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Comparison of the Existing Methods in Determination of the Characteristic Polynomial

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***Abstract*—**This paper presents comparison among methods of determination of the characteristic polynomial coefficients. First, the resultant systems from the methods are compared based on frequency criteria such as the closed loop bandwidth, gain and phase margins. Then the step responses of the resultant systems are compared on the basis of the transient behavior criteria including overshoot, rise time, settling time and error (via IAE, ITAE, ISE and ITSE integral indices). Also relative stability of the systems is compared together. Finally the best choices in regards to the above diverse criteria are presented.

***Keywords*—**Characteristic Polynomial, Transient Response, Filters, Stability.

I. Introduction

To define the desired closed loop transient response in terms of the characteristic polynomials is one of the most important issues in the control system design. There are different methods for determination of characteristic polynomial that results this desired transient response. Our aim in this paper is to compare the resulted systems from these methods based on time and frequency parameters. Selected methods consist of:

- ITAE standard, which is an optimization method.
- Binomial method, Kessler method and CDM, which alight on methods that determine characteristic polynomial coefficients directly. We call them "direct methods" in this paper.
- Design methods of Bessel, Butterworth and Chebyshev filters, which are noticed in the recent methods like CRA.

At first, these methods are introduced. Then they are compared on the basis of time and frequency characteristics. They are also compared from relative stability point of view. At the end, the best choices among these methods in regards of different criteria are presented.

II. Introduction to the Existing Methods

In this section we begin with introduction of Binomial polynomial. This polynomial that was formerly used in the control aircraft systems is presented as (1) [1]:

$$p(s) = (s + \omega_0)^n \tag{1}$$

The relation between characteristic polynomial coefficients and transient response has been studied, for the first time in 1953, by Graham and his colleague who developed the Integral Time Absolute Error (ITAE) standard form [2]. In 1960, Kessler did some changes in the defined form to get step responses with less oscillations and overshoots [3]. His proposed form causes 8 percent overshoot. It is also more stable (has less oscillatory response) in comparison with IATE. One of the most important methods in the category is the Coefficients Diagram Method (CDM) which has been introduced by Manabe [4].

To introduce the two recent methods, we use Naslin's definitions [5]. Suppose that $p(s)$ is a Hurwitz polynomial with positive real coefficients.

$$p(s) = a_n s^n + a_{n-1} s^{n-1} + ... + a_1 s + a_0 \tag{2}$$

Characteristic ratios are defined as ($1 \le i < n$):

$$\alpha_i = \frac{a_i^2}{a_{i-1} a_{i+1}} \tag{3}$$

Also generalized time constant is:

$$\tau = a_1 / a_0 \tag{4}$$

Having a_0, α_i s and τ, the coefficients of the polynomial are uniquely determined as:

$$a_1 = \tau a_0, \quad a_i = \frac{\tau^i a_0}{\alpha_{i-1} \alpha_{i-2}^2 \alpha_{i-3}^3 ... \alpha_1^{i-1}} \tag{5}$$

In Kessler method and CDM, Characteristics ratios are selected as Table 1.

TABLE I
CHARACTERISTICS RATIOS OF KESSLER METHOD AND CDM

Method	$\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_{n-1}$
CDM	2.5, 2, 2, $\cdots$, 2
Kessler	2, 2, 2, $\cdots$, 2

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CRA is a new method that uses characteristic ratios of Butterworth filter with some changes [6]. In addition to Butterworth filter, we select Bessel filter design method and Chebyshev filter design method (with 0.5 dB ripple) for comparison. Among low pass filters, Butterworth filter has been known as a maximally-flat and Bessel filter has been known as maximally-flat time delay [7].

In all of the stated methods, we assume generalized constant time equals to 1 ($\tau = 1$). With this assumption, we find resulted systems from the methods and compare them in regards of time and frequency criteria.

III. Frequency Comparison

In this section we compare existing methods in determination of characteristic polynomials in regards of frequency methods. For this purpose, we select three parameters bandwidth, gain and phase margin from frequency parameters.

A. Bandwidth

In Fig. 1, the bandwidths of resulted systems from the methods are depicted for three different degrees $(n = 2, 5, 7)$.

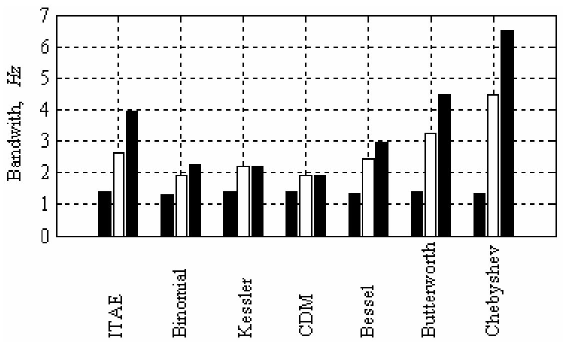

Fig. 1 Bandwidths of different methods for $n = 2, 5, 7$ (Columns are ascending to degrees)

In minimum degree$(n = 2)$, there isn't discernible difference but with increasing the degree, differences are observed. Chebyshev filter has maximum bandwidth and Butterworth filter, ITAE method and Bessel filter stand in the next positions respectively. From this aspect, filter design methods have noticeable priority over other methods. Among methods that determine characteristic polynomial coefficients directly, in low degrees the Kessler method and in high degrees the Binomial method has upper bandwidths. In these methods, bandwidth converges to specific amount with increasing the degree.

B. Gain and Phase Margin

To compare the gain and phase margins, we find them for the open loop system that converts to the resulted closed loop system from different methods with unit feedback. Fig. 2 shows gain margin of these methods for three different degrees $(n = 4, 7, 8)$. In the low degrees, Binomial method has maximum gain margin and in the high degrees CDM exposes its predominance.

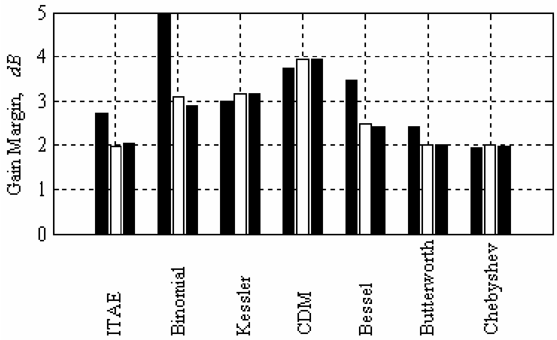

Fig. 2 Gain margins of different methods for $n = 4, 7, 8$ (Columns are ascending to degrees)

Fig. 3 shows phase margin of the methods for three different degrees ($n = 3, 6, 8$). Like in the gain margin, Binomial method is the first for low degrees and for high degrees CDM is above all. In filters, Bessel has the biggest phase margin. Also Butterworth filter has minimum phase margin. The phase margin converges to specific amount by increasing the degree.

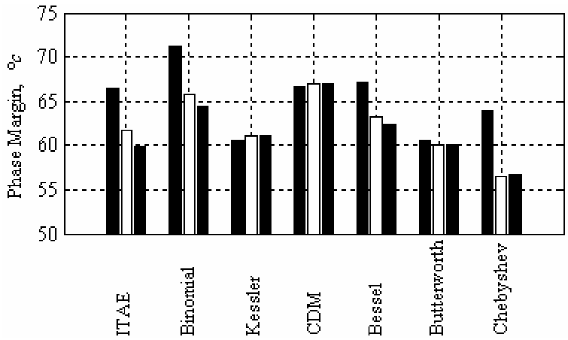

Fig. 3 Phase margins of different methods for $n = 3, 6, 8$ (Columns are ascending to degrees)

IV. Time Comparison

In this section we consider and compare step response of the resulted systems from these methods. For this purpose, we compare these responses due to overshoot at first. Then speeds of systems are compared. For speed comparison, we use two parameters, rise time and settling time. At the end, we consider error due to IAE, ITAE, ISE and ITSE criteria.

TABLE II
OVERSHOOTS OF RESPONSES FOR FOUR DIFFERENT DEGREES

n	3	4	7	8
ITAE	1.02%	1.03%	14.68%	12.83%
Binomial	0%	0%	0%	0%
Kessler	8.15%	6.24%	5.54%	5.54%
CDM	0.96%	0.02%	0%	0%
Bessel	0.75%	0.84%	0.49%	0.34%
Butterworth	8.15%	10.83%	15.41%	16.34%
Chebyshev	8.93%	18.10%	15.27%	23.00%

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A. Overshoot

Table 2 shows overshoots of the responses for four degrees. It is considerable that Binomial method has no overshoot. Also CDM has no overshoot for $n \geq 5$. Chebyshev and Butterworth filters have most overshoot among existing methods. Hence based on this criterion the direct methods have considerable priority over other methods.

B. Rise Time

In Fig. 4, 10% to 90% rise times of the resulted systems from the methods are depicted for three different degrees $(n = 3, 6, 8)$. The figure shows that Chebyshev filter has minimum rise time and Butterworth filter and ITAE standard alight after it. Direct methods indicate slow responses according to the rising time. In all the existing methods, the rise time decreases by increasing the degree. However, in CDM and Kessler methods, the increase is not noticeable.

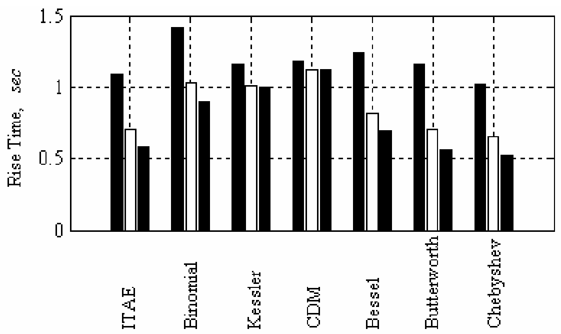

Fig. 4 Rise times of different methods for $n = 3, 6, 8$ (Columns are ascending to degrees)

C. Settling Time

Fig. 5 shows 2% settling times obtained by the existing methods for three different degrees $(n = 3, 6, 8)$. There is a remarkable point here. Chebyshev and Butterworth filter responses that are fastest responses according to rise time, have the greatest settling times. ITAE standard has erratic response; it has comparably fast response for some degrees and comparably slow response for other degrees. Among the existing methods, Bessel filter has minimum settling time and CDM and Binomial method alight after it.

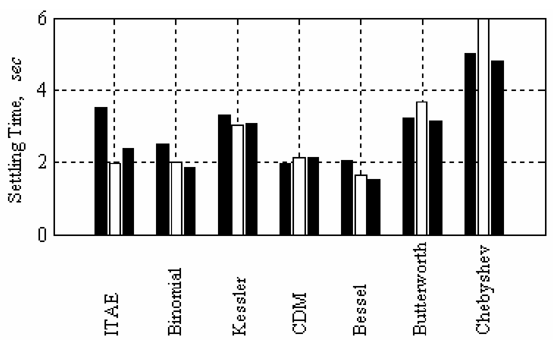

Fig. 5 Settling times of different methods for $n = 3, 6, 8$ (Columns are ascending to degrees)

D. Error

In this section we compare responses of the methods according to IAE, ITAE, ISE and ITSE criteria.

In Fig. 6, errors of the responses according to IAE criterion are depicted for three different degrees $(n = 3, 6, 8)$. For low degree, Binomial has minimum error. After it, CDM and Bessel filter alight. In greater degrees, Bessel filter has lowest error and CDM and Binomial method alight after it. Also the maximum error belongs to Chebyshev filter and after it, belongs to Butterworth filter.

Fig. 7 shows errors according to ITAE criterion for three different degrees $(n = 3, 6, 8)$. CDM has the minimum error for the low degrees and for the greater degrees Bessel filter has the minimum error. Same as in the previous section, Chebyshev filter has the maximum error.

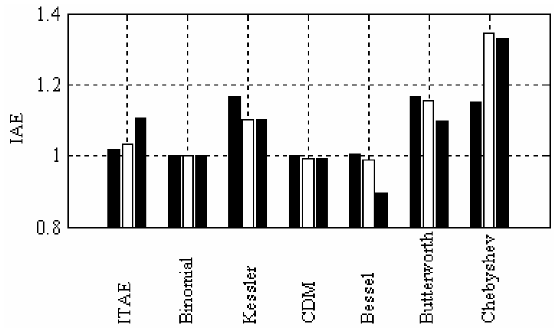

Fig. 6 Error (IAE criterion) of different methods for $n = 3, 6, 8$ (Columns are ascending to degrees)

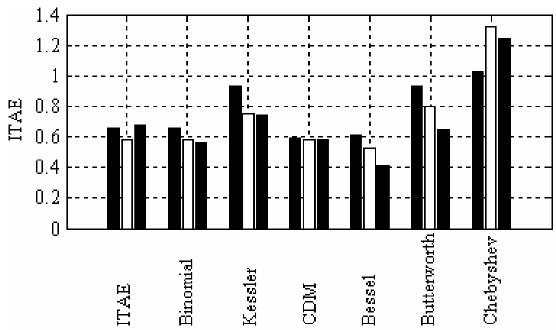

Fig. 7 Error (ITAE criterion) of different methods for $n = 3, 6, 8$ (Columns are ascending to degrees)

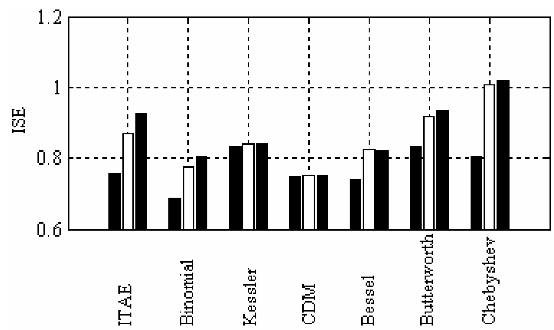

Fig. 8 Error (ISE criterion) of different methods for $n = 3, 6, 8$ (Columns are ascending to degrees)

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In Fig. 8, the ISE errors of the responses are depicted for three different degrees $(n = 3, 6, 8)$. For the lower degrees, Binomial has the minimum error. But in the higher degrees CDM is the best choice regarding to ISE criterion.

Fig. 9 shows the errors according to ITSE criterion for three different degrees $(n = 3, 6, 8)$. Binomial method has the minimum error for the lower degrees and for the higher degrees CDM has the minimum error. After CDM, Bessel filter and Binomial method alight. Also the maximum errors belong to Chebyshev and Butterworth filters.

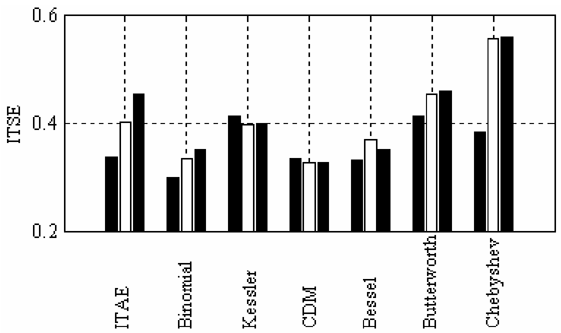

Fig. 9 Error (ITSE criterion) of different methods for $n = 3, 6, 8$ (Columns are ascending to degrees)

V. Relative Stability

In this section our aim is to compare the exiting methods of determining the characteristic polynomial with respect to the relative stability. We know that if the curvature of the a_i becomes larger, the system becomes more stable [8].

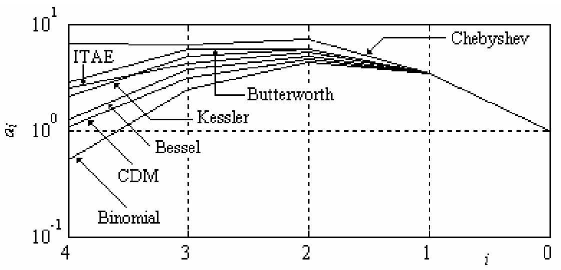

Fig. 10 a_i graph of different methods for $n = 4$

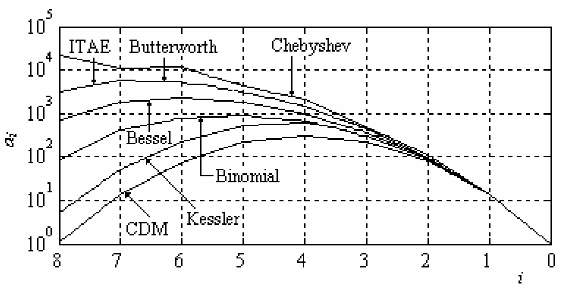

Fig. 11 a_i graph of different methods for $n = 8$

Fig. 10 shows a_i graph of the resulted systems from the existing methods for $n = 4$ and $\tau = 3.4$. The similar graphs are presented in Fig. 11 for $n = 8$ and $\tau = 14$. When $n = 4$, Binomial is more stable than other methods. CDM and Bessel filter alight after this method. Also Chebyshev filter has the minimum stability and after it, the next minimum stability belongs to Butterworth filter and ITAE standard. For $n = 8$, CDM is more stable than other methods. After CDM, Kessler method alights. Chebyshev filter has the minimum stability again and after it, ITAE and Butterworth filter alight.

VI. Conclusion

Table 3 indicates best choices among the exiting methods based on different criteria. As we see, none of the existing methods transcends other methods regarding to all criteria. Therefore, a designer by prioritizing the favorites that he seeks, may select a method that most congruent with his will.

In this paper, we have considered criteria that are important from the control design point of view. To compare the methods based on other criteria such as robustness and sensitivity, we need to know controller structure or other information. This matter requires long debate that would be assessed in the next papers.

Table III
Best Choices According to Different Parameters

	Low degrees $n = 2, 3, 4$	High degrees $n \geq 5$
Bandwidth	Chebyshev	Chebyshev
Gain margin	Binomial	CDM
Phase margin	Binomial	CDM
Overshoot	Binomial	CDM & Binomial
Rise time	Chebyshev	Chebyshev
Settling time	Bessel	Bessel
IAE	Binomial	CDM
ITAE	CDM	Bessel
ISE	Binomial	CDM
ITSE	Binomial	CDM
Stability	Binomial	CDM

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