International Journal of Architectural, Civil and Construction Sciences
ISSN: 2415-1734
Vol:4, No:10, 2010

Method of Moments for Analysis of Multiple Crack Interaction in an Isotropic Elastic Solid

Weifeng Wang, Xianwei Zeng, and Jianping Ding

***Abstract*—**The problem of N cracks interaction in an isotropic elastic solid is decomposed into a subproblem of a homogeneous solid without crack and N subproblems with each having a single crack subjected to unknown tractions on the two crack faces. The unknown tractions, namely pseudo tractions on each crack are expanded into polynomials with unknown coefficients, which have to be determined by the consistency condition, i.e. by the equivalence of the original multiple cracks interaction problem and the superposition of the N+1 subproblems. In this paper, Kachanov's approach of average tractions is extended into the method of moments to approximately impose the consistence condition. Hence Kachanov's method can be viewed as the zero-order method of moments. Numerical results of the stress intensity factors are presented for interactions of two collinear cracks, three collinear cracks, two parallel cracks, and three parallel cracks. As the order of moment increases, the accuracy of the method of moments improves.

***Keywords*—**Crack interaction, stress intensity factor, multiple cracks, method of moments.

I. Introduction

THE study of multiple crack interaction has attracted consideration attention in both the mechanics and the materials science communities. The solution to the problem of multiple crack interaction is key to the evaluation of the effective elastic properties of brittle solids damaged by micro cracks, a review of this topic was presented by Kachanov [1]. The solution to multiple crack interaction is also important for the understanding of maincrack-microcrack interaction in elastic solids [2]. Due to the interaction of maincrack-microcrack, microcracks around the tip of a maincrack can change the elastic field around the maincrack and consequently affect the propagation of the main crack [3]. A variety of analytical methods have been proposed for the solution of an elastic solid with multiple cracks, majority of these analytical method can be classified into the group of the singular integral equation method with different kernel functions. A thorough review of the singular integral equation method for analysis of multiple cracks was given by Chen [4].

In the singular integral equation method, one approach is to replace each crack by distributed dislocations with unknown densities [5], another approach is to assume the normal and shear tractions on each crack as the primary unknowns [6]. The latter approach is more appealing for engineers for its clear physical meanings. In the literature, the second approach is referred as the method of pseudo tractions [6]. These pseudo tractions can be expanded into Chebyshev polynomial [7], simple polynomial [6], or Legendre polynomial [8]. Different methods can be applied to obtain the coefficients in these polynomials. A popular method to determine the zero-th order of polynomial of the pseudo traction is the Kachanov's method [9]. The key assumption in the Kachnov's method is that the pseudo traction is assumed to be the summation of a uniform component (i.e. zero-th order polynomial) and a non-uniform component. In evaluation the interaction between a crack and the rest of cracks in an elastic solid, only the contributions of the uniform components of the pseudo-tractions on these cracks to that particular crack is considered, with the contribution of non-uniform components of the pseudo-tractions neglected.

The accuracy of the Kachanov method deteriorates when the crack tips approach each other very closely, or the cracks are stacked together parallel to the direction of applied load. To overcome the deficiencies of the Kachanov method, Li et al. [10] extended the Kachanov method by decomposing the pseudo traction into a linear component and a non-linear part. In evaluation the interaction between a crack and other cracks in an elastic solid, only the contribution of the uniform component of the pseudo-traction on other cracks to that particular crack is considered, with the non-uniform components of the pseudo-tractions ignored. It is also assumed that the resultant of the linear components be in equilibrium with the corresponding Kachanov traction.

In this study, Kachanov idea of average traction is further extended to the method of moments. The idea of the method of moments is to decompose the pseudo traction into a m-th polynomial and a high order term. In determine the interaction between a crack and other cracks, only the contributions of the m-th order polynomial of the pseudo-traction on those cracks to that particular crack is considered, with the high-order term component of the pseudo-traction ignored. A method of moments is used to determine the coefficients of the m-th order polynomial for each crack in the solid. Therefore, the

Weifeng Wang is a professor with the School of Civil Engineering and Transportation, South China University of Technology, Guangzhou, 510640, China (e-mail: ctwfwang@scut.edu.cn).

Xianwei Zeng is an associate professor with the School of Civil Engineering and Transportation, South China University of Technology, Guangzhou, 510640, China (phone: +86.20.87110804; fax: +86.20.87114460; e-mail: zengxw@scut.edu.cn).

Jianping Ding is with the College of Materials Science and Engineering , South China University of Technology, Guangzhou, 510640, China (e-mail: jpding@scut.edu.cn).

Kachanov method is essentially the zero-th order method of moments. The proposed method of moments is used to calculate the stress intensity factors of some typical multiple crack interaction problems, e.g. two collinear cracks, three collinear cracks, two parallel cracks and three parallel cracks. The numerical results are compared with available analytical results in the literature to study the effect of the order of moment on the accuracy of solution.

II. Method of Moments for Multiple Cracks in an Elastic Solid

Consider the deformation of an infinite isotropic elastic plane containing N cracks subjected to remote loading σ_x^∞ and σ_y^∞ (Fig. 1). This problem is equivalent to the superposition of N+1 subproblems with: (1) a homogeneous infinite plane without crack subjected to remote loading σ_x^∞ and σ_y^∞; (2) an infinite plane containing the i-th (i=1, 2, …, N) crack with zero stress at infinity, but the crack faces are subjected to unknown normal and shear tractions, namely pseudo tractions,

$$p_i(t) = \sigma_{nn}(t) - i\sigma_{nt}(t) \tag{1}$$

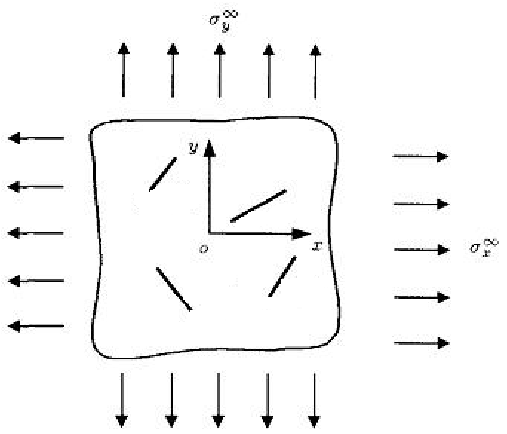

Fig. 1 An elastic solid with multiple cracks subjected to remote loading.

The equivalence of the summation of the N+1 subproblems and the original crack interaction problem requires that the remote loading conditions are satisfied and the traction-free boundary condition on each of the N cracks are also satisfied. This leads to the consistency condition such that the pseudo-traction on the i-th crack can be expressed as,

$$p_i = -n_i \cdot \sigma^\infty + \sum_{j \neq i} \Delta p_{ji} \tag{2}$$

where n_i is unit normal of the *i*-th crack line and Δp_{ji} is the traction on the line of the *i*-th crack due to the action of the pseudo-traction p_j on the *j*-th crack in the corresponding subproblem.

In the present study, the pseudo-traction on each crack is expanded into a polynomial as,

$$p_i(t) = p_{0i} + p_{1i} \cdot t + p_{2i} \cdot t^2 + \ldots \tag{3}$$

where the coefficients in the polynomial have to be determined by the consistence condition (2). The substitution of eqn. (3) into eqn. (2) leads to,

$$p_i(t) = p_0(t) + \sum_{j \neq i} [G_{ji}^0(t) p_{0j} + G_{ji}^1(t) p_{1j} + G_{ji}^2(t) p_{2j} + \ldots] \tag{4}$$

where $p_0(t) = -n_i \cdot \sigma^0$, $G_{ji}^0(t)$, $G_{ji}^1(t)$, and $G_{ji}^2(t)$ are influence functions which can be determined analytically by the complex potential functions in the following section.

2.1 Complex potential functions for a center crack with its faces subjected to tractions

The problem of a center crack located at [-*a*, *a*] on the *x*-axis with its upper and lower surfaces subjected to normal and shear tractions can be solved by Muskhelishvili's complex potential functions φ and ψ [11],

$$\sigma_x + \sigma_y = 2(\Phi + \bar{\Phi}) \tag{5}$$

$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2(\bar{z}\Phi' + \Psi) \tag{6}$$

where $z = x + iy$ and $i = \sqrt{-1}$. Assuming the tractions $p(x) = \sigma_y - i\tau_{xy}$ acting on the upper and lower crack surfaces, the stress potential functions φ and ψ are obtained as [11],

$$\Phi(z) = \frac{1}{2\pi i\sqrt{z^2 - a^2}} \int_{-a}^{+a} \frac{\sqrt{t^2 - a^2}\, p(t)}{t - z} dt \tag{7}$$

$$\Psi(z) = \overline{\Phi(\bar{z})} - \Phi(z) - z\Phi(z) \tag{8}$$

If the tractions on the crack faces are polynomials in the form of (3), the explicit solution of eqn. (7) can be derived by the theorem of residue as (e.g. second order),

$$\begin{aligned} \Phi(z) &= \frac{1}{2\pi i\sqrt{z^2 - a^2}} \int_{-a}^{+a} \frac{\sqrt{t^2 - a^2}\left(p_0 + p_1 \cdot t + p_2 \cdot t^2\right)}{t - z} dt \\ &= \frac{p_0}{2}\left(1 - \frac{z}{\sqrt{z^2 - a^2}}\right) + \frac{p_1}{2}\left(z - \frac{z^2}{\sqrt{z^2 - a^2}} + \frac{a^2}{2\sqrt{z^2 - a^2}}\right) \\ &+ \frac{p_2}{2}\left(z^2 - \frac{z^3}{\sqrt{z^2 - a^2}} + \frac{a^2 \cdot z}{2\sqrt{z^2 - a^2}}\right) \end{aligned} \tag{9}$$

Therefore the analytical solution to the subproblem containing the *i*-th crack is readily available, and the influence functions appearing in eqn. (4) can be determined accordingly.

International Journal of Architectural, Civil and Construction Sciences
ISSN: 2415-1734
Vol:4, No:10, 2010

2.2 Method of moments to impose the consistency condition

To impose the consistence equation (4) exactly, one has to expand the pseudo-traction into a polynomial of infinite orders. However, in the present study, the pseudo-traction is expanded into a polynomial up to the *m*-th order and the method of moments is used to impose the consistence equation (4) approximately, namely by equating the *k*-th (k=0, 1, 2, …, m) moment of the left and the right sides of eqn. (4), when m=2 and a crack extending from $-a$ to $+a$ on its local x-axis,

$$\int_{-a}^{+a} p_i(t)\,dt = \int_{-a}^{+a} \Big(p_0(t) + \sum_{j\neq i} G_{if}^{0}(t)p_{0j} + G_{ij}^{1}(t)\,p_{1j} + G_{ij}^{2}(t)\,p_{2j}\Big)dt \qquad (10)$$

$$\int_{-a}^{+a} t\cdot p_i(t)\,dt = \int_{-a}^{+a} t\cdot\Big(p_0(t) + \sum_{j\neq i} G_{if}^{0}(t)p_{0j} + G_{ij}^{1}(t)\,p_{1j} + G_{ij}^{2}(t)\,p_{2j}\Big)dt \qquad (11)$$

$$\int_{-a}^{+a} t^2\cdot p_i(t)\,dt = \int_{-a}^{+a} t^2\cdot\Big(p_0(t) + \sum_{j\neq i} G_{if}^{0}(t)p_{0j} + G_{ij}^{1}(t)\,p_{1j} + G_{ij}^{2}(t)\,p_{2j}\Big)dt \qquad (12)$$

The right sides of the above equations have to be evaluate by numerical quadrature, while the left rights can be evaluated analytically,

$$\int_{-a}^{+a} p_i(t)dt = p_{0i}\cdot 2a + p_{2i}\cdot\frac{2}{3}a^3 \qquad (13)$$

$$\int_{-a}^{+a} t\cdot p_i(t)dt = p_{1i}\cdot\frac{2}{3}a^3 \qquad (14)$$

$$\int_{-a}^{+a} t^2\cdot p_i(t)dt = p_{0i}\cdot\frac{2}{3}a^3 + p_{2i}\cdot\frac{2}{5}a^5 \qquad (15)$$

This leads to a linear equations system with p_{0i}, p_{1i} and p_{2i} (i=1, 2, …, N) as unknowns. After solving the system of linear simultaneous equations, the pseudo tractions on each crack can be obtained and the mode I and mode II stress intensities on the left and the right tips of the i-th crack are determined by,

$$K = K_{\mathrm{I}}(\pm a) - iK_{\mathrm{II}}(\pm a) = \frac{1}{\sqrt{\pi a}}\int_{-a}^{+a}\sqrt{\frac{a\pm t}{a\mp t}}\,p(t)dt \qquad (16)$$

The above equation can be integrated approximately by the Chebyshev integration as [4],

$$K = K_{\mathrm{I}}(\pm a) - iK_{\mathrm{II}}(\pm a) = \frac{\sqrt{\pi a}}{M}\sum_{m=1}^{M} p\Big(a\cos\frac{(2m-1)\pi}{2M}\Big)*\Big(1\pm\cos\frac{(2m-1)\pi}{2M}\Big) \qquad (17)$$

III. Numerical Results for Two and Three Collinear Cracks

In the ensuing sections, the method of moments is applied to study a variety of multiple crack interaction problems. Fig. 2 shows the interaction of two collinear cracks on the x-axis subjected to remote tension. The coordinates for the points A, B, C and D are (-1, 0), (-k, 0), (k, 0) and (1, 0) respectively.

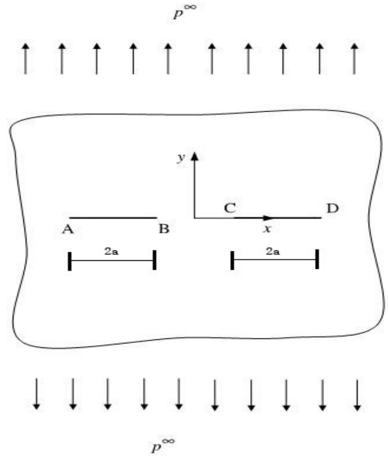

Fig. 2 Two collinear cracks with equal length.

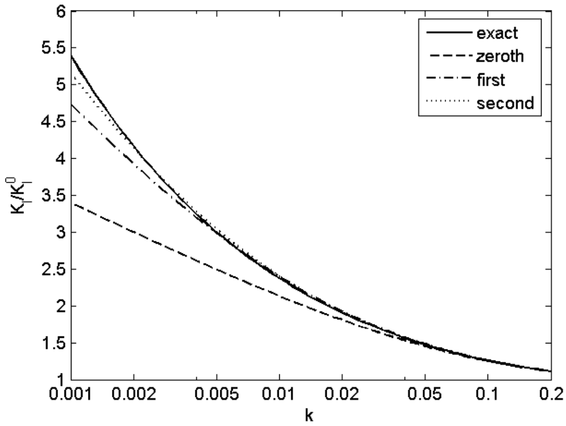

Fig. 3 Variation of stress intensity factors of inner tips with k

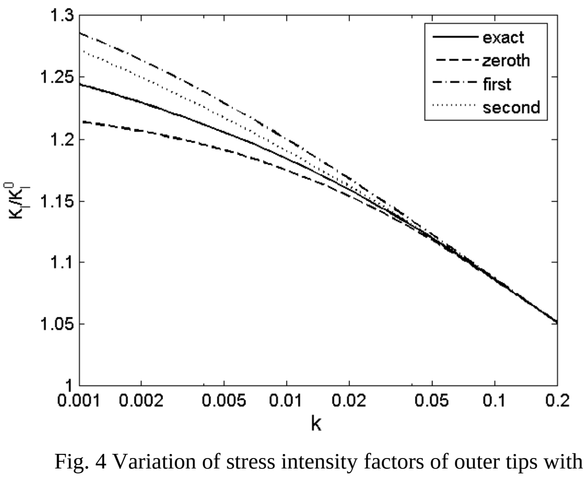

Fig. 4 Variation of stress intensity factors of outer tips with k

Table 1 and Table 2 show the stress intensity factors at the inner and outer crack tips, respectively. The variation of the stress intensity factors with the parameter *k* are shown in Fig. 3 and Fig. 4 graphically. The exact solutions to this problem are obtained from [12]. In the results, K_I^0 denotes the SIF of a center crack subjected to the same loading condition. As the two inner crack tips increasingly approach each other, the SIFs at the inner crack tips increase quickly, however they increases gradually at the two outer crack tips. As the order of the method of moments increases, the accuracy of SIF improves, e.g. when k=0.001, the relative error of SIF at the inner tips is 37.0% by the zero-th order method (i.e. the Kachanov method), 12.2% by the first order method and 4.4% by the second order method, respectively.

TABLE I STRESS INTENSITY FACTORS OF INNER CARCK TIPS

k	K_I / K_I^0			
	Exact	Zero-th order	First order	Second order
0.2	1.1127	1.1120	1.1128	1.1125
0.1	1.2551	1.2509	1.2573	1.2555
0.05	1.4726	1.4524	1.4811	1.4756
0.02	1.9044	1.8084	1.9268	1.9188
0.01	2.3715	2.1335	2.3953	2.4020
0.005	2.2992	2.4933	2.9825	3.0399
0.002	4.1645	3.0020	3.9245	4.1407
0.001	5.3947	3.3996	4.7350	5.1586

TABLE II STRESS INTENSITY FACTORS OF OUTER CRACK TIPS

k	K_I / K_I^0			
	Exact	Zero-th order	First order	Second order
0.2	1.0520	1.0516	1.0518	1.0517
0.1	1.0870	1.0858	1.0872	1.0864
0.05	1.1215	1.1180	1.1228	1.1204
0.02	1.1628	1.1538	1.1677	1.1617
0.01	1.1895	1.1748	1.1996	1.1904
0.005	1.2057	1.1910	1.2292	1.2172
0.002	1.2294	1.2063	1.2639	1.2499
0.001	1.2443	1.2142	1.2860	1.2722

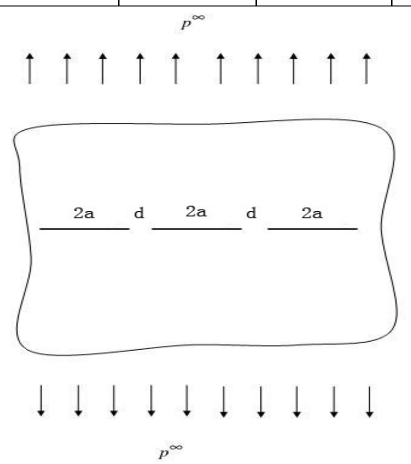

Fig.5 Three collinear cracks with equal length.

Fig. 5 shows the interaction of three collinear cracks. The analytical solution to this problem was presented by Sih GC [13]. Tables 3, 4 and 5 show the SIFs of the center and the side cracks. The numerical results by the second-order method of moments agree well with the analytical solution.

TABLE III STRESS INTENSITY FACTOR (CENTER CRACK)

d/a	K_I / K_I^0			
	Exact	Zero-th order	First order	Second order
1.0	1.1674	1.1669	1.1680	1.1675
0.5	1.3214	1.3179	1.3244	1.3218
0.2	1.6542	1.6308	1.6706	1.6581
0.1	2.0325	1.9601	2.0723	2.0474
0.05	2.5537	2.3656	2.6229	2.5935

TABLE IV STRESS INTENSITY FACTOR (SIDE CRACK, INNER TIP)

d/a	K_I / K_I^0			
	Exact	Zero-th order	First order	Second order
1.0	1.1387	1.1393	1.1394	1.1388
0.5	1.2836	1.2857	1.2870	1.2837
0.2	1.6119	1.6176	1.6291	1.6148
0.1	1.9923	1.9903	2.0305	2.0064
0.05	2.5185	2.4684	2.5767	2.5619

TABLE V STRESS INTENSITY FACTOR (SIDE CRACK, OUTER TIP)

d/a	K_I / K_I^0			
	Exact	Zero-th order	First order	Second order
1.0	1.0687	1.0689	1.0686	1.0686
0.5	1.1103	1.1114	1.1119	1.1104
0.2	1.1714	1.1755	1.1787	1.1722
0.1	1.2167	1.2244	1.2334	1.2196
0.05	1.2587	1.2587	1.2896	1.2662

IV. NUMERICAL RESULTS FOR TWO AND THREE PARALLEL CRACKS

Fig. 6 shows the interaction of two parallel cracks stacked along the direction of applied load. Table 6 and Table 7 show the mode I and mode II SIFs respectively. Remote tension can induce both mode I and mode II stress intensity factors. The interaction between the two parallel cracks is strong. The differences between the numerical solutions corresponding to different orders of moment are distinct.

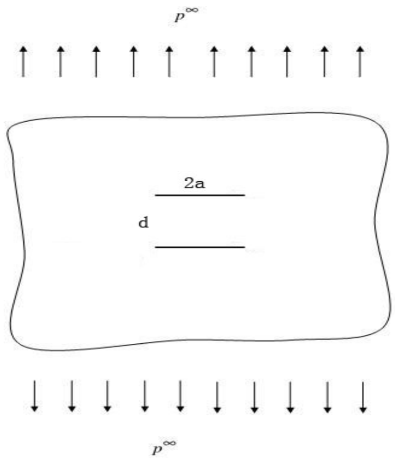

Fig. 6 Two parallel cracks with equal length.

TABLE VI MODE I STRESS INTENSITY FACTOR

d/a	K_I / K_I^0		
	Zero-th order	First order	Second order
1.0	0.7679	0.7671	0.7737
0.5	0.7410	0.7599	0.7537
0.2	0.7351	0.8047	0.7686
0.1	0.7367	0.8421	0.8006
0.05	0.7392	0.8693	0.8351

TABLE VII MODE II STRESS INTENSITY FACTOR

d/a	K_{II} / K_I^0		
	Zero-th order	First order	Second order
1.0	0.1272	0.1512	0.1416
0.5	0.1626	0.2399	0.2170
0.2	0.1726	0.3345	0.3150
0.1	0.1709	0.3803	0.3753
0.05	0.1680	0.4052	0.4133

Fig. 7 shows the interaction of three parallel cracks stacked along the direction of applied load. The remote tension induced only mode I SIFs at the middle crack, where it induces both mode I and mode II SIFs at the outer cracks. In this configuration, the interactions between the three parallel cracks are strong. The differences between the numerical solutions with different orders of moment are clear.

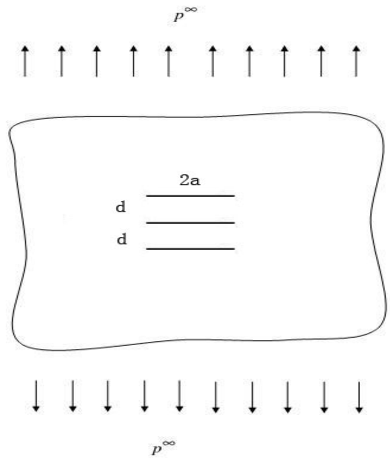

Fig. 7 Three parallel cracks with equal length.

TABLE VIII MODE I STRESS INTENSITY FACTOR (MIDDLE CRACK)

d/a	K_I / K_I^0		
	Zero-th order	First order	Second order
1.0	0.5560	0.5467	0.5333
0.5	0.5251	0.5443	0.4264
0.2	0.5476	0.7010	0.3425
0.1	0.5662	0.8512	0.3198
0.05	0.5799	0.9723	0.3167

TABLE IX MODE I STRESS INTENSITY FACTOR (OUTER CRACKS)

d/a	K_I / K_I^0		
	Zero-th order	First order	Second order
1.0	0.7282	0.7320	0.7549
0.5	0.6841	0.6938	0.7259
0.2	0.6687	0.6979	0.7094
0.1	0.6699	0.7239	0.7141
0.05	0.6739	0.7523	0.7251

TABLE X MODE II STRESS INTENSITY FACTOR (OUTER CRACKS)

d/a	K_{II} / K_I^0		
	Zero-th order	First order	Second order
1.0	0.1288	0.1304	0.1074
0.5	0.1831	0.2088	0.1328
0.2	0.2133	0.3376	0.1679
0.1	0.2147	0.4307	0.1912
0.05	0.2111	0.4979	0.2070

V. Conclusions

The Kachanov method for multiple crack interaction is generalized into the method of moments. The basic idea of the method of moments is to decompose the pseudo traction into a *m*-th order polynomial and a high order term. In determine the interaction between multiple cracks, only the contribution of the *m*-th order polynomial of the pseudo-traction is considered, with the high-order term component of the pseudo-traction neglected. The method of moments is used to obtain a system of linear simultaneous equation involving the coefficients of the *m*-th order polynomial for each crack in a solid. The moments of the pseudo-traction usually have to be evaluated by numerical quadrature. The Kachanov method is identical to the method of moments with zero-th order. The proposed method of moments is used to obtain the stress intensity factors of some typical configurations of multiple crack interaction, e.g. elastic solids with two collinear cracks, three collinear cracks, two parallel cracks and three parallel cracks. Numerical results are compared with available analytical results in the literature to study the effect of the order of moment on the accuracy of the proposed method for analysis of many cracks.

References

[1] M. Kachanov, "Effective elastic properties of cracked solids: critical review of some basic concepts," *Appl. Mech. Review*, vol. 45, pp. 304-335, 1992.
[2] N. Horri, and S. Nemat-Nasser, "Interacting microcracks near the tip in the process zone of a macro-crack", *J. Mech. Phys. Solids*, vol. 35, pp. 601-629, 1987.
[3] J W. Hutchinson, "Crack tip shielding by microcracking in brittle soilds", *Acta. Metall.*, vol. 35, pp. 1606-1619.
[4] Y. Z. Chen, "Integral equation methods for multiple crack problems and related topics," *Appl. Mech. Review*, vol. 60, pp. 172-194, 2007.
[5] Y. Z. Chen, "Complex potentials in plane elasticity by distribution of dislocation or force doublet along a curve," *Int. J. Eng. Sci.*, vol. 36, pp. 23-31, 1998.
[6] H. Horri, and S. Nemat-Nasser, "Elastic fields of interacting imhomogeneities," *Int. J. Solids Struct.*, vol. 21, pp. 731-745, 1985.
[7] D. Gross, "Stress intensity factors of system of cracks," *Ing. Archs*., vol. 51, pp. 301-310, 1982.
[8] Y. Benveniste, G. J. Dvorak, J. Zarzour , and C. J. Wung, "On interacting cracks and complex crack configurations in linear elastic media," *Int. J. Solids Struct*., vol. 25, pp. 1279-1293, 1989.
[9] M. Kachanov, "Elastic soilds with many cracks: a simple method of analysis," *Int. J. Solids and Struct.,* vol.23, pp. 23-43, 1987.
[10] Y. P. Li, L. G. Tham, Y. H. Hwang, and Y. Tsui, "A modified Kachanov method for analysis of solids with multiple cracks," *Eng. Frac. Mech*., vol. 70, pp. 1115-1129, 2003.
[11] N. I. Muskhelishvili, *Some Problems in the Mathematical Theory of Elasticity*. Noordhoff, Groningen, 1953.
[12] C. Hwu, "Collinear cracks in anisotropic bodies," *Int. J. Frac.,* vol. 52, pp. 239-251, 1991.
[13] G. C. Sih, Boundary problems for longitudinal shear cracks. *Proceedings, Second Conference on Theoretical and Applied Mechanics,* New York: Pergamon, 1964.