

The Finite Difference Scheme for the Suspended String Equation with the Nonlinear External Forces

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Abstract—This paper presents the finite difference scheme and the numerical simulation of suspended string. The vibration solutions when the various external forces are taken into account are obtained and compared with the solutions without external force. In addition, we also investigate how the external forces and their powers and coefficients affect the amplitude of vibration.

Keywords—Nonlinear external forces, Numerical simulation, Suspended string equation.

I. INTRODUCTION

THE vibration equation of the suspended string was first studied by [1]. Author in [2] considered this equation taking into account the external force having the blow up term and showed the existence of the global weak solutions. Authors in [3] considered this equation with the nonlinear factor known as absorbing term and showed the existence of the global classical solution of nonlinear suspended string equation. To fully understand the characteristics of the suspended string vibration according to mathematical theory, the numerical simulations of the suspended string vibration have been studied. Authors in [4] and [5] showed the numerical simulations of the suspended string vibration without the external force using the finite difference and Crank-Nicolson methods, respectively. The characteristics of the vibration under the same various initial shapes and initial velocities were considered. To study the effect of damping force to the vibration characteristic, authors in [6] and [7] showed the numerical simulation with the linear and nonlinear damping terms respectively.

In this work, the numerical simulation of a heavy and flexible vibrating suspended string with a finite length a with the nonlinear external force is studied. The suspended string equation with the initial and boundary conditions can be represented by

$$\begin{aligned} u_{tt} - ((m\Delta x)u_{xx} + u_x) &= \beta |u|^{d-1} u, & \beta \in R, d \geq 1, \\ u(0, t) &= 0, & t \in [0, T], \\ u(x, 0) &= \phi(x), \quad u_t(x, 0) = \psi(x), & x \in [0, a]. \end{aligned} \quad (1)$$

where $u(x, t)$ is the horizontal displacement of the string at (x, t) , β is the real number denoting the coefficient of the external force term and the power d is the positive integer.

It is known that when β is positive and $d=3$, this is called the blow up case. When β is negative and $d=3$, it is called the absorbing case as studied in [2] and [3] respectively. In this work, we investigate the effect of the nonlinear external force accounting for both blow up and absorbing terms to see how they affect the amplitude vibration. In addition, we will consider how various other values of d and different values of the coefficients β of the external force play roles in dictating the vibration characteristics, e.g., the increase and decrease of the vibration amplitude and the resonance of the suspended string vibration. Finite difference method is employed to show the numerical simulation of the suspended string.

II. THE METHOD OF SOLUTION

To apply the finite difference method to (1), the initial condition has been modified as follows

$$\begin{aligned} u_{tt} - ((m\Delta x)u_{xx} + u_x) &= \beta |u|^{d-1} u, & \beta < 0, d \geq 1, \\ u(0, t) &= 0, \quad u_x(a, t) = 0, & t \in [0, T], \\ u(x, 0) &= \phi(x), \quad u_t(x, 0) = \psi(x), & x \in [0, a]. \end{aligned} \quad (2)$$

The solution domain $(0 < x < 1, t > 0)$ is divided into subintervals Δx and Δt in the direction of the position x and of the time t , respectively. The numerical solution at the grid point is found by substituting u_{tt} , u_{xx} , u_x and u_t in (2) using the central finite difference as

$$\begin{aligned} &\left(\frac{u_m^{n+1} - 2u_m^n + u_m^{n-1}}{k^2} \right) - \left[mh \left(\frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{h^2} \right) + \left(\frac{u_{m+1}^n - u_{m-1}^n}{2h} \right) \right] \\ &= \beta |u_m^n|^{d-1} u_m^n, \end{aligned} \quad (3)$$

where $u_m^{n+1} = u(x, t+k)$, $u_m^n = u(x, t)$, $u_m^{n-1} = u(x, t-k)$, $u_{m+1}^n = u(x+h, t)$, $u_{m-1}^n = u(x-h, t)$, m is the position step ($m = 1, \dots, M$) and n is the time step ($n = 1, \dots, N$), while h and k are the mesh size in x and t , respectively.

For the third power nonlinear damping case ($c = 3$), the last term of left hand side of (3) can be shown as

$$\left(\frac{u_m^{n+1} - 2u_m^n + u_m^{n-1}}{k^2} \right) - \left[\frac{mh \left(\frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{h^2} \right)}{+ \left(\frac{u_{m+1}^n - u_{m-1}^n}{2h} \right)} \right] \quad (4)$$

$$= \beta \left| u_m^n \right|^{d-1} u_m^n$$

Multiplying (4) by $2k^2$, we have

$$\left(2u_m^{n+1} - 4u_m^n + 2u_m^{n-1} \right) - \left[2mp \left(\frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{h^2} \right) + p \left(\frac{u_{m+1}^n - u_{m-1}^n}{h} \right) \right] \quad (5)$$

$$= 2k^2 \beta \left| u_m^n \right|^{d-1} u_m^n$$

where $p = k^2/h$, $\beta \geq R$ and $d \geq 1$.

Equation (5) can be classified into 4 different cases depending on the values of n and m . We then obtain the finite difference schemes for the numerical solution as follows:

Case 1: $n = 0$ and $m = 1, 2, 3, \dots, M-1$

$$u_m^1 = \frac{1}{2} \left[(2-4mp)u_m^0 + (2mp+p)u_{m+1}^0 + (2mp-p)u_{m-1}^0 + 2k\psi(x) + 2k\beta \left| u_m^0 \right|^d \right] \quad (6)$$

Case 2: $n > 0$ and $m = 1, 2, 3, \dots, M-1$

$$u_m^{n+1} = \frac{1}{2} \left[(4-4mp)u_m^n + (2mp+p)u_{m+1}^n + (2mp-p)u_{m-1}^n - 2u_m^{n-1} + 2k\beta \left| u_m^n \right|^d \right] \quad (7)$$

Case 3: $n = 0$ and $m = M$

$$u_M^1 = \frac{1}{2} \left[(2-4Mp)u_M^0 + 4Mp u_{M-1}^0 + 2k\psi(x) + 2k\beta \left| u_M^0 \right|^d \right] \quad (8)$$

Case 4: $n > 0$ and $m = M$

$$u_M^{n+1} = \frac{1}{2} \left[(4-4Mp)u_M^n + 4Mp u_{M-1}^n - 2u_M^{n-1} + 2k\beta \left| u_M^n \right|^d \right] \quad (8)$$

where $p = k^2/h$, $\beta \geq R$ and $d \geq 1$.

The finite difference schemes (6)-(8) have been programmed in MATLAB and the numerical solutions are shown graphically (to be discussed in the next section).

III. RESULTS AND DISCUSSION

We show the graphical comparison of the vibration displacement when no external force and nonlinear external force are accounted for provided that there is no initial velocity and various different external force coefficients (β) are considered as follows.

1. The comparison of the vibration without external force and with external force to the first power ($d=1$) under the various values of β as shown in Figs. 1- 4.
2. The comparison of the vibration without external force and with external force to the second power ($d=2$) under the various values of β as shown in Figs. 5 and 6.
3. The comparison of the vibration with external force to the first ($d=1$) and the third power ($d=3$), external force to the first ($d=1$) and the fifth power ($d=5$), external force to the first ($d=1$) and seventh ($d=7$) under the same β as shown in Figs. 7-9.
4. The comparison of the vibration with external force to the first ($d=1$) and the third power ($d=3$), external force to the third ($d=3$) and the fifth power ($d=5$), external force to the fifth ($d=5$) and seventh ($d=7$), using the same β as shown in Figs. 10-14.
5. The comparison of the vibration with external force to the second ($d=2$) and the forth power ($d=4$), external force to the fourth ($d=4$) and the sixth power ($d=6$) and external force to the sixth ($d=6$) and the eighth power ($d=8$) using $\beta = -4, 4, 10$ and -25 as shown in Figs. 15-18.

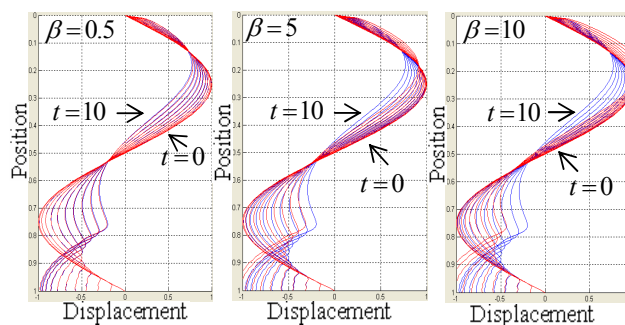


Fig. 1 Graphical comparison of the vibration displacements between the no external force (blue line) and the first power nonlinear external force (red line) for different $\beta = 0.5, 5$, and 10

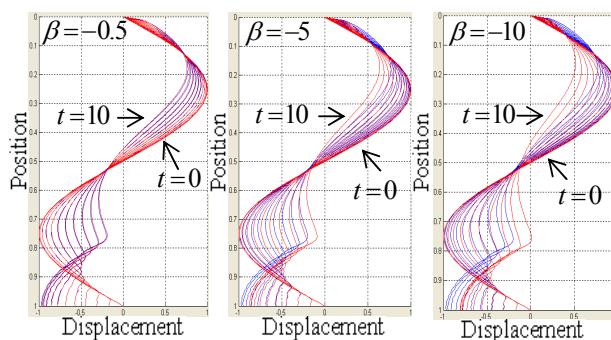


Fig. 2 Graphical comparison of the vibration displacements between the no external force (blue line) and the first power nonlinear external force (red line) for different $\beta = -0.5, -5$, and -10

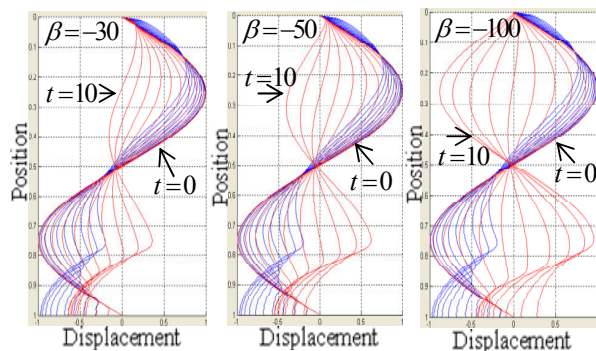


Fig. 3 Graphical comparison of the vibration displacements between the no external force and the first power nonlinear external force (red line) for different $\beta = -30, -50$, and -100

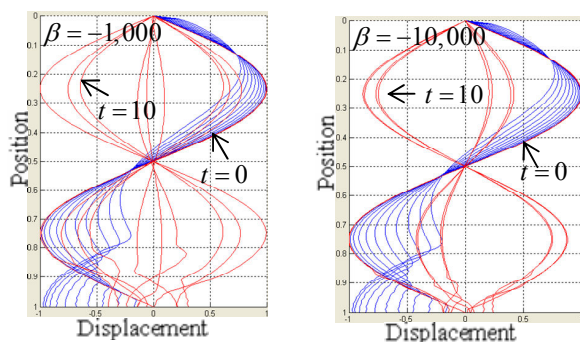


Fig. 4 Graphical comparison of the vibration displacements between the no external force and the first power nonlinear external force (red line) for $\beta = -1000$

By comparing the vibration characteristics without external force and with the external force to the first power, we found that the vibration amplitude increases and finally the resonance takes place when β is positive and increases as illustrated in Fig. 1. The amplitude of vibration will decrease when β is less than zero and decreases as illustrated in Fig. 2. Additionally, when β is negative and further decreases, the amplitude of vibration will also decrease within the interval -1 to 1 as illustrated in Figs. 3 and 4.

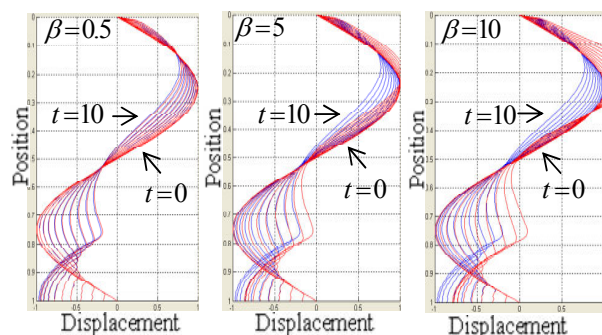


Fig. 5 Graphical comparison of the vibration displacements between the no external force and the second power nonlinear external force (red line) for different betas, $\beta = -0.5, -5$, and -10

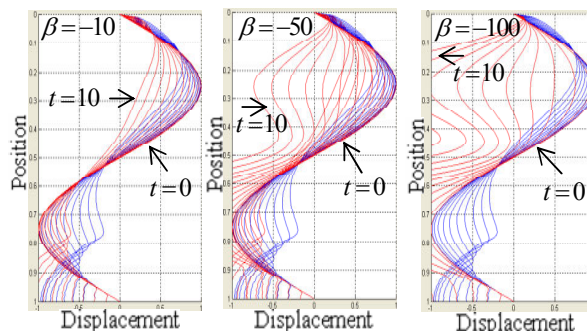


Fig. 6 Graphical comparison of the vibration displacements between the no external force and the second power nonlinear external force (red line) for different betas, $\beta = -10, -50$, and -100

By comparing the vibration characteristics without external force and with the external force to the second power, we found that the amplitude of vibration increases and the resonance takes place when β is positive and increases as shown in Fig. 5. In addition, the vibration amplitude decreases and the resonance also take place when β is negative and further decreases as shown in Fig. 6.

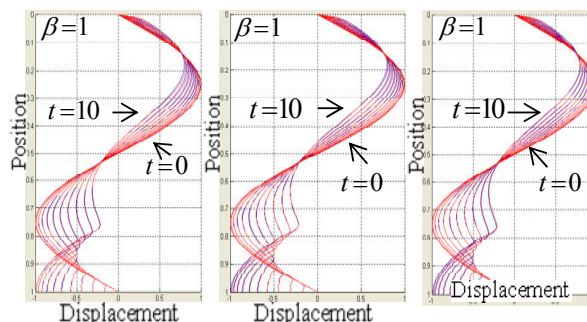


Fig. 7 Graphical comparison of the vibration displacements between the first and the third power, the fifth power, the seventh power nonlinear external force (red line) for $\beta = 1$.

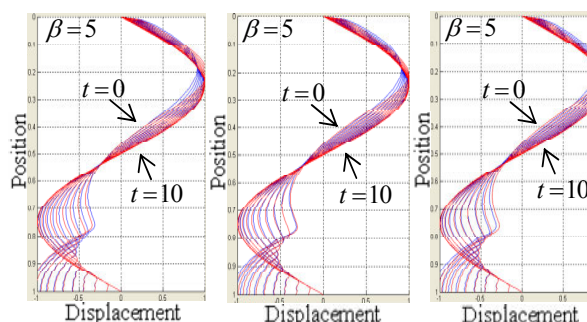


Fig. 8 Graphical comparison of the vibration displacements between the first and the third power, the fifth power, the seventh power nonlinear external force (red line) for $\beta = 5$

Figs. 7 and 8 indicate that when β is positive and small, the vibration with the external force to the power of $1, 3, 5$ and 7 are almost identical but clearly demonstrates differences when β increases further, namely, the amplitude of vibration will

start to blow up with increasing power of the external force as illustrated in Figs. 9 and 10.

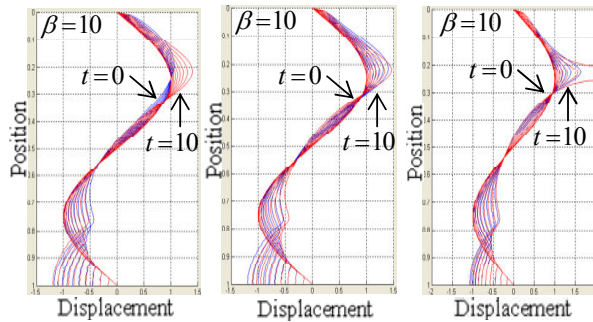


Fig. 9 Graphical comparison of the vibration displacements between the first and the third power, the third power and the fifth power, the seventh power and the ninth power nonlinear external force (red line) for $\beta = 10$

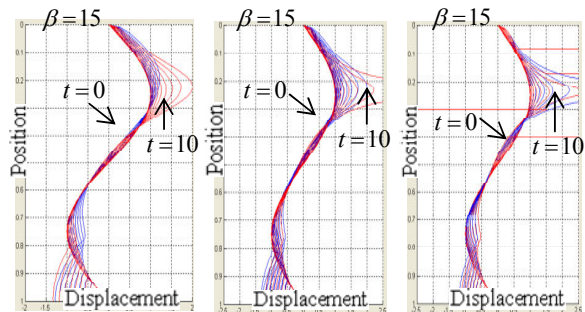


Fig. 10 Graphical comparison of the vibration displacements between the first and the third power, the third power and the fifth power, the fifth power and the seventh power nonlinear external force (red line) for $\beta = 15$

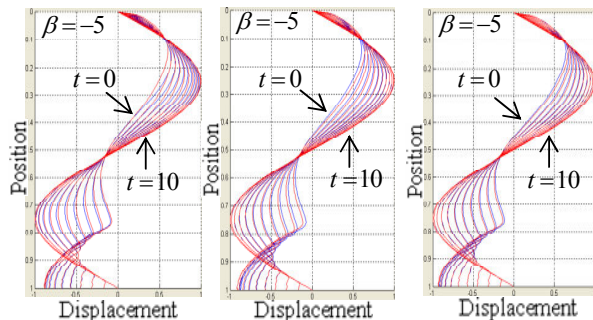


Fig. 11 Graphical comparison of the vibration displacements between the first and the third power, the third power and the fifth power, fifth power and the seventh power nonlinear external force (red line) for $\beta = -5$

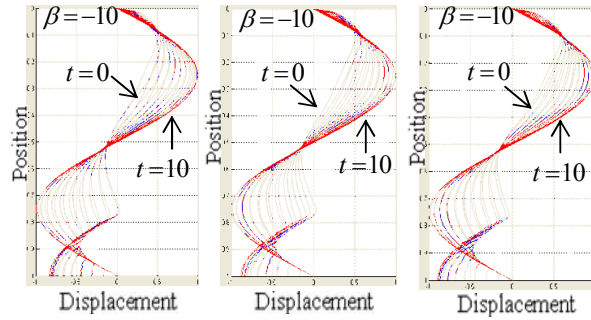


Fig. 12 Graphical comparison of the vibration displacements between the first and the third power, the third power and the fifth power, the fifth power and the seventh power nonlinear external force (red line) for $\beta = -10$

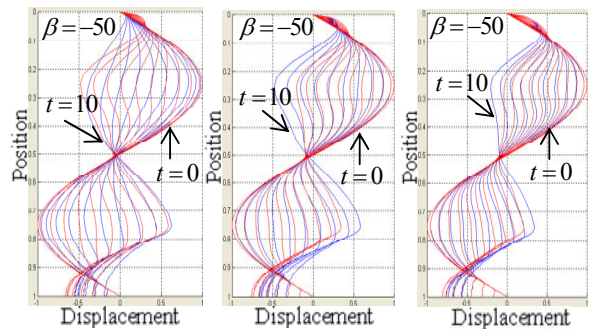


Fig. 13 Graphical comparison of the vibration displacements between the first and the third power, the third power and the fifth power, the fifth power and the seventh power nonlinear external force (red line) for $\beta = -50$

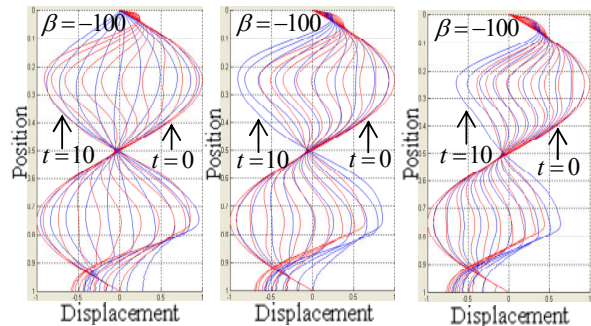


Fig. 14 Graphical comparison of the vibration displacements between the first and the third power, the third power and the fifth power, the fifth power and the seventh power nonlinear external force (red line) for $\beta = -100$

Figs. 11-14 show that the vibration with the external force to the power 1, 3, 5 and 7 which exhibit similar characteristics and show no resonance.

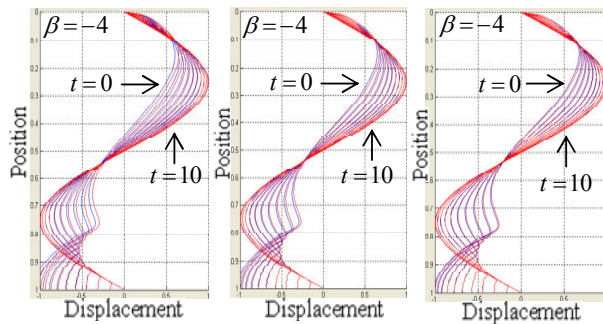


Fig. 15 Graphical comparison of the vibration displacements between the second and the fourth power, the fourth power and the sixth power, the sixth power and the eighth power nonlinear external force for $\beta = -4$

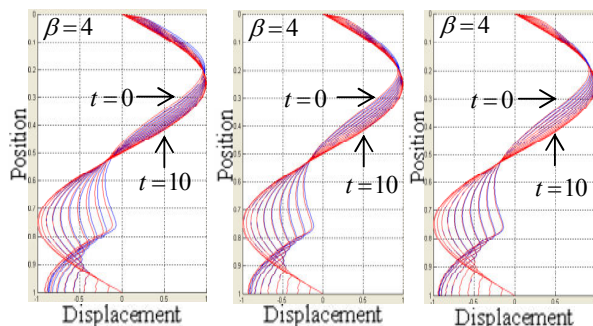


Fig. 16 Graphical comparison of the vibration displacements between the second and the fourth power, the fourth power and the sixth power, the sixth power and the eighth power nonlinear external force for $\beta = 4$

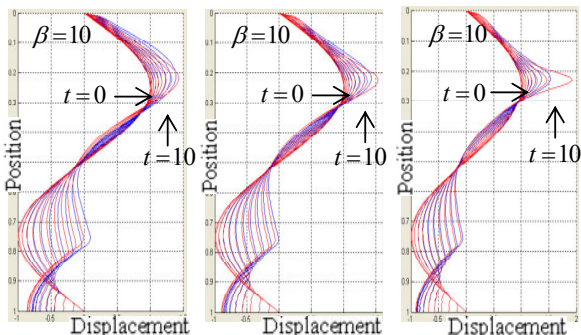


Fig. 17 Graphical comparison of the vibration displacements between the second and the fourth power, the fourth power and the sixth power, the sixth power and the eighth power nonlinear external force for $\beta = 10$

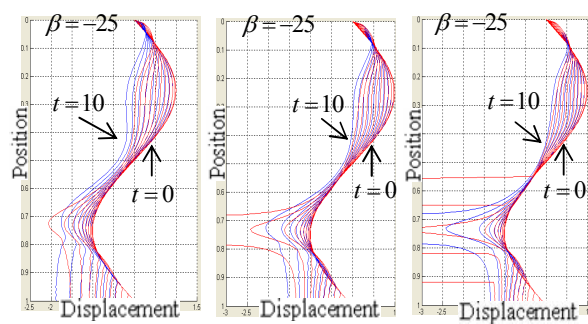


Fig. 18 Graphical comparison of the vibration displacements between the second and the fourth power, the fourth power and the sixth power, the sixth power and the eighth power nonlinear external force for $\beta = -25$

It is also noted that the vibrations with external force with the odd power (2, 4, 6, 8) demonstrate similar characteristic when β is in the range from -4 to 4 as shown in Figs. 15 and 16. However, the vibration amplitudes are different and demonstrate resonance when β is out of above interval as shown in Figs. 17 and 18.

From Figs. 1-18, it is obvious that the power and the coefficients of the external force directly affect the vibration characteristics. We can classify the vibration into 2 different cases, namely, the vibration with the external force to the odd and even powers and they are summarized as shown in the following Table I and II.

TABLE I THE AMPLITUDE OF THE SUSPENDED STRING VIBRATION AS A FUNCTION OF THE POWERS AND THE COEFFICIENTS OF THE EXTERNAL FORCE			
The power of the external force	The coefficient of the external force	Amplitude increasing	Amplitude decreasing
The first power	$\beta > 0$	✓	
	$\beta < 0$		✓
The second power	$\beta > 0$	✓	
	$\beta < 0$		✓

TABLE II THE COMPARISON OF THE AMPLITUDE OF THE SUSPENDED STRING VIBRATION WITH VARIOUS POWERS AND COEFFICIENTS OF THE EXTERNAL FORCE			
The power of the external force	The coefficient of the external force	Amplitude similar	Amplitude dissimilar
Odd power (1, 3, 5, 7)	$0 < \beta < 5$	✓	
	$\beta > 5$		✓
Even power (2, 4, 6, 8)	$\beta < 0$	✓	
	$-5 < \beta < 5$	✓	
	$\beta < -5, \beta > 5$		✓

Table III summarizes the resonance characteristic. We can see that the resonance will not take place when the external force has the odd power and when β is negative or small positive values (i.e., $0 < \beta < 5$).

TABLE III
THE RESONANCE OF THE SUSPENDED STRING VIBRATION AS A FUNCTION OF
THE POWERS AND THE COEFFICIENTS OF THE EXTERNAL FORCE

The power of the external force	The coefficient of the external force	Resonance	No resonance
Odd power (1, 3, 5, 7)	$0 < \beta < 5$		✓
	$\beta > 5$	✓	
	$\beta < 0$		✓
Even power (2, 4, 6, 8)	$\beta < 0$	✓	
	$-5 < \beta < 5$	✓	

From result Table III, we can study more the damping term how effect to resonance occurring. It prevents the resonance from occurring or not.

IV. CONCLUSION

The vibration characteristics can be divided into two different cases, namely, when the power of the external force is the odd power and the even power. In both cases, the vibration amplitude is proportional to the coefficient β of the external force. When β is negative or small positive values (i.e., $0 < \beta < 5$), the vibrations characteristics with the external force to the odd power are quite similar. However, when $\beta > 5$, the vibration characteristics demonstrate differently. In addition, when β is small and in the interval $-5 < \beta < 5$, the vibration characteristics with the external force to the even power are again quite similar and demonstrates different characteristics when β is not in the mentioned range. The numerical simulation of the suspended string shows that the vibration characteristics agree well with the mathematical theory provided in [2] and [3].

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