International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:7, No:8, 2013

The relationship of eigenvalues between backward MPSD and Jacobi iterative matrices

Zhuan-de Wang, Hou-biao Li and Zhong-xi Gao

Abstract—In this paper, the backward MPSD (Modified Preconditioned Simultaneous Displacement) iterative matrix is firstly proposed. The relationship of eigenvalues between the backward MPSD iterative matrix and backward Jacobi iterative matrix for block p-cyclic case is obtained, which improves and refines the results in the corresponding references.

Keywords—Backward MPSD iterative matrix, Jacobi iterative matrix, eigenvalue, p-cyclic matrix

I. Introduction

To solve the equations

$$Ax = b, \tag{1}$$

where $A = [a_{ij}]$ is a given $n \times n$ complex matrix and nonsingular, $n \geq 2$, which is partitioned in the form

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,p} \\ A_{2,1} & A_{2,2} & \dots & A_{2,p} \\ \vdots & \vdots & \ddots & \vdots \\ A_{p,1} & A_{p,2} & \cdots & A_{p,p} \end{bmatrix},$$

iterative methods are considered.

Let $A = D - C_L - C_U$ where $D = \operatorname{diag}(A)$ is a block diagonal matrix obtained from A and nonsingular, $-C_L$ and $-C_U$ are strictly lower and upper triangular matrices obtained from A, respectively. We also let $L = D^{-1}C_L$, $U = D^{-1}C_U$. The equation (1) becomes the equivalent one

$$(I - L - U)x = D^{-1}Ax = D^{-1}b.$$

The Jacobi iterative matrix is

$$B = L + U = I - D^{-1}A.$$

The MPSD (Modified Preconditioned Simultaneous Displacement) iterative method is studied in [2-5]. Here, we give the backward MPSD iterative matrix as follows:

$$\begin{aligned} \tilde{S}_{\tau,\omega_1,\omega_2} &= (I - \omega_1 U)^{-1}(I - \omega_2 L)^{-1}[(1-\tau)I + (\tau - \omega_1)U \\ &\quad + (\tau - \omega_2)L + \omega_1\omega_2 LU], \end{aligned}$$

with special values of ω_1, ω_2 and τ, we have

(1) When $\omega_1 = 0$, $\omega_2 = 0$ and $\tau = 1$, we obtain the Jacobi iterative method;

(2) When $\omega_1 = 0$, $\omega_2 = 0$ and $\tau = \omega$, we obtain the backward JOR iterative method;

(3) When $\omega_1 = 1$, $\omega_2 = 0$ and $\tau = 1$, we obtain the backward G-S iterative method;

(4) When $\omega_1 = \omega$, $\omega_2 = 0$ and $\tau = \omega$, we obtain the backward SOR iterative method;

(5) When $\omega_1 = \omega$, $\omega_2 = 0$ and $\tau = \alpha$, we obtain the backward AOR iterative method;

(6) When $\omega_1 = \omega$, $\omega_2 = \omega$ and $\tau = \omega(2-\omega)$, we obtain the backward SSOR iterative method;

(7) When $\omega_1 = \omega$, $\omega_2 = \omega$ and $\tau = \omega$, we obtain the backward EMA iterative method;

(8) When $\omega_1 = \omega$, $\omega_2 = \omega$ and $\tau = \alpha$, we obtain the backward PSD iterative method;

(9) When $\omega_1 = \omega$, $\omega_2 = \omega$ and $\tau = 1$, we obtain the backward PJ iterative method.

If A has the following block form

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} & 0 & \cdots & 0 \\ 0 & A_{2,2} & A_{2,3} & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & A_{p-1,p} \\ A_{p,1} & 0 & \cdots & \cdots & A_{p,p} \end{bmatrix},$$

then A is called a p-cyclic matrix [1]. Such matrices naturally arise, e.g., for $p = 2$ in the discretization of second-order elliptic or parabolic PDEs by finite differences, finite element or collocation methods, for $p = 3$ in the case of large scale least-squares problems, and for any $p \geq 2$ in the case of Markov chain analysis. The p-cyclic matrix is considered in many papers [1, 6-13]. The eigenvalue relationship between the SOR iterative matrix and the Jacobi iterative matrix for p-cyclic case is studied in Theorem 4.5 in [1], and the eigenvalue relationship between the USAOR iterative matrix and the Jacobi iterative matrix for the p-cyclic case is studied in [6]. In the following we will consider the eigenvalue relationship between the backward MPSD iterative matrix and the Jacobi iterative matrix for the p-cyclic case.

II. Preliminary

If A is a p-cyclic matrix, then

$$D = \begin{bmatrix} A_{1,1} & & & \\ & A_{2,2} & & \\ & & \ddots & \\ & & & A_{p,p} \end{bmatrix}, \quad C_L = \begin{bmatrix} 0 & & & \\ 0 & 0 & & \\ \vdots & & \ddots & \\ -A_{p,1} & & \cdots & 0 \end{bmatrix},$$

$$C_U = \begin{bmatrix} 0 & -A_{1,2} & & & \\ & 0 & -A_{2,3} & & \\ & & \ddots & \ddots & \\ & & & 0 & -A_{p-1,p} \\ & & & & 0 \end{bmatrix},$$

Zhuan-De Wang is with the School of Mathematical Science, University of Electronic Science and Technology, Chengdu, Sichuan, 610054 P. R. China e-mail: zhdwang@126.com.

Hou-biao Li and Zhong-xi Gao are with the School of Mathematical Science, University of Electronic Science and Technology, Chengdu, Sichuan, 610054 P. R. China e-mail: lihoubiao0189@163.com, zhongxigao@126.com.

and

$$L = D^{-1}C_L = \begin{bmatrix} 0 & & & \\ 0 & 0 & & \\ \vdots & & \ddots & \\ B_{p,1} & & \cdots & 0 \end{bmatrix},$$

$$U = D^{-1}C_U = \begin{bmatrix} 0 & B_{1,2} & & & \\ & 0 & B_{2,3} & & \\ & & \ddots & \ddots & \\ & & & 0 & B_{p-1,p} \\ & & & & 0 \end{bmatrix}.$$

Let λ be the eigenvalue of $\tilde{S}_{\tau,\omega_1,\omega_2}$, x be the corresponding eigenvector. Then

$$\tilde{S}_{\tau,\omega_1,\omega_2}x = \lambda x,$$

equivalently,

$$(I-\omega_1 U)^{-1}(I-\omega_2 L)^{-1}[(1-\tau)I+(\tau-\omega_1)L+(\tau-\omega_2)L+\omega_1\omega_2 LU] = \lambda x,$$

or

$$[(1-\tau)I+(\tau-\omega_1)U+(\tau-\omega_2)L+\omega_1\omega_2 LU]x = \lambda(I-\omega_2 L)(I-\omega_1 U)x,$$

that is to say,

$$\begin{bmatrix} (1-\tau)I_1 & (\tau-\omega_1)B_{1,2} & & \\ & (1-\tau)I_2 & & \\ & \ddots & \ddots & \\ & & & (\tau-\omega_1)B_{p-1,p} \\ (\tau-\omega_2)B_{p,1} & \omega_1\omega_2 B_{p,1}B_{1,2} & & (1-\tau)I_p \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}$$

$$= \begin{bmatrix} I_1 & & & \\ & I_2 & & \\ & & \ddots & \\ -\omega_2 B_{p,1} & & & I_p \end{bmatrix} \begin{bmatrix} x_1 - \omega_1 B_{1,2}x_2 \\ x_2 - \omega_1 B_{2,3}x_3 \\ \vdots \\ x_{p-1} - \omega_1 B_{p-1,p}x_p \\ x_p \end{bmatrix}, \tag{2}$$

where

$$LU = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & \ddots & \\ 0 & B_{p,1}B_{1,2} & \cdots & 0 \end{bmatrix}.$$

If μ is an eigenvalue of B and x is the corresponding eigenvector, that is,

$$\begin{bmatrix} 0 & B_{1,2} & & & \\ & 0 & B_{2,3} & & \\ & & \ddots & \ddots & \\ & & & 0 & B_{p-1,p} \\ B_{p,1} & & & & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} = \mu \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}. \tag{3}$$

III. Main results

The relationship of eigenvalues between backward MPSD and Jacobi iterative matrices is given as follows:

Theorem 3.1 *Let A be a p-cyclic matrix, B be the corresponding the block Jacobi iterative matrix. If $\mu \neq 0$ is an eigenvalue of B and λ satisfies*

$$\begin{aligned}\mu^p\{(\tau-\omega_1+\lambda\omega_1)^{p-2}[(\lambda+\tau-1)(1-\lambda)\omega_1\omega_2 \\ +(\tau-\omega_1+\lambda\omega_1)(\tau-\omega_2+\lambda\omega_2)]\} = (\lambda+\tau-1)^p.\end{aligned} \tag{4}$$

Then λ is an eigenvalue of the backward MPSD iterative matrix $\tilde{S}_{\tau,\omega_1,\omega_2}$. Conversely, if λ is an eigenvalue of $\tilde{S}_{\tau,\omega_1,\omega_2}$ with $\lambda+\tau-1 \neq 0$, then there exists an eigenvalue μ of B satisfying (4).

Proof. Let λ be the eigenvalue of $\tilde{S}_{\tau,\omega_1,\omega_2}$, x be the corresponding eigenvector. By (2), we have

$$\begin{cases} (1-\tau)x_1 + (\tau-\omega_1)B_{1,2}x_2 = \lambda x_1 - \lambda\omega_1 B_{1,2}x_2, \\ (1-\tau)x_2 + (\tau-\omega_1)B_{2,3}x_3 = \lambda x_2 - \lambda\omega_1 B_{2,3}x_3, \\ \quad\vdots \\ (1-\tau)x_{p-1} + (\tau-\omega_1)B_{p-1,p}x_p = \lambda x_{p-1} - \lambda\omega_1 B_{p-1,p}x_p, \\ (\tau-\omega_2)B_{p,1}x_1 + \omega_1\omega_2 B_{p,1}B_{1,2}x_2(1-\tau)x_p \\ \qquad = \lambda\left[-\omega_2 B_{p,1}(x_1-\omega_1 B_{1,2}x_2)+x_p\right], \end{cases}$$

equivalently, let $\eta = \lambda+\tau-1$, $\xi_i = \tau-\omega_i+\lambda\omega_i, i=1,2,$ we have

$$\begin{cases} \eta x_1 = \xi_1 B_{1,2}x_2, \\ \eta x_2 = \xi_1 B_{2,3}x_3, \\ \quad\vdots \\ \eta x_{p-1} = \xi_1 B_{p-1,p}x_p, \\ \eta x_p = (1-\lambda)\omega_1\omega_2 B_{p,1}B_{1,2}x_2 + \xi_2 B_{p,1}x_1. \end{cases} \tag{5}$$

From the first $n-2$ equations, we have

$$\begin{cases} \eta^2 x_1 = \xi_1^2 B_{1,2}B_{2,3}x_3, \\ \eta^3 x_1 = \xi_1^3 B_{1,2}B_{2,3}B_{3,4}x_4, \\ \quad\vdots \\ \eta^{p-2}x_1 = \xi_1^{p-2}B_{1,2}B_{2,3}\cdots B_{p-2,p-1}x_{p-1}, \end{cases} \tag{6}$$

and from the last two equations and the first equation, we have

$$\begin{aligned} \eta^2 x_{p-1} &= \xi_1 B_{p-1,p}\eta x_p \\ &= \xi_1 B_{p-1,p}[(1-\lambda)\omega_1\omega_2 B_{p,1}B_{1,2}x_2 + \xi_2 B_{p,1}x_1] \\ &= \xi_1\xi_2 B_{p-1,p}B_{p,1}x_1 + (1-\lambda)\omega_1\omega_2 B_{p-1,p}B_{p,1} \\ &\quad \xi_1 B_{1,2}x_2 \\ &= [\eta(1-\lambda)\omega_1\omega_2 + \xi_1\xi_2]B_{p-1,p}B_{p,1}x_1. \end{aligned} \tag{7}$$

Combining (6) with (7), we have

$$\begin{aligned} \eta^2\eta^{p-2}x_1 &= \eta^2\xi_1^{p-2}B_{1,2}B_{2,3}\cdots B_{p-3,p-2}B_{p-2,p-1}x_{p-1} \\ &= \xi_1^{p-2}B_{1,2}B_{2,3}\cdots B_{p-3,p-2}B_{p-2,p-1}\eta^2 x_{p-1} \\ &= \xi_1^{p-2}B_{1,2}B_{2,3}\cdots B_{p-3,p-2}B_{p-2,p-1}[\eta(1-\lambda)\omega_1\omega_2 \\ &\quad +\xi_1\xi_2]B_{p-1,p}B_{p,1}x_1 \\ &= \xi_1^{p-2}[\eta(1-\lambda)\omega_1\omega_2+\xi_1\xi_2]B_{1,2}B_{2,3}\cdots B_{p-3,p-2} \\ &\quad B_{p-2,p-1}B_{p-1,p}B_{p,1}x_1, \\ \eta^p x_1 &= \xi_1^{p-2}[\eta(1-\lambda)\omega_1\omega_2+\xi_1\xi_2]B_{1,2}B_{2,3}\cdots B_{p-3,p-2} \\ &\quad B_{p-2,p-1}B_{p-1,p}B_{p,1}x_1. \end{aligned} \tag{8}$$

Assuming that $\lambda+\tau-1 \neq 0$.

If $x_1 = 0$, then, by (5), we have $x_p = x_{p-1} = \cdots = x_2 = x_1 = 0$. But x is an eigenvector, so $x_1 \neq 0$. By (8), we know that

$$\frac{\eta^p}{\xi_1^{p-2}[\eta(1-\lambda)\omega_1\omega_2+\xi_1\xi_2]}$$

is an eigenvalue of $B_{1,2}B_{2,3}\cdots B_{p-2,p-1}B_{p-1,p}B_{p,1}$ and x_1 is the corresponding eigenvector.

If μ is an eigenvalue of B and x is the corresponding eigenvector, by (3) we obtain

$$\begin{aligned} B_{1,2}x_2 &= \mu x_1, \\ B_{2,3}x_3 &= \mu x_2, \\ &\vdots \\ B_{p-1,p}x_p &= \mu x_{p-1}, \\ B_{p,1}x_1 &= \mu x_p. \end{aligned}$$

From the above equations, we have

$$B_{1,2}B_{2,3}B_{3,4}x_4 = \mu^3 x_1,$$
$$\vdots$$
$$B_{1,2}B_{2,3}B_{3,4}\cdots B_{p-1,p}x_p = \mu^{p-1}x_1.$$

So,

$$\begin{cases} \mu B_{1,2}B_{2,3}B_{3,4}\cdots B_{p-1,p}x_p = \mu^p x_1, \\ B_{1,2}B_{2,3}B_{3,4}\cdots B_{p-1,p}\mu x_p = \mu^p x_1, \\ B_{1,2}B_{2,3}B_{3,4}\cdots B_{p-1,p}B_{p,1}x_1 = \mu^p x_1. \end{cases} \quad (9)$$

Hence, μ^p is an eigenvalue of $B_{1,2}B_{2,3}B_{3,4}\cdots B_{p-1,p}B_{p,1}$.

Combining (8) with (9), we obtain that

$$\mu^p = \frac{\eta^p}{\xi_1^{p-2}[\eta(1-\lambda)\omega_1\omega_2 + \xi_1\xi_2]},$$

i.e.,

$$\mu^p\{\xi_1^{p-2}[\eta(1-\lambda)\omega_1\omega_2 + \xi_1\xi_2]\} = \eta^p. \quad (10)$$

So, if λ is an eigenvalue of $\tilde{S}_{\tau,\omega_1,\omega_2}$, then there exists an eigenvalue μ of B satisfying (10). Conversely, if μ is an eigenvalue of B and λ satisfies (10), we can easily prove that λ is an eigenvalue of $\tilde{S}_{\tau,\omega_1,\omega_2}$. Thus, we proved the Theorem 3.1. ■

With the special values of ω_1, ω_2 and τ, we have several corollaries, which improve and refine the results in the corresponding references.

Corollary 3.1 *Let A be a p-cyclic matrix, B be the corresponding Jacobi iterative matrix. If $\mu \neq 0$ is an eigenvalue of B and λ satisfies*

$$\mu^p\omega^p = (\lambda + \omega - 1)^p. \quad (11)$$

Then λ is an eigenvalue of the backward JOR iterative matrix $S_{\omega,0,0}$. Conversely, if λ is an eigenvalue of $S_{\omega,0,0}$ with $\lambda + \omega - 1 \neq 0$, then there exists an eigenvalue μ of B satisfying (11).

Corollary 3.2 *Let A be a p-cyclic matrix, B be the corresponding Jacobi iterative matrix. If μ is an eigenvalue of B, $\mu \neq 0$, and λ satisfies*

$$\mu^p = \lambda. \quad (12)$$

Then λ is an eigenvalue of the backward Gauss-Seidel iterative matrix $S_{1,1,0}$. Conversely, if λ is an eigenvalue of $S_{1,1,0}$ for which $\lambda \neq 0$, then there must exist an eigenvalue μ of B satisfying (12).

Corollary 3.3 *Let A be a p-cyclic matrix, B be the corresponding Jacobi iterative matrix. If $\mu \neq 0$ is an eigenvalue of B, and λ satisfies*

$$\lambda^{p-1}\mu^p\omega^p = (\lambda + \omega - 1)^p. \quad (13)$$

Then λ is an eigenvalue of the backward SOR iterative matrix $S_{\omega,\omega,0}$. Conversely, if λ is an eigenvalue of $S_{\omega,\omega,0}$ with $\lambda + \omega - 1 \neq 0$, then there exists an eigenvalue μ of B satisfying (13).

Corollary 3.4 *Let A be a p-cyclic matrix, B be the corresponding Jacobi iterative matrix. If $\mu \neq 0$ is an eigenvalue of B and λ satisfies*

$$\mu^p\alpha(\alpha - \omega + \lambda\omega)^{p-1} = (\lambda + \alpha - 1)^p. \quad (14)$$

Then λ is an eigenvalue of the backward AOR iterative matrix $S_{\alpha,\omega,0}$. Conversely, if λ is an eigenvalue of $S_{\alpha,\omega,0}$ with $\lambda + \alpha - 1 \neq 0$, then there exists an eigenvalue μ of B satisfying (14).

Corollary 3.5 *Let A be a p-cyclic matrix, B be the corresponding Jacobi iterative matrix. If $\mu \neq 0$ is an eigenvalue of B and λ satisfies*

$$\begin{aligned}[\lambda - (\omega-1)^2]^p &= \mu^p\{[\omega((1+\lambda)-\omega)]^{p-2}[(\lambda+\omega-1)(1-\lambda)\omega^2 \\ &\quad + [\omega((1+\lambda)-\omega)]^2\}.\end{aligned} \quad (15)$$

Then λ is an eigenvalue of the backward SSOR iterative matrix $S_{\omega(2-\omega),\omega,\omega}$. Conversely, if λ is an eigenvalue of $S_{\omega(2-\omega),\omega,\omega}$ for which $\lambda + (\omega-1)^2 \neq 0$, then there exists an eigenvalue μ of B satisfying (15).

Corollary 3.6 *Let A be a p-cyclic matrix, B be the corresponding Jacobi iterative matrix. If $\mu \neq 0$ is an eigenvalue of B and λ satisfies*

$$(\lambda+\omega-1)^p = \mu^p\{(\lambda\omega)^{p-2}[(\lambda+\omega-1)(1-\lambda)\omega^2 + (\lambda\omega)^2]\}. \quad (16)$$

Then λ is an eigenvalue of the backward EMA iterative matrix $S_{\omega,\omega,\omega}$. Conversely, if λ is an eigenvalue of $S_{\omega,\omega,\omega}$ with $\lambda + \omega - 1 \neq 0$, then there exists an eigenvalue of B satisfying (16).

Corollary 3.7 *Let A be a p-cyclic matrix, B be the corresponding Jacobi iterative matrix. If $\mu \neq 0$ is an eigenvalue of B and λ satisfies*

$$(\lambda+\alpha-1)^p = \mu^p\{(\alpha-\omega+\lambda\omega)^{p-2}[(\lambda+\alpha-1)(1-\lambda)\omega^2 + (\alpha-\omega+\lambda\omega)^2]\}. \quad (17)$$

Then λ is the eigenvalue of the backward PSD iterative matrix $S_{\alpha,\omega,\omega}$. Conversely, if λ is an eigenvalue of $S_{\alpha,\omega,\omega}$ with $\lambda + \alpha - 1 \neq 0$, then there exists an eigenvalue μ of B satisfying (17).

Corollary 3.8 *Let A be a p-cyclic matrix, B be the corresponding Jacobi iterative matrix. If $\mu \neq 0$ is an eigenvalue of B and λ satisfies*

$$\lambda^p = \mu^p\{(1-\omega+\lambda\omega)^{p-2}[\lambda(1-\lambda)\omega^2 + (1-\omega+\lambda\omega)^2]\}. \quad (18)$$

Then λ is an eigenvalue of the backward PJ iterative matrix $S_{\omega,\omega,1}$. Conversely, if λ is an eigenvalue of $S_{\omega,\omega,1}$ for which $\lambda \neq 0$, then there exists an eigenvalue μ of B satisfying (18).

IV. Numerical example

Example 4.1 Let the coefficient matrix A of (1) be

$$A = \begin{bmatrix} 1 & 0 & -0.125 & -0.125 & 0 & 0 \\ 0 & 1 & -0.125 & -0.125 & 0 & 0 \\ 0 & 0 & 1 & 0 & -0.125 & -0.125 \\ 0 & 0 & 0 & 1 & -0.125 & -0.125 \\ -0.125 & -0.125 & 0 & 0 & 1 & 0 \\ -0.125 & -0.125 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

It is obvious that A is 3-cyclic matrix. By caculation, we obtain $\mu = \frac{1}{4}$ is an eigenvalue of the Jacobi matrix B.

(1)By calculation, we can get that $\lambda = \frac{1}{64}$ is an eigenvalue of the backward Gauss-Seidel matrix, and

$$\mu^3 = (\frac{1}{4})^3 = \frac{1}{64} = \lambda.$$

So, the asymptotic rate of convergence of the backward Gauss-Seidel iteration is triple that of the Jacobi iteration.

(2)With $\omega_2 = 0$, $\omega_1 = \frac{1}{2}$,$\tau = 1$, we obtain the backward AOR iterative method. By calculation, $\lambda = \frac{669}{3814}$ is an eigenvalue of the backward AOR matrix. Meanwhile the equation $\mu^3\alpha(\alpha - \omega + \lambda\omega)^{3-1} = (\lambda + \alpha - 1)^3$ equals to $256\lambda^3 - \lambda^2 - 2\lambda - 1 = 0$, and $\lambda = \frac{669}{3814}$ is just the root of it.

(3)The numerical results between other iterative method and the Jacobi iterative method is analogous to the above, and is omitted.

V. Conclusion

The eigenvalue relationship is vital for the convergence of iterative methods. In this paper, the backward MPSD iterative matrix is proposed, and the relationship of eigenvalues between backward MPSD and Jacobi iterative matrices for p-cyclic case is obtained, which is useful to some issues such as Markov Chains,etc. These results involves some special iterative methods which are proposed in the references.

Acknowledgment

The work is part supported by National Nature Science Foundation of China (1117105, 11101071, 51175443) and the Fundamental Research Funds for China Scholarship Council.

References

[1] R. S. Varga, *Matrix Iterative Analysis*, 2nd Endition, Springer, Berlin, 2000.
[2] X. P. Liu, Convergence of some iterative methods, *Numerical Computing and Computer Applications*, 1(1992)58-64 (in Chinese).
[3] H. X. Chen, Convergent and divergent relationship between MPSD iterative method and Jacobi method, *J. Appl. Math. and Computational Math.*, 14(1)(2000)1-8 (in Chinese).
[4] N. M. Missirlis, D. J. Evans, The modified preconditioned simultaneous displacement (MPSD) method, *Math. Comp. Simulations*, XXVI (1984) 257-262.
[5] Z. D. Wang, T. Z. Huang, Comparison results between Jacobi and other iterative methods, *J. Comp. Appl. Math.* 169(2004)45-51.
[6] R. M. Li, Relationship of eigenvalue for USAOR iterative method applied to a class of p-cyclic matrices, *Linear Algebra Appl.*362(2003)101-108.
[7] A. Hadjidimos, D. Noutsos, M. Tzoumas, Torwards the determination of optimal p-cyclic SSOR, *J. Comp. Appl. Math.* 90(1996)1-14.
[8] S. Galanis, A. Hadjidimos, D. Noutsos, A Young-Eidson's type algorithm for complex p-cyclic SOR spectra, *Linear Algebra Appl.* 286(1999)87-106.
[9] A. Hadjidimos, D. Noutsos, M. Tzoumas, On the exact p-cyclic SSOR convergence domains, *Linear Algebra Appl.* 232(2003)213-236.
[10] A. Hadjidimos, D. Noutsos, M. Tzoumas, On the convergence domains of the p-cyclic SOR, *J. Comp. Appl. Math.* 72(1996)63-83.
[11] S. Galanis, A. Hadjidimos, D. Noutsos, Optimal p-cyclic SOR for complex spectra, *Linear Algebra Appl.* 263(1997)233-260.
[12] D. M. Young, Iterative Solution of Large Linear Systems, NewYork-London, Academic Press, 1971.
[13] A. Hadjidimos, M. Neumann, Superior convergence domains for the p-cyclic SSOR majorizer, *J. Comp. Appl. Math.* 62(1995)27-40.

Zhuan-de Wang is with the School of Mathematics Sciences, University of Electronic Science and Technology of China (UESTC) and received the Ph.D. degrees in computational and applied mathematics from University of Electronic Science and Technology of China (UESTC), China, in 2009. His current research interests include numerical linear algebra, preconditioning technology ,Matrix computation and numerical optimization, special matrices,etc. Email: zhdwang@126.com.

Hou-biao Li received the M.Sc. and Ph.D. degrees in computational and applied mathematics from University of Electronic Science and Technology of China (UESTC), China, in 2005 and 2007, respectively. He currently is an Associate Professor with the School of Mathematics Sciences, UESTC. His research interests involve numerical linear algebra, preconditioning technology and computational mathematics, etc. Email: lihoubiao0189@163.com

Zhong-xi Gao is with the School of Mathematics Sciences, University of Electronic Science and Technology of China (UESTC), China. His current research interests include linear algebra and its application, matrix theory, especially combinatorics and graph theory. Email: zhongxigao@126.com.