int_{t-\tau_1}^{t} (x(s) - x_0(s))(a_1(s) - b_1(s)x_0(s-\tau_1)) \\
&\quad - c_1(s)y_0^m(s-\tau_2))ds \\
&+ \int_{t-\tau_1}^{t} x(s)\{b_1(s)(x_0(s-\tau_1) - x(s-\tau_1)) \\
&\quad + c_1(s)(y_0^m(s-\tau_2) - y^m(s-\tau_2))\}ds
\end{aligned}
\tag{4}
$$

and

$$
\begin{aligned}
&\int_{t-\tau_3}^{t} (\dot{y}(s) - \dot{y}_0(s))ds \\
&= \int_{t-\tau_3}^{t} (y(s) - y_0(s))(-a_2(s) - b_2(s)y_0(s-\tau_3) \\
&\quad + c_2(s)x_0(t)y_0^{m-1}(s))ds \\
&- \int_{t-\tau_3}^{t} b_2(s)y(s)(y(s-\tau_3) - y_0(s-\tau_3))ds \\
&+ \int_{t-\tau_3}^{t} c_2(s)y(s)(x(s)(y^{m-1}(s) - y_0^{m-1}(s)) \\
&\quad + y_0^{m-1}(s)(x(s) - x_0(s)))ds.
\end{aligned}
\tag{5}
$$

Substitution of (3), (4) and (5) into (2) yields

$$
\begin{aligned}
D^+ V_1(t)|_{(PP)} &\leqslant (c_2(t)M_2^{m-1} - \beta b_1(t))|x(t) - x_0(t)| \\
&- b_2(t)|y(t) - y_0(t)| - M_3 c_2(t)|y^{m-1}(t) - y_0^{m-1}(t)| \\
&+ \beta c_1(t)|y^m(t-\tau_2) - y_0^m(t-\tau_2)| \\
&+ \beta b_1(t)\left\{ \int_{t-\tau_1}^{t} |x(s) - x_0(s)| \right. \\
&\times (a_1(s) + M_1 b_1(s) + M_2^m c_1(s))ds \\
&+ M_1 \int_{t-\tau_1}^{t} (b_1(s)|x_0(s-\tau_1) - x(s-\tau_1)| \\
&\left. + c_1(s)|y_0^m(s-\tau_2) - y^m(s-\tau_2)|)ds \right\} \\
&+ b_2(t)\left\{ \int_{t-\tau_3}^{t} |y(s) - y_0(s)| \right. \\
&\times (a_2(s) + M_2 b_2(s) + M_1 M_4^{m-1} c_1(s))ds \\
&+ M_2 \int_{t-\tau_3}^{t} b_2(s)|y(s-\tau_3) - y_0(s-\tau_3)|ds \\
&+ M_1 M_2 \int_{t-\tau_3}^{t} c_2(s)|y^{m-1}(s) - y_0^{m-1}(s)|ds \\
&\left. + M_2 M_4^{m-1} \int_{t-\tau_3}^{t} c_2(s)|x(s) - x_0(s)|ds \right\}.
\end{aligned}
\tag{6}
$$

Note that

$$V_2'(t)$$
$$= \beta c_1(t+\tau_2)|y^m(s) - y_0^m(t)|$$
$$- \beta c_1(t)|y^m(t-\tau_2) - y_0^m(t-\tau_2)|$$
$$+ \beta(a_1(t) + M_1 b_1(t) + M_2^m c_1(t))$$
$$\times \int_t^{t+\tau_1} b_1(l)dl\,|x(t) - x_0(t)|$$
$$- \beta b_1(t) \int_{t-\tau_1}^t (a_1(s) + M_1 b_1(s) + M_2^m c_1(s))$$
$$\times |x(s) - x_0(s)|ds$$
$$+ \beta M_1 b_1(t) \int_t^{t+\tau_1} b_1(l)dl\,|x(t-\tau_2) - x_0(t-\tau_2)|$$
$$+ \beta M_1 c_1(t)b_1(t) \int_t^{t+\tau_1} b_1(l)dl|y^m(t-\tau_2) - y_0^m(t-\tau_2)|$$
$$- \beta M_1 b_1(t) \int_{t-\tau_1}^t (b_1(s)|x(s-\tau_2) - x_0(s-\tau_2)|$$
$$+ c_1(s)|y^m(s-\tau_2) - y_0^m(s-\tau_2)|)ds$$
$$+ (a_2(t) + M_2 b_2(t) + M_1 M_4^{m-1} c_1(t))$$
$$\times \int_t^{t+\tau_3} b_2(l)dl\,|y(t) - y_0(t)|$$
$$- b_2(t) \int_{t-\tau_3}^t |y(s) - y_0(s)|(a_2(s) + M_2 b_2(s)$$
$$+ M_1 M_4^{m-1} c_1(s))ds$$
$$+ M_2 b_2(t) \int_t^{t+\tau_3} b_2(l)dl\,|y(t-\tau_3) - y_0(t-\tau_3)|$$
$$- M_2 b_2(t) \int_{t-\tau_3}^t b_2(s)|y(s-\tau_3) - y_0(s-\tau_3)|ds$$
$$+ M_1 M_2 c_2(t) \int_t^{t+\tau_3} b_2(l)dl|y^{m-1}(t) - y_0^{m-1}(t)|$$
$$- M_1 M_2 \int_{t-\tau_3}^t b_2(l)c_2(s)|y^{m-1}(s) - y_0^{m-1}(s)|ds$$
$$+ M_2 M_4^{m-1} c_2(t) \int_t^{t+\tau_3} b_2(l)dl\,|x(t) - x_0(t)|$$
$$- M_2 M_4^{m-1} b_2(t) \int_{t-\tau_3}^t c_2(s)|x(s) - x_0(s)|ds$$

(7)

and

$$V_3'(t)$$
$$= \beta M_1 \int_{t+\tau_2}^{t+\tau_1+\tau_2} b_1(l)b_1(t+\tau_2)(|x(t) - x_0(t)|$$
$$+ c_1(t+\tau_2)|y^m(t) - y_0^m(t)|)dl$$
$$- \beta M_1 b_1(t) \int_t^{t+\tau_1} b_1(l)(|x(t-\tau_2) - x_0(t-\tau_2)|$$
$$+ c_1(t)|y^m(t-\tau_2) - y_0^m(t-\tau_2)|)dl$$
$$+ M_2 b_2(t+\tau_3) \int_{t+\tau_3}^{t+2\tau_3} b_2(l)dl|y(t) - y_0(t)|$$
$$- M_2 b_2(t) \int_t^{t+\tau_3} b_2(l)dl|y(t-\tau_3) - y_0(t-\tau_3)|.$$

(8)

From (6)-(8), we obtain

$$D^+ V(t)$$
$$= D^+ V_1(t) + V_2'(t) + V_3'(t).$$
$$\leqslant \big(c_2(t)\,M_2^{m-1} - \beta b_1(t) \big)\,|x(t) - x_0(t)|$$
$$- b_2(t)|y(t) - y_0(t)| - M_3 c_2(t)|y^{m-1}(t) - y_0^{m-1}(t)|$$
$$+ \beta c_1(t+\tau_2)|y^m(t) - y_0^m(t)|$$
$$+ \beta(a_1(t) + M_1 b_1(t) + M_2^m c_1(t))$$
$$\times \int_t^{t+\tau_1} b_1(l)dl\,|x(t) - x_0(t)|$$
$$+ \big(a_2(t) + M_2 b_2(t) + M_1 M_4^{m-1} c_1(t)\big)$$
$$\times \int_t^{t+\tau_3} b_2(l)dl\,|y(t) - y_0(t)|$$
$$+ \beta M_1 b_1(t+\tau_2) \int_{t+\tau_2}^{t+\tau_1+\tau_2} b_1(l)dl(|x(t) - x_0(t)|$$
$$+ c_1(t+\tau_2)|y^m(t) - y_0^m(t)|)$$
$$+ M_2 b_2(t+\tau_3) \int_{t+\tau_3}^{t+2\tau_3} b_2(l)dl\,|y(t) - y_0(t)|$$
$$+ M_1 M_2 c_2(t) \int_t^{t+\tau_3} b_2(l)dl|y^{m-1}(t) - y_0^{m-1}(t)|$$
$$+ M_2 M_4^{m-1} c_2(t) \int_t^{t+\tau_3} b_2(l)dl\,|x(t) - x_0(t)|$$
$$\leqslant - \Big(\beta b_1(t) - c_2(t)\,M_2^{m-1} - \beta(a_1(t) + M_1 b_1(t)$$
$$+ M_2^m c_1(t)) \int_t^{t+\tau_1} b_1(l)dl - M_2 M_4^{m-1} c_2(t)$$
$$\times \int_t^{t+\tau_3} b_2(l)dl - \beta M_1 b_1(t+\tau_2) \int_{t+\tau_2}^{t+\tau_1+\tau_2} b_1(l)dl \Big)$$
$$\times |x(t) - x_0(t)| - \Big[b_2(t) - (a_2(t) + M_2 b_2(t)$$
$$+ M_1 M_4^{m-1} c_1(t)) \int_t^{t+\tau_3} b_2(l)dl - M_2 b_2(t+\tau_3)$$
$$\times \int_{t+\tau_3}^{t+2\tau_3} b_2(l)dl \Big] |y(t) - y_0(t)|$$
$$- c_2(t) \Big(M_3 - M_1 M_2 \int_t^{t+\tau_3} b_2(l)dl \Big) |y^{m-1}(t) - y_0^{m-1}(t)|$$
$$+ \beta c_1(t+\tau_2) \Big(1 + M_1 b_1(t+\tau_2) \int_{t+\tau_2}^{t+\tau_1+\tau_2} b_1(l)dl \Big)$$
$$\times |y^m(t) - y_0^m(t)|$$
$$= - g_1(t)|x(t) - x_0(t)| - g_2(t)|y(t) - y_0(t)|$$
$$- g_3(t)|y^{m-1}(t) - y_0^{m-1}(t)| + g_4(t)|y^m(t) - y_0^m(t)|.$$

(9)

In view of $g_4^L > 0$ and $g_3^L \geqslant 0$, we get from the mean value theorem that

$$g_4(t)|y^m(t) - y_0^m(t)| \leqslant mM_4^{m-1}g_4(t)|y(t) - y_0(t)|, \quad (10)$$

$$g_3(t)|y^{m-1}(t) - y_0^{m-1}(t)| \geqslant (1-m)M_2^{m-2}g_3(t)|y(t) - y_0(t)|. \quad (11)$$

Combination of (9)-(4.11) yields

$$\begin{aligned}
&D^+V(t) \\
&\leqslant -g_1(t)|x(t) - x_0(t)| - (g_2(t) + (1-m)M_2^{m-2}g_3(t) \\
&\quad - mM_4^{m-1}g_4(t))|y(t) - y_0(t)|.
\end{aligned}$$

It follows from assumptions that there must exist two positive constants λ and γ such that

$$D^+V(t) \leqslant -\lambda|x(t) - x_0(t)| - \gamma|y(t) - y_0(t)| \quad \text{for } t > T.$$

Thus, $V(t)$ is non-increasing on $[0, +\infty)$. Integrating the above inequality from T to t we obtain

$$\begin{aligned}
&V(t) + \lambda \int_T^t |x(t) - x_0(t)|dt + \gamma \int_T^t |y(t) - y_0(t)|\,dt \\
&\leqslant V(0) < +\infty \quad \text{for } t > T.
\end{aligned}$$

By applying Barbălat's Lemma [22], we have

$$\lim_{t \to +\infty} |x(t) - x_0(t)| = 0 \quad \text{and} \quad \lim_{t \to +\infty} |y(t) - y_0(t)| = 0.$$

The proof is now finished. ∎

Remark 1. It follows from the mean value theorem and $g_3^U \leqslant 0$ that

$$g_3(t)|y^{m-1}(t) - y_0^{m-1}(t)| \geqslant (1-m)M_4^{m-2}g_3(t)|y(t) - y_0(t)|. \quad (12)$$

If (11) is replaced by (12), then the proof of Theorem 4.2 is obtained immediately. Similarly, in view of $g_3^L < 0$ we have

$$g_3(t)|y^{m-1}(t) - y_0^{m-1}(t)| \geqslant (1-m)M_4^{m-2}g_3^L|y(t) - y_0(t)|,$$

which yields that Theorem 4.3 holds.

V. EXAMPLES AND SIMULATION

In system (PP) we let

$$a_1(t) = 12 + 0.01\sin t, \; b_1(t) = 6, \; c_1(t) = 0.3 + 0.29\sin t,$$
$$a_2(t) = 5 - 0.01\sin t, \; b_2(t) = 3.4, \; c_2(t) = 1.2 + 0.1\sin t.$$

Example 5.1. Take $\tau_1 = 0.01$, $\tau_2 = 0$, $\tau_3 = 0.1$ and $m = 1/3$, and choose $\beta = 2$. We obtain

$$M_1 \doteq 2.25709856, \; M_2 \doteq 0.450910134, \; K_1 \doteq 11.55757282,$$

$$M_3 \doteq 1.88840248, \; K_2 \doteq 1.175749967, \; M_4 \doteq 0.113688327,$$

$$g_1^L \doteq 0.86559480, \; g_3^L \doteq 1.696604746, \; G_1^L \doteq 0.735131706.$$

According to Theorems 3.1 and 3.2 we claim that the system is permanent and has 2π-periodic positive solution, and according to Theorem 4.1 we assert that the system is globally attractive, see Figure 1 for more details.
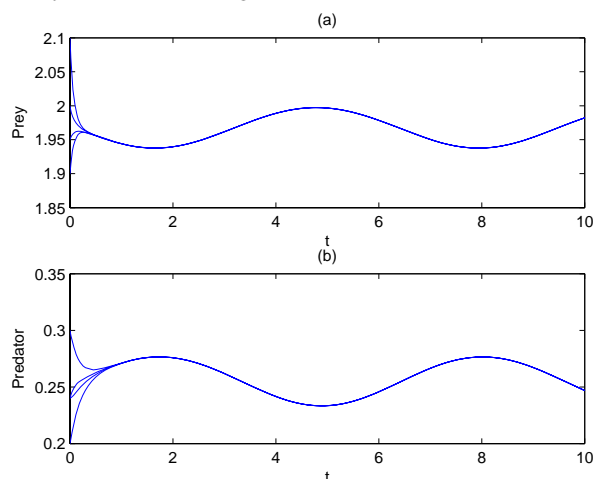

Figure 1. Evolution of the solutions of Example 5.1 with initial values: $(\phi(t), \psi(t)) = (1.9, 0.24; 1.95, 0.3; 2, 0.2; 2.1, 0.24)$, $\forall t \in [-1, 0]$.

Example 5.2. Let $\tau_1 = 0.01$, $\tau_2 = 0$, $\tau_3 = 0.1$ and $m = 0.9$ and $\beta = 1$. We obtain

$$M_1 \doteq 2.2570986, \; M_2 \doteq 0.004942348, \; K_1 \doteq 11.98991596,$$

$$M_3 \doteq 1.9675316, \; K_2 \doteq 2.154403597, \; M_4 \doteq 0.000216219,$$

$$g_1^L \doteq 0.6258010, \; g_3^L \doteq 2.160112597, \; G_1^L \doteq 75.95693025.$$

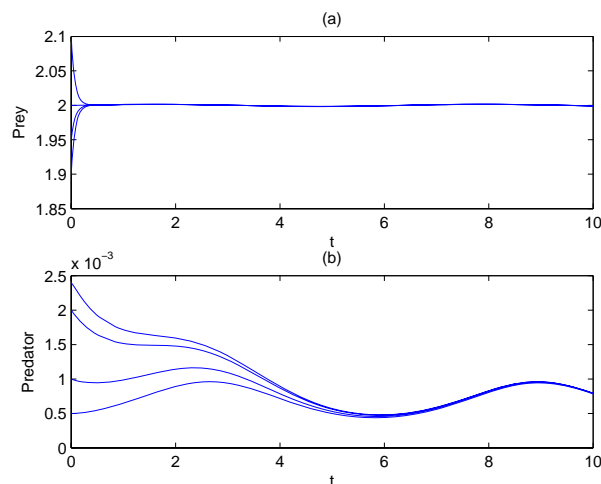One can see from Figure 2 that the prey and predator are permanent, which is the same as Example 5.1.

Figure 2. Evolution of the solutions of Example 5.2 with initial values: $(\phi(t), \psi(t)) = (1.9, 0.5e\text{-}3; 2, 0.2e\text{-}2; 1.95, 0.1e\text{-}2; 2.1, 0.24e\text{-}2)$, $\forall t \in [-1, 0]$.

Example 5.3. Let $\tau_1 = 0.01$, $\tau_2 = 0$, $\tau_3 = 0.1$ and $m = 1$. We see from Figure 3 that the predator is extinct finally. Comparison of Examples 5.1, 5.2 and 5.3 yields that the mutual interference constant m can influence the permanence of the predator. But it doesn't influence the permanence of the prey, which is only dependent on the coefficients a_1, b_1 and c_1.
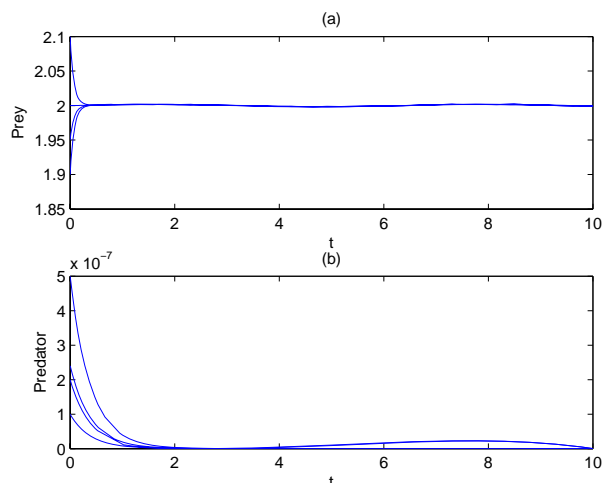
Figure 3. Evolution of the solutions of Example 5.3 with initial values: $(\phi(t), \psi(t)) = (1.9, 0.5e\text{-}6; 2, 0.2e\text{-}6; 1.95, 0.1e\text{-}6; 2.1, 0.24e\text{-}6), \forall t \in [-1, 0]$.

Example 5.4. In this example we consider the special case that $\tau_1 = \tau_3 = 0$, $m = 1/2, \beta = 1$ and $\tau_2 = 0$ and 100, respectively. By calculating we have

$$M_1 \doteq 2.00166667, \ M_2 \doteq 0.271937517, \ K_1 \doteq 11.70232898,$$
$$M_3 \doteq 1.95038816, \ K_2 \doteq 1.663276495, \ M_4 \doteq 0.110218234,$$
$$g_1^L \doteq 2.08423493, \ g_3^L \doteq 2.145426980, \ G_1^L \doteq 10.94943628.$$

Comparison of Figures 4 and 5 shows that the delay τ_2 has no intrinsic influence on the permanence of the species in these examples. In fact, it follows from $b_1(t) = 6$ that g_1, g_2 and g_3 are independent of the delay τ_2.
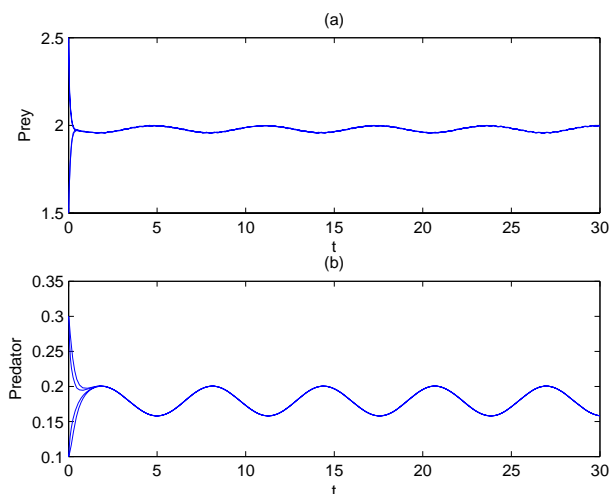

Figure 4. Evolution of the solutions of Example 5.4 with $\tau_2 = 0$ and initial values: $(\phi(t), \psi(t)) = (1.5, 0.3; 2.5, 0.1; 1.5, 0.1; 2.5, 0.3), \forall t \in [-1, 0]$.
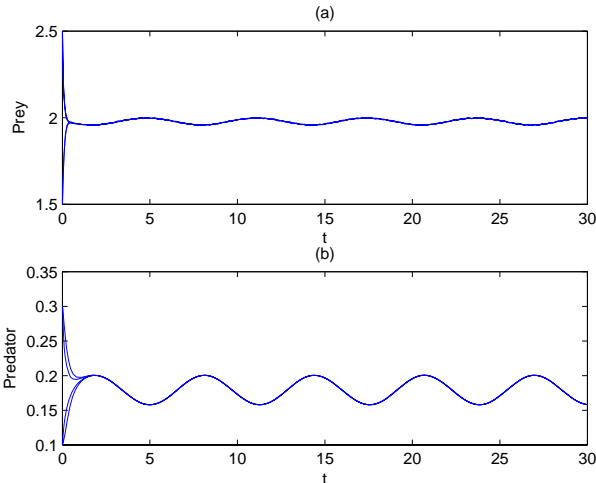

Figure 5. Evolution of the solutions of Example 5.4 with $\tau_2 = 100$ and initial values: $(\phi(0), \psi(0)) = (1.5, 0.3; 2.5, 0.1; 1.5, 0.1; 2.5, 0.3), \forall t \in [-1, 0]$.

VI. Conclusions

In this paper we study a predator-prey model with mutual interference and delays. By applying the comparison theorem and constructing suitable Lyapunov functional we present some sufficient conditions for the permanence and global attractivity of the model. The results obtained are both delay-dependent and mutual interference-dependent. Some interesting phenomenons are found.

Conclusion 1. If $m = 1$, that is there is no mutual inference between the prey and predator, then from figure 6 one can easily see that the prey is permanent but the predator is extinct eventually. So the mutual interference can effect the population of the predator. In the real world we must consider the predator-prey model under the influence of mutual interference.

Conclusion 2. One can see from the formula of $g_i, i = 1, 2, .., 4$ that the delays τ_1 and τ_3 can not only influence the permanence of the model (PP) but also the global attractivity of the model. In order to guarantee the permanence and global attractivity of the model (PP), the delays τ_1 and τ_3 should be small enough.

Conclusion 3. Theorem 3.1 yields that the delay τ_2 doesn't influence the permanence of the model (PP). Furthermore, we claim that the delay τ_2 has no essential influence on the global attractivity of the model (PP).

Actually, if $\tau_1 = \tau_3 = 0$, then Theorem 3.1 is delay-independent. Moreover,

$$g_1(t) = \beta b_1(t) - M_4^{m-1} c_2(t), \ g_2(t) = b_2(t),$$
$$g_3(t) = M_3 c_2(t), \ g_4(t) = \beta c_1(t + \tau_2).$$

Obviously, $g_3^L \geqslant 0$. If $c_1(t) \equiv$ const., then Theorems 4.1-4.3 are also delay-independent.

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