

The positive solution for singular eigenvalue problem of one-dimensional p-Laplace operator

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Abstract—In this paper, by constructing a special cone and using fixed point theorem and fixed point index theorem of cone, we get the existence of positive solution for a class of singular eigenvalue value problems with p-Laplace operator, which improved and generalized the result of related paper.

Keywords—Cone, fixed point index, eigenvalue problem, p-Laplace operator, positive solutions.

I. INTRODUCTION

THE eigenvalue problems with p-Laplace operator arises in a variety of applied mathematics and physics, and they are widely applied in studying for non-newtonian fluid mechanics, cosmological physics, plasma physics, and theory of elasticity, etc. In recent years, some important results have been obtained by a variety of method(see[1-9]). In paper [10], Wang and Ge study for the following problem

$$\begin{cases} (\phi_p(u'))' + a(t)f(t, u(t)) = 0, t \in (0, 1) \\ u(0) = u(1) = 0, \end{cases}$$

by using fixed point theorem of cone, they get the existence of multiple positive solution. Motivated by paper [4,6,10], we consider the following problems:

$$\begin{cases} -(\varphi_p(x'(t)))' = \lambda h(t)f(x(t)), & t \in (0, 1) \\ \alpha\varphi_p(x(0)) - \beta\varphi_p(x'(0)) = 0, \\ \gamma\varphi_p(x(1)) + \delta\varphi_p(x'(1)) = 0, \end{cases} \quad (1)$$

where $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, and λ is a positive parameter, $h(t)$ is nonnegative measurable function in $(0, 1)$, $h(t)$ may be singular at $t = 0, 1$, $\alpha > 0, \beta \geq 0, \gamma > 0, \delta \geq 0$ $f(x)$ is nonnegative continuous function in $[0, +\infty)$, f is sup-linear and sub-linear at 0 and ∞ .

We first list the following conditions:

(H₁) $h(t)$ is nonnegative function in $(0, 1)$, for any closed subinterval of $(0, 1)$, $h(t) \neq 0$ and $0 < \int_0^1 h(t)dt < +\infty$;

(H₂) $f \in C([0, +\infty), [0, +\infty))$ and $f(0) = 0$; for $u > 0$, $f(u) > 0$;

(H₃) $\lim_{x \rightarrow 0} \frac{f(x)}{x^{p-1}} = a$, where $a \in [0, +\infty]$;

(H₄) $\lim_{x \rightarrow +\infty} \frac{f(x)}{x^{p-1}} = +\infty$; (f is sup-linear at $x = +\infty$.)

(H₅) $\lim_{x \rightarrow 0} \frac{f(x)}{x^{p-1}} = 0$; (f is sub-linear at $x = 0$.)

(H₆) $\lim_{x \rightarrow +\infty} \frac{f(x)}{x^{p-1}} = 0$. (f is sub-linear at $x = +\infty$.)

For the sake of convenience, we list the following definitions and lemmas:

Definition 1.1 If $x \in C[0, 1] \cap C^1(0, 1)$ and satisfy (1), $\varphi_p(x'(t))$ is absolutely continuous in $(0, 1)$, $-(\varphi_p(x'(t)))' = \lambda h(t)f(x(t))$ hold almost everywhere in $(0, 1)$, we call x is positive solution for problem (1).

Definition 1.2 Let E be a real Banach space, if K is a nonempty convex closed set in E , and satisfy the following conditions:

(1) $x \in K, \lambda \geq 0 \Rightarrow \lambda x \in K$; (2) $x \in K, -x \in K \Rightarrow x = \theta$, θ is zero element in E ; we call K is a cone in E .

Let $E = C[0, 1] \cap C^1[0, 1]$, we induce the order $x < y$: for all $t \in [0, 1]$, we have $x(t) < y(t)$. If we denote the norm $\|x\| = \max\{\max_{0 \leq t \leq 1} |x(t)|, \max_{0 \leq t \leq 1} |x'(t)|\}$, then $(E, \|\cdot\|)$ is a Banach space.

Let $K = \{x \in E : x(t) \geq 0, \alpha\varphi_p(x(0)) - \beta\varphi_p(x'(0)) = 0, \gamma\varphi_p(x(1)) + \delta\varphi_p(x'(1)) = 0, x \text{ is concave function in } [0, 1]\}$, then K is a cone in E .

Lemma 1.1 For any $0 < \varepsilon < \frac{1}{2}$, $x \in K$ has the following properties:

- (1) $x(t) \geq \|x\|t(1-t)$, $\forall t \in [0, 1]$;
- (2) $x(t) \geq \varepsilon^2\|x\|$, $\forall t \in [\varepsilon, 1-\varepsilon]$. (the proof is elementary.)

lemma 1.2 Suppose H_3, H_4 hold, and $a = \infty$, then there exists $R > 0$, such that $\frac{f(R)}{R^{p-1}} = \min_{t>0} \frac{f(t)}{t^{p-1}}$, suppose H_5, H_6 hold, then there exists $L > 0$, such that $\frac{f(L)}{L^{p-1}} = \max_{t>0} \frac{f(t)}{t^{p-1}} = C'$.

Lemma 1.3 (see[11]) Let E be Banach space, K is a cone in E , for $r > 0$, we define $K_r = \{x \in K : \|x\| \leq r\}$. Suppose $T : K_r \rightarrow K$ is completely continuous, such that $\forall u \in \partial K_r = \{x \in K : \|x\| = r\}$, we have $Tx \neq x$, If $\|x\| \leq \|Tx\|, x \in \partial K_r$, then $i(T, K_r, K) = 0$; if $\|x\| \geq \|Tx\|, x \in \partial K_r$, then $i(T, K_r, K) = 1$.

Lemma 1.4 (see [12]) Let Ω_1, Ω_2 is a bounded open set in E , $\theta \in \Omega_1, \bar{\Omega}_1 \subset \Omega_2, A : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \rightarrow K$ is completely continuous. If $\|Ax\| \leq \|x\|, \forall x \in K \cap \partial\Omega_1; \|Ax\| \geq \|x\|, \forall x \in K \cap \partial\Omega_2$. or $\|Ax\| \leq \|x\|, \forall x \in K \cap \partial\Omega_2; \|Ax\| \geq \|x\|, \forall x \in K \cap \partial\Omega_1$, then A has fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

II. CONCLUSION

Theorem 2.1 If conditions $(H_1), (H_2), (H_3), (H_4)$ hold, and $a = +\infty$.

(a) If there exists $\lambda^* > 0$ such that $(\lambda^*)^{\frac{1}{p-1}} + \max\{(\frac{\beta}{\alpha})^{\frac{1}{p-1}}, (\frac{\delta}{\gamma})^{\frac{1}{p-1}}\} (\frac{f(\bar{R})}{R^{p-1}})^{\frac{1}{p-1}} \psi_p(\int_0^1 h(t)dt) \leq 1$

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where $\psi_p(t) = |t|^{\frac{1}{p-1}} \text{sgn}(t)$ is converse function of φ_p , $\bar{R} \in (0, R]$ is the maximum point of f in $(0, R]$, then for $0 < \lambda < \lambda^*$, Problem (1) has two positive solutions $x_1(t), x_2(t)$, and satisfy $0 < \|x_1\| < R < \|x_2\|$.

(b) There exists λ^{**} , when $\lambda > \lambda^{**}$, the problem (1) has no positive solution.

(a) **Proof** For any $x \in K$, we have $x'(0) \geq 0, x'(1) \leq 0$, so there exists a constant $\sigma (= \sigma_x)$ such that $x'(\sigma) = 0$, we define $T_\lambda : K \rightarrow E$ as follow

$$(T_\lambda x)(t) = \begin{cases} \psi_p(\frac{\beta}{\alpha} \int_0^\sigma \lambda h(r) f(u(r)) dr) + \\ \int_0^t \psi_p(\int_s^\sigma \lambda h(r) f(u(r)) dr) ds, 0 \leq t \leq \sigma, \\ \psi_p(\frac{\delta}{\gamma} \int_\sigma^1 \lambda h(r) f(u(r)) dr) + \\ \int_t^1 \psi_p(\int_\sigma^s \lambda h(r) f(u(r)) dr) ds, \sigma \leq t \leq 1, \end{cases}$$

By the definition of T_λ , we know $\forall x \in K, T_\lambda x \in C^1[0, 1]$ is nonnegative and satisfy the boundary condition, furthermore,

$$(T_\lambda x)'(t) = \begin{cases} \psi_p \int_t^\sigma \lambda h(r) f(u(r)) dr > 0, & 0 \leq t \leq \sigma, \\ -\psi_p(\int_\sigma^t \lambda h(r) f(u(r)) dr) < 0, & \sigma \leq t \leq 1, \end{cases}$$

is continuous and non-increasing in $[0, 1]$, and $(T_\lambda x)'(\sigma) = 0$, so $(T_\lambda x)(\sigma)$ is the maximum value of $T_\lambda x$ in $[0, 1]$. Since $(T_\lambda x)'$ is continuous and non-increasing in $[0, 1]$, we have $T_\lambda x \in K$, this imply $T_\lambda K \subset K$, furthermore, $-(\varphi_p(T_\lambda x'(t)))' = \lambda h(t) f(x(t))$, so the fixed point of T_λ in K is solution for problem (1).

Similar to the method of [4,5], we know $T_\lambda : K \rightarrow K$ is completely continuous.

By (H_1) , $\forall \varepsilon > 0$, we have $0 < \int_\varepsilon^{1-\varepsilon} h(t) dt < +\infty$, and when $\varepsilon \leq x \leq 1 - \varepsilon$, $y(x) = \int_\varepsilon^x \psi_p(\int_s^x h(r) dr) ds + \int_x^{1-\varepsilon} \psi_p(\int_x^s h(r) dr) ds$ is nonnegative continuous.

Let $P = \min_{\varepsilon \leq x \leq 1-\varepsilon} y(x) > 0$, by (H_3) and $a = \infty$, i.e. $\lim_{x \rightarrow 0} \frac{f(x)}{x^{p-1}} = \infty$, we know there exists $0 < r' < R$, such that when $0 \leq x \leq r'$, $f(x) \geq (Mx)^{p-1}$, where $M > 2(\lambda^{\frac{1}{p-1}} \varepsilon^2 P)$, for $x \in \partial K_{r'} = \{x \in K : \|x\| = r'\}$, we have

$$\begin{aligned} 2\|T_\lambda x\| &\geq \int_\varepsilon^\sigma \psi_p(\int_s^\sigma \lambda h(r) f(u(r)) dr) ds + \\ &\int_\sigma^{1-\varepsilon} \psi_p(\int_\sigma^s \lambda h(r) f(u(r)) dr) ds \\ &\geq \lambda^{\frac{1}{p-1}} M \varepsilon^2 r' (\int_\varepsilon^\sigma \psi_p(\int_s^\sigma h(r) dr) ds + \\ &\int_\sigma^{1-\varepsilon} \psi_p(\int_\sigma^s h(r) dr) ds) \\ &= \lambda^{\frac{1}{p-1}} M \varepsilon^2 r' y(\sigma) \\ &\geq \lambda^{\frac{1}{p-1}} M \varepsilon^2 r' P \\ &\geq 2r' = 2\|x\|, \quad \sigma \in [\varepsilon, 1 - \varepsilon] \end{aligned}$$

$$\begin{aligned} \|T_\lambda x\| &\geq \int_\varepsilon^{1-\varepsilon} \psi_p(\int_s^{1-\varepsilon} \lambda h(r) f(u(r)) dr) ds \\ &\geq \lambda^{\frac{1}{p-1}} M \varepsilon^2 r' (\int_\varepsilon^{1-\varepsilon} \psi_p(\int_s^{1-\varepsilon} h(r) dr) ds) \\ &= \lambda^{\frac{1}{p-1}} M \varepsilon^2 r' y(1 - \varepsilon) \\ &\geq \lambda^{\frac{1}{p-1}} M \varepsilon^2 r' P \\ &> 2r' > r' = \|x\|, \quad \sigma > 1 - \varepsilon, \end{aligned}$$

$$\begin{aligned} \|T_\lambda x\| &\geq \int_\varepsilon^{1-\varepsilon} \psi_p(\int_\varepsilon^s \lambda h(r) f(u(r)) dr) ds \\ &\geq \lambda^{\frac{1}{p-1}} M \varepsilon^2 r' y(\varepsilon) \\ &\geq \lambda^{\frac{1}{p-1}} M \varepsilon^2 r' P \\ &> 2r' > r' = \|x\|, \quad \sigma < \varepsilon, \end{aligned}$$

so for $x \in \partial K_{r'}$, we have $\|T_\lambda x\| \geq \|x\|$, by lemma 1.3,

$$i(T_\lambda, K_{r'}, K) = 0. \quad (2)$$

By (H_4) $\lim_{x \rightarrow +\infty} \frac{f(x)}{x^{p-1}} = +\infty$, there exists $R_1 > 0, \forall x \geq R_1$, we have $f(x) \geq (Mx)^{p-1}$, take $\tilde{R} > \max\{R, R_1\}$, for $x \in \partial \tilde{R}, \|x\| = \tilde{R}$, by lemma 1.1, we have

$$\begin{aligned} 2\|T_\lambda x\| &\geq \int_\varepsilon^\sigma \psi_p(\int_s^\sigma \lambda h(r) f(u(r)) dr) ds + \\ &\int_\sigma^{1-\varepsilon} \psi_p(\int_\sigma^s \lambda h(r) f(u(r)) dr) ds \\ &\geq \lambda^{\frac{1}{p-1}} M \varepsilon^2 \tilde{R} (\int_\varepsilon^\sigma \psi_p(\int_s^\sigma h(r) dr) ds + \\ &\int_\sigma^{1-\varepsilon} \psi_p(\int_\sigma^s h(r) dr) ds) \\ &\geq \lambda^{\frac{1}{p-1}} M \varepsilon^2 \tilde{R} y(\sigma) \\ &\geq \lambda^{\frac{1}{p-1}} M \varepsilon^2 \tilde{R} P \\ &\geq 2\tilde{R} = 2\|x\|, \quad \sigma \in [\varepsilon, 1 - \varepsilon], \end{aligned}$$

$$\begin{aligned} \|T_\lambda x\| &\geq \int_\varepsilon^{1-\varepsilon} \psi_p(\int_s^{1-\varepsilon} \lambda h(r) f(u(r)) dr) ds \\ &\geq \lambda^{\frac{1}{p-1}} M \varepsilon^2 \tilde{R} (\int_\varepsilon^{1-\varepsilon} \psi_p(\int_s^{1-\varepsilon} h(r) dr) ds) \\ &= \lambda^{\frac{1}{p-1}} M \varepsilon^2 \tilde{R} y(1 - \varepsilon) \\ &\geq \lambda^{\frac{1}{p-1}} M \varepsilon^2 r' P \\ &> 2\tilde{R} > \tilde{R} = \|x\|, \quad \sigma > 1 - \varepsilon, \end{aligned}$$

$$\begin{aligned} \|T_\lambda x\| &\geq \int_\varepsilon^{1-\varepsilon} \psi_p(\int_\varepsilon^s \lambda h(r) f(u(r)) dr) ds \\ &\geq \lambda^{\frac{1}{p-1}} M \varepsilon^2 \tilde{R} y(\varepsilon) \\ &\geq \lambda^{\frac{1}{p-1}} M \varepsilon^2 \tilde{R} P \\ &> 2\tilde{R} > \tilde{R} = \|x\|, \quad \sigma < \varepsilon, \end{aligned}$$

so for $x \in \partial K_{\tilde{R}}$, we have $\|T_\lambda x\| \geq \|x\|$, by lemma 1.3,

$$i(T_\lambda, K_{\tilde{R}}, K) = 0. \quad (3)$$

On the other hand, for $x \in \partial K_R$, we have

$$\begin{aligned}
\|T_\lambda x\| &\leq \psi_p\left(\int_0^1 \lambda h(r)f(u(r))dr\right)ds + \\
&\quad \max\left\{\psi_p\left(\frac{\beta}{\alpha}\int_0^1 \lambda h(r)f(u(r))dr\right),\right. \\
&\quad \left.\psi_p\left(\frac{\delta}{\gamma}\int_0^1 \lambda h(r)f(u(r))dr\right)\right\} \\
&\leq \lambda^{\frac{1}{p-1}}(1 + \max\left\{\left(\frac{\beta}{\alpha}\right)^{\frac{1}{p-1}}, \left(\frac{\delta}{\gamma}\right)^{\frac{1}{p-1}}\right\}) \times \\
&\quad \psi_p\left(\int_0^1 h(r)f(\bar{R})dr\right) \\
&= \lambda^{\frac{1}{p-1}}(1 + \max\left\{\left(\frac{\beta}{\alpha}\right)^{\frac{1}{p-1}}, \left(\frac{\delta}{\gamma}\right)^{\frac{1}{p-1}}\right\}) \times \\
&\quad \psi_p\left(\frac{f(\bar{R})}{\varphi_p(\bar{R})}\varphi_p(\bar{R})\int_0^1 h(r)dr\right) \\
&= \lambda^{\frac{1}{p-1}}(1 + \max\left\{\left(\frac{\beta}{\alpha}\right)^{\frac{1}{p-1}}, \left(\frac{\delta}{\gamma}\right)^{\frac{1}{p-1}}\right\}) \times \\
&\quad \left(\frac{f(\bar{R})}{R^{p-1}}\right)^{\frac{1}{p-1}} \psi_p\left(\int_0^1 h(r)dr\right)\bar{R} \\
&< \bar{R} \leq R = \|x\|,
\end{aligned}$$

by lemma 1.3,

$$i(T_\lambda, K_R, K) = 1. \quad (4)$$

by (2),(3),(4) and the additivity of fixed point index

$$i(T_\lambda, K_{\bar{R}} \setminus \dot{K}_R) = -1, i(T_\lambda, K_R \setminus \dot{K}_r) = 1.$$

So T_λ has fixed point x_1 in $K_{\bar{R}} \setminus \dot{K}_R$ and x_2 in $K_R \setminus \dot{K}_r$.

Next we show $x_1 \neq x_2$, we only need to show when $x_i \in \partial K_R, i = 1, 2, T_\lambda x_i \neq x_i$ hold.

If it is not true, when $x_i \in \partial K_R, i = 1, 2, T_\lambda x_i = x_i$, so $\|T_\lambda x_i\| = \|x_i\|$. Since x_i satisfy (1), we have

$$\begin{aligned}
\|T_\lambda x_i\| &\leq \psi_p\left(\int_0^1 \lambda h(r)f(u(r))dr\right)ds + \\
&\quad \max\left\{\psi_p\left(\frac{\beta}{\alpha}\int_0^1 \lambda h(r)f(u(r))dr\right),\right. \\
&\quad \left.\psi_p\left(\frac{\delta}{\gamma}\int_0^1 \lambda h(r)f(u(r))dr\right)\right\} \\
&\leq \lambda^{\frac{1}{p-1}}(1 + \max\left\{\left(\frac{\beta}{\alpha}\right)^{\frac{1}{p-1}}, \left(\frac{\delta}{\gamma}\right)^{\frac{1}{p-1}}\right\}) \times \\
&\quad \psi_p\left(\int_0^1 h(r)f(\bar{R})dr\right) \\
&= \lambda^{\frac{1}{p-1}}(1 + \max\left\{\left(\frac{\beta}{\alpha}\right)^{\frac{1}{p-1}}, \left(\frac{\delta}{\gamma}\right)^{\frac{1}{p-1}}\right\}) \times \\
&\quad \psi_p\left(\frac{f(\bar{R})}{\varphi_p(\bar{R})}\varphi_p(\bar{R})\int_0^1 h(r)dr\right) \\
&= \lambda^{\frac{1}{p-1}}(1 + \max\left\{\left(\frac{\beta}{\alpha}\right)^{\frac{1}{p-1}}, \left(\frac{\delta}{\gamma}\right)^{\frac{1}{p-1}}\right\}) \times \\
&\quad \left(\frac{f(\bar{R})}{R^{p-1}}\right)^{\frac{1}{p-1}} \psi_p\left(\int_0^1 h(r)dr\right)\bar{R} \\
&\leq \lambda^{\frac{1}{p-1}}(1 + \max\left\{\left(\frac{\beta}{\alpha}\right)^{\frac{1}{p-1}}, \left(\frac{\delta}{\gamma}\right)^{\frac{1}{p-1}}\right\}) \times \\
&\quad \left(\frac{f(\bar{R})}{R^{p-1}}\right)^{\frac{1}{p-1}} \psi_p\left(\int_0^1 h(r)dr\right)\|x_i\|
\end{aligned}$$

this imply

$$1 \leq \lambda^{\frac{1}{p-1}}(1 + \max\left\{\left(\frac{\beta}{\alpha}\right)^{\frac{1}{p-1}}, \left(\frac{\delta}{\gamma}\right)^{\frac{1}{p-1}}\right\}) \times \left(\frac{f(\bar{R})}{R^{p-1}}\right)^{\frac{1}{p-1}} \psi_p\left(\int_0^1 h(r)dr\right). \quad (5)$$

this is a contradiction, so $x_1 \neq x_2$.

Last, obviously $0 < \|x_1\| < R < \|x_2\|$.

(b) Proof Suppose there exists a subsequence $\{\lambda_n\}$, and $\lambda_n > n$ such that for any n , problem (1) has a positive solution $x_n \in K$, by $H_3, \forall \varepsilon > 0$, we have $f(x) \geq \bar{C}x^{p-1}$, where $\bar{C} = \frac{f(\bar{R})}{R^{p-1}}$, when $\sigma < \varepsilon$, by lemma 1.1, we have

$$\begin{aligned}
\|x_n\| &\geq \int_\varepsilon^{1-\varepsilon} \psi_p\left(\int_\varepsilon^s \lambda_n h(r)f(u(r))dr\right)ds \\
&\geq \lambda_n^{\frac{1}{p-1}} \int_\varepsilon^{1-\varepsilon} \psi_p\left(\int_\varepsilon^s h(r)\bar{C}(u_n)^{\frac{1}{p-1}}dr\right)ds \\
&\geq (\lambda_n \bar{C})^{\frac{1}{p-1}} \varepsilon^2 \|x_n\| \int_\varepsilon^{1-\varepsilon} \psi_p\left(\int_\varepsilon^s h(r)dr\right)ds \\
&= (\lambda_n \bar{C})^{\frac{1}{p-1}} \varepsilon^2 \|x_n\| y(\varepsilon) \\
&\geq (\lambda_n \bar{C})^{\frac{1}{p-1}} \varepsilon^2 \|x_n\| P,
\end{aligned}$$

so

$$1 \geq n \bar{C} \varepsilon^{2(p-1)} P^{p-1}. \quad (6)$$

Since n is sufficient large, so we get a contradiction.

When $\sigma > 1 - \varepsilon$ and $\sigma \in [\varepsilon, 1 - \varepsilon]$, we can get the similar result.

So there exists λ^{**} , when $\lambda > \lambda^{**}$, problem (1) has no positive solution, the proof is finished.

Theorem 2.2 If $(H_1), (H_3), (H_4)$ hold, and $0 < a < +\infty$, if there exists $\lambda^{***} > 0$ and $\lambda^{***\frac{1}{p-1}}(1 + \max\left\{\left(\frac{\beta}{\alpha}\right)^{\frac{1}{p-1}}, \left(\frac{\delta}{\gamma}\right)^{\frac{1}{p-1}}\right\})\psi_p\left(a\int_0^1 h(t)dt\right) \leq 1$, where $\psi_p(t) = |t|^{\frac{1}{p-1}} \text{sgn}(t)$ is converse function of φ_p , so for $0 < \lambda < \lambda^{***}$, problem (1) has a positive solution.

Proof Take $\epsilon > 0$, such that $(\lambda^{\frac{1}{p-1}} + \max\left\{\left(\frac{\beta}{\alpha}\right)^{\frac{1}{p-1}}, \left(\frac{\delta}{\gamma}\right)^{\frac{1}{p-1}}\right\})(a + \epsilon)^{\frac{1}{p-1}}\psi_p\left(\int_0^1 h(t)dt\right) < 1$. by H_3 , there exists $\eta > 0$ such that when $0 \leq x \leq \eta$, $f(x) \leq x^{p-1}(a + \epsilon)$. so for $x \in \partial K_\eta$, we have

$$\begin{aligned}
\|T_\lambda x\| &\leq \psi_p\left(\int_0^1 \lambda h(r)f(u(r))dr\right)ds + \\
&\quad \max\left\{\psi_p\left(\frac{\beta}{\alpha}\int_0^1 \lambda h(r)f(u(r))dr\right),\right. \\
&\quad \left.\psi_p\left(\frac{\delta}{\gamma}\int_0^1 \lambda h(r)f(u(r))dr\right)\right\} \\
&\leq \lambda^{\frac{1}{p-1}}(1 + \max\left\{\left(\frac{\beta}{\alpha}\right)^{\frac{1}{p-1}}, \left(\frac{\delta}{\gamma}\right)^{\frac{1}{p-1}}\right\}) \times \\
&\quad \psi_p\left(\int_0^1 h(r)x^{p-1}(r)(a + \epsilon)dr\right) \\
&\leq \lambda^{\frac{1}{p-1}}(1 + \max\left\{\left(\frac{\beta}{\alpha}\right)^{\frac{1}{p-1}}, \left(\frac{\delta}{\gamma}\right)^{\frac{1}{p-1}}\right\}) \times \\
&\quad (a + \epsilon)^{\frac{1}{p-1}} \psi_p\left(\int_0^1 h(r)dr\right)\|x\| \\
&\leq \lambda^{\frac{1}{p-1}}(1 + \max\left\{\left(\frac{\beta}{\alpha}\right)^{\frac{1}{p-1}}, \left(\frac{\delta}{\gamma}\right)^{\frac{1}{p-1}}\right\}) \times \\
&\quad (a + \epsilon)^{\frac{1}{p-1}} \psi_p\left(\int_0^1 h(r)dr\right)\eta \\
&< \eta = \|x\|.
\end{aligned}$$

By H_4 , there exists $\varrho > 0$, such that when $x \geq \varrho, f(x) \geq (Mx)^{p-1}$, choose $\mu > \max\{\varrho, \eta\}$, by the similar method with theorem 2.1, we can show when $x \in \partial K_\mu, \|T_\lambda x\| \geq \|x\|$, so if we define

$\Omega_1 = \{x \in K : \|x\| < \eta\}$, $\Omega_2 = \{x \in K : \|x\| < \mu\}$, by lemma 1.4, T_λ has at least one fixed point $x \in K$, and $\mu > \|x\| > \eta$, the proof is finished.

Corollary In condition H_3 , let $a = 0$, then $\forall \lambda > 0$, problem (1) has at least one positive solution.

Theorem 2.3 If H_1, H_2, H_5, H_6 hold, then

(a) $\forall \varepsilon \in (0, \frac{1}{2})$, there exists $\lambda_* = \lambda_*(\varepsilon) > 0$, such that for all $\lambda > \lambda_*$, problem (1) has at least two x_1, x_2 and $0 < \|x_1\| < L < \|x_2\|$.

(b) If there exist $\lambda_{**} > 0$ such that $\lambda_{**}^{\frac{1}{p-1}}(1 + \max\{(\frac{\beta}{\alpha})^{\frac{1}{p-1}}, (\frac{\delta}{\gamma})^{\frac{1}{p-1}}\})(C' \int_0^1 h(t)dt) \leq 1$, then for all $\lambda < \lambda_{**}$, problem (1) has no positive solution, where $C' = \frac{f(L)}{L^{p-1}}$.

(a) **Proof** For any $0 < \varepsilon < \frac{1}{2}$, $\forall x \in K$ and $\|x\| = L$, Let $v = \min_{\varepsilon \leq t \leq 1-\varepsilon} \frac{f(u(t))}{u(t)^{p-1}}$, by (H_2) and lemma 1.1, $v > 0$, let $\lambda_* = \frac{(\varepsilon^2 Q)^{p-1}}{v}$, where $Q = \min_{\varepsilon \leq x \leq 1-\varepsilon} y(x) > 0$, then for $\lambda > \lambda_*$ we have

$$\begin{aligned} 2\|T_\lambda x\| &\geq \int_\sigma^\sigma \psi_p(\int_\sigma^\sigma \lambda h(r)f(u(r))dr)ds + \\ &\int_{\frac{1-\varepsilon}{\sigma}}^{\frac{1-\varepsilon}{\sigma}} \psi_p(\int_\sigma^s \lambda h(r)f(u(r))dr)ds \\ &\geq (\lambda v)^{\frac{1}{p-1}} \varepsilon^2 (\int_\sigma^\sigma \psi_p(\int_\sigma^\sigma h(r)dr)ds + \\ &\int_\sigma^{1-\varepsilon} \psi_p(\int_\sigma^s h(r)dr)ds) \\ &= (\lambda v)^{\frac{1}{p-1}} \varepsilon^2 Q L \\ &> 2L = 2\|x\|, \sigma \in [\varepsilon, 1-\varepsilon], \end{aligned}$$

$$\begin{aligned} \|T_\lambda x\| &\geq \int_\varepsilon^{1-\varepsilon} \psi_p(\int_\sigma^s \lambda h(r)f(u(r))dr)ds \\ &\geq (\lambda v)^{\frac{1}{p-1}} \varepsilon^2 Q L \\ &> \|x\|, \sigma > 1-\varepsilon, \end{aligned}$$

$$\begin{aligned} \|T_\lambda x\| &\geq \int_\varepsilon^{1-\varepsilon} \psi_p(\int_\varepsilon^s \lambda h(r)f(u(r))dr)ds \\ &\geq (\lambda v)^{\frac{1}{p-1}} \varepsilon^2 Q L \\ &\geq \|x\|, \sigma < \varepsilon, \end{aligned}$$

so for $x \in \partial K_L$, we have $\|T_\lambda x\| > \|x\|$.

For the same λ , choose $\varepsilon' > 0$ such that $\varepsilon'(\lambda^{\frac{1}{p-1}} + \max\{(\frac{\beta}{\alpha})^{\frac{1}{p-1}}, (\frac{\delta}{\gamma})^{\frac{1}{p-1}}\})\psi_p(\int_0^1 h(r)dr) < 1$, by (H_5) , there exists $0 < l < L$, such that when $0 \leq x \leq l$, $f(x) \leq (\varepsilon'x)^{p-1}$,

so for $x \in \partial K_l$, we have

$$\begin{aligned} \|T_\lambda x\| &\leq \psi_p(\int_0^1 \lambda h(r)f(u(r))dr)ds + \\ &\max\{\psi_p(\frac{\beta}{\alpha} \int_0^1 \lambda h(r)f(u(r))dr), \\ &\psi_p(\frac{\delta}{\gamma} \int_0^1 \lambda h(r)f(u(r))dr)\} \\ &\leq \lambda^{\frac{1}{p-1}}(1 + \max\{(\frac{\beta}{\alpha})^{\frac{1}{p-1}}, (\frac{\delta}{\gamma})^{\frac{1}{p-1}}\}) \times \\ &\psi_p(\int_0^1 h(r)f(x(r))dr) \\ &\leq \lambda^{\frac{1}{p-1}}(1 + \max\{(\frac{\beta}{\alpha})^{\frac{1}{p-1}}, (\frac{\delta}{\gamma})^{\frac{1}{p-1}}\}) \times \\ &\psi_p \int_0^1 (\varepsilon'x)^{p-1} h(r)dr \\ &\leq \lambda^{\frac{1}{p-1}}(1 + \max\{(\frac{\beta}{\alpha})^{\frac{1}{p-1}}, (\frac{\delta}{\gamma})^{\frac{1}{p-1}}\}) \times \\ &\varepsilon' \psi_p(\int_0^1 h(r)dr)l \\ &< l = \|x\|. \end{aligned}$$

We define a new function $\bar{f}(x) = \max_{0 \leq s \leq x} f(s)$, so $\bar{f}(x)$ is nondecreasing monotonously, by (H_6) $\lim_{x \rightarrow +\infty} \frac{f(x)}{x^{p-1}} = 0$, we can get $\lim_{x \rightarrow +\infty} \frac{\bar{f}(x)}{x^{p-1}} = 0$, for the same $\varepsilon' > 0$, there exists $S > 0$ such that when $x \leq S$, $\bar{f}(x) \leq (\varepsilon'x)^{p-1}$, choose $L' = \max\{L, S\}$, so for $x \in K_{L'}$, we have

$$\begin{aligned} \|T_\lambda x\| &\leq \psi_p(\int_0^1 \lambda h(r)f(u(r))dr)ds + \\ &\max\{\psi_p(\frac{\beta}{\alpha} \int_0^1 \lambda h(r)f(u(r))dr), \\ &\psi_p(\frac{\delta}{\gamma} \int_0^1 \lambda h(r)f(u(r))dr)\} \\ &\leq \lambda^{\frac{1}{p-1}}(1 + \max\{(\frac{\beta}{\alpha})^{\frac{1}{p-1}}, (\frac{\delta}{\gamma})^{\frac{1}{p-1}}\}) \times \\ &\psi_p(\int_0^1 h(r)\bar{f}(L')dr) \\ &\leq \lambda^{\frac{1}{p-1}}(1 + \max\{(\frac{\beta}{\alpha})^{\frac{1}{p-1}}, (\frac{\delta}{\gamma})^{\frac{1}{p-1}}\}) \times \\ &\varepsilon' \psi_p \int_0^1 h(r)dr L' \\ &< L' = \|x\|. \end{aligned}$$

we define $\Omega_1 = \{x \in K : \|x\| < L\}$, $\Omega_2 = \{x \in K : \|x\| < L'\}$, by lemma 1.4, T_λ has at least two fixed points $x_1(t), x_2(t)$ in K , and satisfy $l \leq \|x_1\| \leq L \leq \|x_2\| \leq L'$.

Similarly to the proof of theorem 2.1, T_λ has no fixed point in ∂K_L , so $x_1(t) \neq x_2(t)$, the proof is finished.

(b) **Proof** Suppose there exists a subsequence $\lambda_n < \lambda_{**}$ and $\lambda_n \in (0, \frac{1}{n})$ such that for $\forall n$ problem (1) has a positive solution $x_n \in K$. since $x > 0$, $f(x) \leq (C'x)^{p-1}$, where

$C' = \frac{f(L)}{L^{p-1}}$, we have

$$\begin{aligned} \|x_{\lambda_n}\| &\leq \psi_p \left(\int_0^1 \lambda_n h(r) f(x_{\lambda_n}(r)) dr \right) ds + \\ &\quad \max \left\{ \psi_p \left(\frac{\beta}{\alpha} \int_0^1 \lambda h(r) f(x_{\lambda_n}(r)) dr \right), \right. \\ &\quad \left. \psi_p \left(\frac{\delta}{\gamma} \int_0^1 \lambda h(r) f(x_{\lambda_n}(r)) dr \right) \right\} \\ &\leq \lambda_n^{\frac{1}{p-1}} \left(1 + \max \left\{ \left(\frac{\beta}{\alpha} \right)^{\frac{1}{p-1}}, \left(\frac{\delta}{\gamma} \right)^{\frac{1}{p-1}} \right\} \right) \times \\ &\quad \psi_p \left(\int_0^1 h(r) C' x_{\lambda_n}^{p-1}(r) dr \right) \\ &\leq \lambda_n^{\frac{1}{p-1}} \left(1 + \max \left\{ \left(\frac{\beta}{\alpha} \right)^{\frac{1}{p-1}}, \left(\frac{\delta}{\gamma} \right)^{\frac{1}{p-1}} \right\} \right) \times \\ &\quad C'^{\frac{1}{p-1}} \psi_p \left(\int_0^1 h(r) dr \right) \|x_{\lambda_n}\|, \end{aligned}$$

i.e.

$$1 \leq \lambda_n^{\frac{1}{p-1}} \left(1 + \max \left\{ \left(\frac{\beta}{\alpha} \right)^{\frac{1}{p-1}}, \left(\frac{\delta}{\gamma} \right)^{\frac{1}{p-1}} \right\} \right) C'^{\frac{1}{p-1}} \psi_p \left(\int_0^1 h(r) dr \right). \quad (7)$$

since $\lambda_n < \lambda_{**}$, so $\lambda_n^{\frac{1}{p-1}} \left(1 + \max \left\{ \left(\frac{\beta}{\alpha} \right)^{\frac{1}{p-1}}, \left(\frac{\delta}{\gamma} \right)^{\frac{1}{p-1}} \right\} \right) C'^{\frac{1}{p-1}} \psi_p \left(\int_0^1 h(r) dr \right) < 1$, this is contradiction, the proof is finished.

Example 1

$$\begin{cases} -(\varphi_p(x'(t)))' = \lambda(1-t)^{p_1} t^{p_2} (cx^{q_1}(t) + x^{q_2}(t)), & t \in (0, 1) \\ x(0) = x(1) = 0, \end{cases}$$

where λ is a positive parameter, $c \in R^+ \cup \{0\}$, $-1 < p_1 < 0$, $-1 < p_2 < 0$, $0 < q_1 \leq p-1 < q_2$.

we consider the following two cases:

$$\begin{aligned} (1) \text{ when } 0 < q_1 < p-1 < q_2 \text{ and } c > 0, \\ R = \bar{R} = \left(c \frac{p-1-q_1}{q_2-p+1} \right)^{\frac{1}{q_2-q_1}}, \text{ and } \lambda^* = \\ \left((q_2-p+1)^{q_2-p+1} (p-1-q_1)^{p-1-q_1} \right)^{\frac{1}{q_2-q_1}} \times (\beta(p_1 + \\ 1, p_2+1))^{-1} \times c^{\frac{q_2-p+1}{q_1-q_2}}. \end{aligned}$$

By theorem 2.1, if $\lambda \in (0, \lambda^*)$, then problem (1) has at least two positive solutions x_1, x_2 satisfy $0 < \|x_1\| < R < \|x_2\|$, there exists λ^{**} sufficient large, when $\lambda > \lambda^{**}$, problem (1) has no positive solution.

(2) $q_1 = p-1$, and $c \geq 0$, if $c > 0$, then $h(t) = (1-t)^{p_1} t^{p_2}$, and $f(x) = cx^{q_1}(t) + x^{q_2}(t)$ satisfy all the conditions of theorem 2, and $\beta = 0, \delta = 0$. Let $\lambda^{***} = (c\beta(p_1+1, p_2+1))^{-1}$, where β is β function, for $0 < \lambda < \lambda^{***}$, problem (1) has at least one positive solution.

If $c = 0$, by corollary, for each $\lambda > 0$, problem (1) has at least one positive solution.

Example 2

$$\begin{cases} -(\varphi_p(x'(t)))' = \lambda h(t)(e^{x(t)} - 1), & t \in (0, 1) \\ x(0) = x(1) = 0 \end{cases}$$

where λ is positive parameter, $h(t)$ is same as above, we consider three cases:

case 1 $p > 2$. let $\lambda^* = \left(\frac{f(\bar{R})}{\bar{R}^{p-1}} \beta(p_1+1, p_2+1) \right)^{-1}$, where $R = \bar{R} \in (p-2, p-1)$ is the only zero point of function

$\chi(x) = e^x(x-p+1)+p-1$, by theorem 2.1, when $\lambda \in (0, \lambda^*)$, problem (1) has at least two solutions, and $0 < \|x_1\| < R < \|x_2\|$. there exists λ^{**} sufficient large, when $\lambda > \lambda^{**}$, problem (1) has no solution.

case 2 $p = 2$, let $\lambda^{***} = (\beta(p_1+1, p_2+1))^{-1}$, where β is β function. by theorem 2.2, for $0 < \lambda < \lambda^{***}$ problem (1) has at least one positive solution.

case 3 $1 < p < 2$, in this case $a = 0$, by corollary, for each $\lambda > 0$, problem (1) has at least one positive solution.

Example 3

$$\begin{cases} -(\varphi_p(x'(t)))' = \lambda t^{-\alpha} x(t)^q e^{-x(t)}, & t \in (0, 1), \\ x(0) = x(1) = 0 \end{cases}$$

where $0 < \alpha < 1, p-1 < q$.

By theorem 2.3, for $\varepsilon \in (0, \frac{1}{2})$, let $\lambda_*(\varepsilon) = \left(\frac{2}{v\varepsilon^2 Q} \right)^{p-1}$, for $0 < \lambda > \lambda_*$, problem (1) has at least two positive solutions, and $0 < \|x_1\| < q-p+1 < \|x_2\|$. there exists λ_{**} sufficient small, when $\lambda < \lambda_{**}$ problem (1) has no solution. Specially, $p = 2$, let $\lambda_*(\varepsilon) = \left(\frac{2}{v\varepsilon^2 Q} \right)^{p-1}$, $v = \varepsilon^{2(q-1)}(q-1)e^{\varepsilon^{2(1-q)}}$, and $Q = \frac{\varepsilon^{-2-\alpha} + (1-\varepsilon)^{2-\alpha} - 2^{\alpha-1}}{(1-\alpha)(2-\alpha)}$, we can get for $0 < \lambda > \lambda_*$, problem (1) has at least two positive solutions, and $0 < \|x_1\| < q-1 < \|x_2\|$.

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