International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:6, No:11, 2012

Rate of convergence for generalized Baskakov-Durrmeyer Operators

Durvesh Kumar Verma and P. N. Agrawal

Abstract—In the present paper, we consider the generalized form of Baskakov Durrmeyer operators to study the rate of convergence, in simultaneous approximation for functions having derivatives of bounded variation.

Keywords—Bounded variation, Baskakov-Durrmeyer operators, Simultaneous approximation, Rate of convergence.

I. Introduction

In the year 2005, Finta [2] considered a new type of Baskakov-Durrmeyer operator by taking the weight functions of Baskakov basis functions and established sufficient conditions for obtaining strong converse inequality. After that Govil and Gupta [3] studied some approximation properties for these operators and estimated local results in terms of modulus of continuity. Also, further properties like pointwise convergence, asymptotic formula and inverse result in simultaneous approximation have been established in [5]. Very recently, Verma et al. [7] Stancu type generalization of the operators $D_n(f,x)$ and studied the direct error estimates and Voronovskaja type asymptotic formula. To approximate the Lebesgue integrable functions on the interval $[0,\infty)$, we introduce Baskakov-Durrmeyer operators in generalized form as:

$$V_{n,r}(f,x)=\frac{(n+r-1)!(n-r)!}{n!(n-1)!}\sum_{k=0}^{\infty}p_{n+r,k}(x)\times\int_0^\infty b_{n-r,k+r}(t)f(t)\,dt$$

where

$$p_{n,k}(x)=\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}},$$
$$b_{n,k}(x)=\frac{1}{B(k,n+1)}\frac{x^{k-1}}{(1+x)^{n+k+1}}.$$

We denote the class of absolutely continuous functions f on $(0,\infty)$ is defined by $DB_q(0,\infty)$, (where q is some positive integer) and satisfying:

(i) $|f(t)|\le C_1t^q,\quad C_1>0,$

(ii) the function f has the first derivative on interval $(0,\infty)$ which coincide a.e. with a function which is of bounded variation on every finite subinterval of $(0,\infty)$. It can be observed that for all $f\in DB_q(0,\infty)$, we can have the representation

$$f(x)=f(c)+\int_c^x\psi(t)dt,\quad x\ge c>0.$$

In the recent years, the rate of convergence for the functions having the derivatives of bounded variation is an interesting area of research, several researchers have studied in this direction we refer some of important paper in this area as [6, 8-10]. Also, Bai et al. [1] worked in this direction and estimated the rate of convergence for the several operators. Gupta [4] estimated the rate of convergence for functions of bounded variation on certain Baskakov-Durrmeyer type operators.

In the present paper we study the rate of convergence for the operators $V_{n,r}$ for functions having the derivatives of bounded variation, we also mention a corollary which provide the result in simultaneous approximation.

II. Auxiliary Results

In the sequel we shall need the following lemmas.

Lemma 1: If we define the central moments as

$$\mu_{n,r,m}(x)=\sum_{k=0}^{\infty}p_{n+r,k}(x)\int_0^\infty b_{n-r,k+r}(t)\,(t-x)^m\,dt$$

Then, $\mu_{n,r,0}(x)=1$, $\mu_{n,r,1}(x)=\frac{r(2x+1)}{n-r}$ and for $n>m$ we have the following recurrence relation:

$$(n-r-m)\mu_{n,r,m+1}(x)=x(1+x)[\mu'_{n,r,m}(x)+2m\mu_{n,r,m-1}(x)]+(m+r)(1+2x).$$

Proof: Taking derivative of above

$$\begin{aligned}\mu'_{nr,,m}(x)&=-m\sum_{k=0}^{\infty}p_{n+r,k}(x)\\&\times\int_0^\infty b_{n-r,k+r}(t)(t-x)^{m-1}dt\\&+\sum_{k=0}^{\infty}p_{n+r,k}(x)\int_0^\infty b_{n-r,k+r}(t)(t-x)^m dt\\&=-m\mu_{n,r,m-1}(x)+\sum_{k=0}^{\infty}p_{n+r,k}(x)\\&\times\int_0^\infty b_{n-r,k+r}(t)(t-x)^m dt\end{aligned}$$

$$x(1+x)[\mu'_{n,r,m}(x)+m\mu_{n,r,m-1}(x)]=\sum_{k=0}^{\infty}x(1+x)p'_{n+r,k}(x)\int_0^\infty b_{n-r,k+r}(t)(t-x)^m dt$$

Durvesh Kumar Verma is a research fellow at the Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee 247667 INDIA, E-mail: durvesh.kv.du@gmail.com.

P. N. Agrawal is a Professor at Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee 247667 INDIA, E-mail: pna_iitr@yahoo.co.in

International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:6, No:11, 2012

using $x(1+x)p'_{n+r,k}(x) = [k-(n+r)x]p_{n+r,k}(x)$, we get

$$\begin{aligned}
&x(1+x)[\mu'_{n,r,m}(x) + m\mu_{n,r,m-1}(x)]\\
&= \sum_{k=0}^{\infty}[k-(n+r)x]p_{n+r,k}(x)\\
&\times \int_0^{\infty} b_{n-r,k+r}(t)(t-x)^m dt\\
&= \sum_{k=0}^{\infty} kp_{n+r,k}(x)\int_0^{\infty} b_{n-r,k+r}(t)(t-x)^m dt\\
&-(n+r)x\mu_{n,m}(x)\\
&= I-(n+r)x\mu_{n,r,m}(x). \qquad (1)
\end{aligned}$$

We can write I as

$$\begin{aligned}
&I = \sum_{k=0}^{\infty} p_{n+r,k}(x)\int_0^{\infty}[(k+r-1)-(n-r+2)t)]\\
&\times b_{n-r,k+r}(t)(t-x)^m dt\\
&+(n-r+2)\sum_{k=0}^{\infty} p_{n+r,k}(x)\int_0^{\infty} b_{n-r,k+r}(t)t\,(t-x)^m dt\\
&-(r-1)\sum_{k=0}^{\infty} p_{n+r,k}(x)\int_0^{\infty} b_{n-r,k+r}(t)\,(t-x)^m dt\\
&= I_1+I_2-(r-1)\mu_{n,r,m}(x), \quad \text{(say)}. \qquad (2)
\end{aligned}$$

First we estimate I_2 as follows

$$\begin{aligned}
I_2 &= (n-r+2)\sum_{k=0}^{\infty} p_{n+r,k}(x)\int_0^{\infty} b_{n-r,k+r}(t)\\
&\times \big[(t-x)^{m+1} + x(t-x)^m\big]dt\\
&= (n-r+2)\big[\mu_{n,r,m+1}(x) + x\mu_{n,r,m}(x)\big]. \qquad (3)
\end{aligned}$$

Next, to estimate I_1, by using the equality $[(k+r-1)-(n-r+2)t)]b_{n-r,k+r}(t) = t(1+t)b'_{n-r,k+r}(t)$, we have

$$\begin{aligned}
I_1 &= \left[\sum_{k=0}^{\infty} p_{n+r,k}(x)\int_0^{\infty} b'_{n-r,k+r}(t)\,t\,(t-x)^m dt\right]\\
&+ \left[\sum_{k=0}^{\infty} p_{n+r,k}(x)\int_0^{\infty} b'_{n-r,k+r}(t)\,t^2\,(t-x)^m dt\right]\\
&= J_1+J_2, \quad \text{(say)}. \qquad (4)
\end{aligned}$$

We can write J_1 as

$$\begin{aligned}
J_1 &= \sum_{k=0}^{\infty} p_{n+r,k}(x)\int_0^{\infty} b'_{n-r,k+r}(t)\\
&\times [(t-x)^{m+1} + x(t-x)^m]dt.
\end{aligned}$$

Now, applying integration by parts, we have

$$\begin{aligned}
J_1 &= -(m+1)\sum_{k=0}^{\infty} p_{n+r,k}(x)\int_0^{\infty} b_{n-r,k+r}(t)\,(t-x)^m dt\\
&- mx\sum_{k=0}^{\infty} p_{n+r,k}(x)\int_0^{\infty} b_{n-r,k+r}(t)\,(t-x)^{m-1} dt\\
&= -(m+1)\mu_{n,r,m}(x) - mx\mu_{n,r,m-1}(x). \qquad (5)
\end{aligned}$$

Proceeding in a similar manner, we obtain the estimate J_2 as

$$\begin{aligned}
J_2 &= -(m+2)\mu_{n,r,m+1}(x)\\
&- 2x(m+1)\mu_{n,r,m}(x) - mx^2\mu_{n,r,m-1}(x). \qquad (6)
\end{aligned}$$

Combining (1)-(6), we get the desired result.

Remark 1: Let $x \in (0,\infty)$ and $C > 2$, then for n sufficiently large, Lemma 1, yields that

$$\mu_{n,r,2}(x,c) \le \frac{Cx(1+x)}{(n-r-1)}$$

Lemma 2: Let $x \in (0,\infty)$ and $C > 2$, then for n sufficiently large, we have

$$\begin{aligned}
\lambda_{n,r}(x,y) &= \sum_{k=0}^{\infty} p_{n+r,k}(x)\int_0^{y} b_{n-r,k+r}(t)dt\\
&\le \frac{Cx(1+x)}{(n-r-1)(x-y)^2}, \quad 0 \le y < x,\\
1-\lambda_{n,r}(x,z) &= \sum_{k=0}^{\infty} p_{n+r,k}(x)\int_z^{\infty} b_{n-r,k+r}(t)dt\\
&\le \frac{Cx(1+x)}{(n-r-1)(z-x)^2}, \quad x < z < \infty.
\end{aligned}$$

Proof: The proof of the lemma follows easily by Remark 1. For instance, for the first inequality for n sufficiently large and $0 \le y < x$, we have

$$\begin{aligned}
\lambda_{n,r}(x,y) &= \sum_{k=0}^{\infty} p_{n+r,k}(x)\int_0^{y} b_{n-r,k+r}(t)dt\\
&\le \sum_{k=0}^{\infty} p_{n+r,k}(x)\int_0^{y} b_{n-r,k+r}(t)\frac{(t-x)^2}{(y-x)^2}dt\\
&= \frac{\mu_{n,r,2}(x)}{(y-x)^2} \le \frac{Cx(1+x)}{(n-r-1)(y-x)^2}.
\end{aligned}$$

The proof of the second inequality follows along the similar lines.

Lemma 3: Let f be s times differentiable on $[0,\infty)$ such that $f^{(r-1)}(t) = O(t^q)$ as $t \to \infty$ where q is a positive integer. Then for any $r, s \in N^0$ and $n > \max\{q, r+s+1\}$, we have

$$D^s V_{n,r}(f,x) = V_{n,r+s}(D^s f,x), \quad D = \frac{d}{dx}.$$

Proof: We prove the result by applying the principle of mathematical induction and using the following identity

$$Dp_{n,k}(x) = n[p_{n+1,k-1}(x) - p_{n+1,k}(x)] \text{ and } \qquad (7)$$

$$Db_{n,k}(t) = (n+1)[b_{n+1,k-1}(t) - b_{n+1,k}(t)]. \qquad (8)$$

The above identity is true even for the case $k = 0$, as we observe that $p_{n+1,negative} = 0$. Using (7), (8) and integrating by parts, we have

$$\begin{aligned} DV_{n,r}(f,x) &= \frac{(n+r-1)!(n-r)!}{n!(n-1)!} \\ &\times \sum_{k=0}^{\infty} Dp_{n+r,k}(x)\int_0^\infty b_{n-r,k+r}(t)f(t)\,dt \\ &= \frac{(n+r)!(n-r)!}{n!(n-1)!}\sum_{k=0}^{\infty}[p_{n+(r+1),k-1}(x) \\ &\quad -p_{n+(r+1),k}(x)]\int_0^\infty b_{n-r,k+r}(t)f(t)\,dt \\ &= \frac{(n+r)!(n-r)!}{n!(n-1)!}\sum_{k=0}^{\infty} p_{n+(r+1),k}(x) \\ &\times \int_0^\infty [b_{n-r,k+r+1}(t)-b_{n-r,k+r}(t)]f(t)\,dt \\ &= -\frac{(n+r)!(n-r-1)!}{n!(n-1)!}\sum_{k=0}^{\infty} p_{n+(r+1),k}(x) \\ &\times \int_0^\infty Db_{n-r-1,k+r+1}(t)f(t)\,dt. \end{aligned}$$

Integrating by parts the last integral, we have

$$\begin{aligned} DV_{n,r}(f,x) &= \frac{(n+r)!(n-r-1)!}{n!(n-1)!}\sum_{k=0}^{\infty} p_{n+(r+1),k}(x) \\ &\times \int_0^\infty b_{n-r-1,k+r+1}(t)Df(t)\,dt \\ &= V_{n,r+1}(Df,x), \end{aligned}$$

which shows that the result holds for $s=1$. Let us suppose that the result holds for $s=m$ i.e.

$$\begin{aligned} & D^m V_{n,r}(f,x) = V_{n,r+m}(D^m f,x) \\ &= \frac{(n+r+m-1)!(n-r-m)!}{n!(n-1)!} \\ &\times \sum_{k=0}^{\infty} p_{n+r+m,k}(x)\int_0^\infty b_{n-r-m,k+r+m}(t,c)D^m f(t)\,dt \end{aligned}$$

Now,

$$\begin{aligned} & D^{m+1}V_{n,r}(f,x) = \frac{(n+r+m-1)!(n-r-m)!}{n!(n-1)!} \\ &\times \sum_{k=0}^{\infty} Dp_{n+r+m,k}(x)\int_0^\infty b_{n-r-m,k+r+m}(t)D^m f(t)\,dt \\ &= \frac{(n+r+m)!(n-r-m)!}{n!(n-1)!}\sum_{k=0}^{\infty}[p_{n+r+m+1,k-1}(x) \\ &\times - p_{n+r+m+1,k}(x)]\int_0^\infty b_{n-r-m,k+r+m}(t)D^m f(t)\,dt \\ &= \frac{(n+r+m)!(n-r-m)!}{n!(n-1)!}\sum_{k=0}^{\infty} p_{n+r+m+1,k}(x) \\ &\times \int_0^\infty [b_{n-r-m,k+r+m+1}(t) \\ &-b_{n-r-m,k+r+m}(t)]D^m f(t)\,dt \\ &= -\frac{(n+r+m)!(n-r-m-1)!}{n!(n-1)!}\sum_{k=0}^{\infty} p_{n+r+m+1,k}(x) \\ &\times \int_0^\infty Db_{n-r-m-1,k+r+m+1}(t)D^m f(t)\,dt \end{aligned}$$

Again integrating by parts the last integral, we have

$$\begin{aligned} & D^{m+1}V_{n,r}(f,x) = \frac{(n+r+m)!(n-r-m-1)!}{n!(n-1)!} \\ &\times \sum_{k=0}^{\infty} p_{n+r+m+1,k}(x) \\ &\times \int_0^\infty b_{n-r-m-1,k+r+m+1}(t)D^{m+1}f(t)\,dt \end{aligned}$$

Therefore,

$$D^{m+1}V_{n,r}(f,x) = V_{n,r+m+1}(D^{m+1}f,x).$$

Thus, the result is true for $s=m+1$, hence by mathematical induction, proof of the lemma is completed.

III. Rate of Convergence

In this section we prove our main results.

Theorem 1: Let $f \in DB_q(0,\infty)$, $q>0$ and $x \in (0,\infty)$. The for $C>2$ and n sufficiently large, we have

$$\begin{aligned} & \left|\frac{(n+r-1)!(n-r)!}{n!(n-1)!}V_{n,r}(f;x)-f(x)\right| \\ &\le \frac{C(1+cx)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-x/k}^{x+x/k}((f')_x)+\frac{x}{\sqrt{n}}\bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}((f')_x) \\ &+\frac{C(1+cx)}{n}(|f(2x)-f(x)-xf'(x^+)|+|f(x)|)+O(n^{-q}) \\ &+\frac{C(1+x)}{n-r-1}|f'(x^+)|+\frac{|f'(x^+)-f'(x^-)|}{2}\sqrt{\frac{Cx(1+x)}{n}} \\ &+\frac{|f'(x^+)+f'(x^-)|}{2}\frac{r(1+2x)}{n-r}, \end{aligned}$$

where $\bigvee_a^b f(x)$ denotes the total variation of f_x on $[a,b]$, the auxiliary function f_x is defined by

$$f_x(t)=\begin{cases} f(t)-f(x^-), & 0\le t<x, \\ 0, & t=x, \\ f(t)-f(x^+), & x<t<\infty. \end{cases}$$

Proof: Using the mean value theorem, we have

$$\begin{aligned} & \left|\frac{(n+r-1)!(n-r)!}{n!(n-1)!}V_{n,r}(f;x)-f(x)\right| \\ &\le \sum_{k=0}^{\infty} p_{n+r,k}(x)\int_0^\infty b_{n-r,k+r}(t)|f(t)-f(x)|\,dt \\ &\le \int_0^\infty \left|\int_x^t \sum_{k=0}^{\infty} p_{n+r,k}(x)b_{n-r,k+r}(t)f'(u)du\right|\,dt \end{aligned}$$

Also, using the identity

$$\begin{aligned} f'(u) &= \frac{f'(x^+)+f'(x^-)}{2} + (f')_x(u) \\ &\quad + \frac{f'(x^+)-f'(x^-)}{2} sgn(u-x) \\ &\quad + \left[f'(x) - \frac{f'(x^+)+f'(x^-)}{2}\right]\chi_x(u), \end{aligned}$$

where

$$\chi_x(u) = \begin{cases} 1, & u = x, \\ 0, & u \neq x. \end{cases}$$

Obviously, we have

$$\begin{aligned} &\sum_{k=0}^{\infty} p_{n+r,k}(x) \int_0^{\infty} \left(\int_x^t \left[f'(x) \right.\right. \\ &\left.\left. -\frac{f'(x^+)+f'(x^-)}{2} \right] \chi_x(u)du \right) p_{n-r,k+r}(t)dt = 0. \end{aligned}$$

Thus, using the above identities, we can write

$$\begin{aligned} &\left| \frac{(n+r-1)!(n-r)!}{n!(n-1)!} V_{n,r}(f;x) - f(x) \right| \\ &\leq \left| \int_0^{\infty} \left(\int_x^t \sum_{k=0}^{\infty} p_{n+r,k}(x) b_{n-r,k+r}(t) \right.\right. \\ &\quad \times \left.\left. \left(\frac{f'(x^+)+f'(x^-)}{2} + (f')_x(u) \right) du \right) \right| dt \\ &\quad + \left| \int_0^{\infty} \left(\int_x^t \sum_{k=0}^{\infty} p_{n+r,k}(x) p_{n-r,k+r}(t) \right.\right. \\ &\quad \times \left.\left. \frac{[f'(x^+)-f'(x^-)]}{2} sgn(u-x) du \right) dt \right|. \end{aligned} \tag{9}$$

Also, it can be verified that

$$\begin{aligned} &\left| \int_0^{\infty} \left(\int_x^t \frac{[f'(x^+)-f'(x^-)]}{2} sgn(u-x) du \right) \right. \\ &\quad \times \left. \sum_{k=0}^{\infty} p_{n+r,k}(x) p_{n-r,k+r}(t) dt \right| \\ &\leq \frac{|f'(x^+)-f'(x^-)|}{2} [\mu_{n,r,2}(x)]^{1/2} \end{aligned} \tag{10}$$

and

$$\begin{aligned} &\int_0^{\infty} \left(\int_x^t \frac{[f'(x^+)+f'(x^-)]}{2} du \right) \\ &\sum_{k=0}^{\infty} p_{n+r,k}(x) p_{n-r,k+r}(t) dt \\ &\leq \frac{|f'(x^+)+f'(x^-)|}{2} \mu_{n,r,1}(x). \end{aligned} \tag{11}$$

Combining (9)-(11), we get

$$\begin{aligned} &\left| \frac{(n+r-1)!(n-r)!}{n!(n-1)!} V_{n,r}(f;x) - f(x) \right| \\ &\leq \left| \int_x^{\infty} \left(\int_x^t (f')_x(u) du \right) \sum_{k=0}^{\infty} p_{n+r,k}(x) p_{n-r,k+r}(t) dt \right. \\ &\quad + \left. \int_0^{x} \left(\int_x^t (f')_x(u) du \right) \sum_{k=0}^{\infty} p_{n+r,k}(x) p_{n-r,k+r}(t) dt \right| \\ &\quad + \frac{|f'(x^+)-f'(x^-)|}{2} [\mu_{n,r,2}(x)]^{1/2} + \frac{|f'(x^+)+f'(x^-)|}{2} \mu_{n,r,1}(x) \\ &= |A_{n,r}(f,x) + B_{n,r}(f,x)| + \frac{|f'(x^+)-f'(x^-)|}{2} [\mu_{n,r,2}(x)]^{1/2} \\ &\quad + \frac{|f'(x^+)+f'(x^-)|}{2} \mu_{n,r,1}(x). \end{aligned} \tag{12}$$

Applying Remark 1 and Lemma 1 in (12), we have

$$\begin{aligned} &\left| \frac{(n+r-1)!(n-r)!}{n!(n-1)!} V_{n,r}(f;x) - f(x) \right| \\ &\quad \leq |A_{n,r}(f,x)| + |B_{n,r}(f,x)| + \frac{|f'(x^+)-f'(x^-)|}{2} \\ &\quad \times \sqrt{\frac{Cx(1+x)}{n-r-1}} + \frac{|f'(x^+)+f'(x^-)|}{2} \frac{r(1+2x)}{n-r}. \end{aligned} \tag{13}$$

In order to complete the proof of the theorem it suffices to estimate the terms $A_{n,r}(f,x)$ and $B_{n,r}(f,x)$.

$$\begin{aligned} &|A_{n,r}(f,x)| \\ &= \left| \int_x^{\infty} \left(\int_x^t (f')_x(u) du \right) \sum_{k=0}^{\infty} p_{n+r,k}(x) p_{n-r,k+r}(t) dt \right| \\ &= \left| \int_{2x}^{\infty} \left(\int_x^t (f')_x(u) du \right) \sum_{k=0}^{\infty} p_{n+r,k}(x) p_{n-r,k+r}(t) dt \right. \\ &\quad + \left. \int_x^{2x} \left(\int_x^t (f')_x(u) du \right) \sum_{k=0}^{\infty} p_{n+r,k}(x) p_{n-r,k+r}(t,c) dt \right| \\ &\leq \left| \sum_{k=0}^{\infty} p_{n+r,k}(x) \int_{2x}^{\infty} (f(t)-f(x)) p_{n-r,k+r}(t) dt \right| \\ &\quad + |f'(x^+)| \left| \sum_{k=0}^{\infty} p_{n+r,k}(x) \int_x^{2x} p_{n-r,k+r}(t)(t-x) dt \right| \\ &\quad + \left| \int_x^{2x} (f')_x(u) du \right| \left| 1 - \lambda_{n,r}(x,2x) \right| \\ &\quad + \left| \int_x^{2x} (f')_x(u) du \right| \left| 1 - \lambda_{n,r}(x,t) \right| \\ &\leq \sum_{k=0}^{\infty} p_{n+r,k}(x) \int_{2x}^{\infty} p_{n-r,k+r}(t) C_1 t^{2q} dt \\ &\quad + \frac{|f(x)|}{x^2} \sum_{k=0}^{\infty} p_{n+r,k}(x) \int_0^{\infty} p_{n-r,k+r}(t)(t-x)^2 dt \\ &\quad + |f'(x^+)| \sum_{k=0}^{\infty} p_{n+r,k}(x) \int_{2x}^{\infty} p_{n-r,k+r}(t) |t-x| dt \\ &\quad + \frac{C(1+x)}{(n-r-1)x} (|f(2x) - f(x) - xf'(x^+)| \\ &\quad + \frac{C(1+x)}{(n-r-1)} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^{x+\frac{x}{k}} ((f')_x) + \frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^{x+\frac{x}{\sqrt{n}}} ((f')_x) \end{aligned} \tag{14}$$

For estimating the integral

$$\sum_{k=0}^{\infty} p_{n+r,k}(x) \int_{2x}^{\infty} p_{n-r,k+r-1}(t) C_1 t^{2q} dt$$

above, we proceed as follows:
Since $t \geq 2x$ implies that $t \leq 2(t-x)$ and it follows from Lemma 1, that

$$\begin{aligned}
&\sum_{k=0}^{\infty} p_{n+r,k}(x) \int_{2x}^{\infty} p_{n-r,k+r}(t) C_1 t^{2q} dt \\
&\leq C_1 2^{2q} \sum_{k=0}^{\infty} p_{n+r,k}(x) \int_{0}^{\infty} p_{n-r,k+r}(t) C_1 (t-x)^{2q} dt \\
&= C_1 2^{2q} \mu_{n,r,2q}(x) = O(n^{-q}), \quad \text{as } n \to \infty. \qquad (15)
\end{aligned}$$

$$\frac{|f(x)|}{x^2} \sum_{k=0}^{\infty} p_{n+r,k}(x) \int_{0}^{\infty} p_{n-r,k+r}(t)(t-x)^2 dt = |f(x)| \frac{C(1+x)}{(n-r-1)x}. \qquad (16)$$

By using the Schwarz inequality and Remark 1, we get the estimate as follows:

$$\begin{aligned}
&|f'(x^+)| \sum_{k=0}^{\infty} p_{n+r,k}(x) \int_{2x}^{\infty} p_{n-r,k+r}(t)|t-x| dt \leq |f'(x^+)| \\
&\times \left(\sum_{k=0}^{\infty} p_{n+r,k}(x) \int_{0}^{\infty} p_{n-r,k+r}(t)(t-x)^2 dt \right)^{1/2} \\
&= |f'(x^+)| \sqrt{\frac{Cx(1+x)}{n-r-1}}. \qquad (17)
\end{aligned}$$

Collecting the estimates from (14)-(17), we obtain

$$\begin{aligned}
&|A_{n,r}(f,x)| \\
&= O(n^{-q}) + |f'(x^+)| \sqrt{\frac{Cx(1+x)}{n-r-1}} \\
&+ \frac{C(1+x)}{(n-r-1)x} (|f(2x) - f(x) - x f'(x^+)| + |f(x)|) \\
&+ \frac{C(1+x)}{n-r-1} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^{x+\frac{x}{k}} ((f')_x) + \frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^{x+\frac{x}{\sqrt{n}}} ((f')_x). \qquad (18)
\end{aligned}$$

On other hand, to estimate $B_{n,r}(f,x)$, applying the Lemma 2 with $y = x - \frac{x}{\sqrt{n}}$ and integration by parts, we have

$$\begin{aligned}
|B_{n,r}(f,x)| &= \left| \int_0^x \int_x^t (f')_x(u) d_t(\lambda_{n,r}(x,t)) \right| \\
&= \int_0^x (f')_x(t) \lambda_{n,r}(x,t) dt \\
&\leq \left(\int_0^y + \int_y^x \right) |(f')_x(t)| |\lambda_{n,r}(x,t)| dt \\
&\leq \frac{Cx(1+x)}{n-r-1} \int_0^y \bigvee_t^x ((f')_x) \frac{1}{(x-t)^2} dt + \int_y^x \bigvee_t^x ((f')_x) dt \\
&\leq \frac{Cx(1+x)}{n-r-1} \int_0^y \bigvee_t^x ((f')_x) \frac{1}{(x-t)^2} dt + \frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^{x} ((f')_x) \\
&= \frac{Cx(1+x)}{n-r-1} \int_1^{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{u}}}^{x} ((f')_x) du + \frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^{x} ((f')_x) \\
&\leq \frac{C(1+x)}{n-r-1} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^{x} ((f')_x) + \frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^{x} ((f')_x), \qquad (19)
\end{aligned}$$

where $u = \frac{x}{x-t}$.

Combining (12), (18) and (19) we get the desired result.

As a consequence of Lemma 3, we can easily prove the following corollary for the derivatives of the operators $V_{n,r}$.

Corollary 1: Let $f^{(s)} \in DB_q(0,\infty), q > 0$ and $x \in (0,\infty)$. The for $C > 2$ and n sufficiently large, we have

$$\begin{aligned}
&\left| \frac{(n+r-1)!(n-r)!}{n!(n-1)!} D^s V_{n,r}(f;x) - f^{(s)}(x) \right| \\
&\leq \frac{C(1+x)}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-x/k}^{x+x/k} ((D^{s+1}f)_x) + \frac{x}{\sqrt{n}} \bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} ((D^{s+1}f)_x) \\
&+ \frac{C(1+x)}{n-r-1} [|D^s f(2x) - D^s f(x) - x D^{s+1} f(x^+)| \\
&+ |D^s f(x)|] + O(n^{-q}) + \frac{C(1+x)}{n-r-1} |D^{s+1} f(x^+)| \\
&+ \frac{|D^{s+1} f(x^+) - D^{s+1} f(x^-)|}{2} \sqrt{\frac{Cx(1+x)}{n-r-1}} \\
&+ \frac{|D^{s+1} f(x^+) + D^{s+1} f(x^-)|}{2} \cdot \frac{r(1+2x)}{n-r},
\end{aligned}$$

where $\bigvee_a^b f(x)$ denotes the total variation of f_x on $[a,b]$, the auxiliary function f_x is defined by

$$D^{s+1} f_x(t) = \begin{cases} D^{s+1} f(t) - D^{s+1} f(x^-), & 0 \leq t < x, \\ 0, & t = x, \\ D^{s+1} f(t) - D^{s+1} f(x^+), & x < t < \infty. \end{cases}$$

Acknowledgment

The first author is thankful to the "Department of Science and Technology, New Delhi, India" and "Ministry of Human Resource and Development, India", for financial support to carry out the above work.

References

[1] G. D. Bai, Y.H. Hua, S.Y. Shaw, Rate of approximation for functions with derivatives of bounded variation, Anal. Math. 28 (3) (2002), 171-196.
[2] Z. Finta, On converse approximation theorems, J. Math. Anal. Appl., 312 (1) (2005), 159-180.
[3] N. K. Govil and V. Gupta, Direct estimates in simultaneous approximation for Durrmeyer type operators, Math. Ineq. Appl., 10 (2) (2007), 371-379.
[4] V. Gupta, Rate of approximation by new sequence of linear positive operators, Comput. Math. Appl. 45(12) (2003), 1895-1904.
[5] V. Gupta, M. A. Noor, M. S. Beniwal and M. K. Gupta, On simultaneous approximation for certain Baskakov Durrmeyer type operators, J. Ineq. Pure Applied Math. 7 (4) (2006) Art. 125, 15 pp.

[6] J. Sinha, V. K. Singh, Rate of convergence on the mixed summation integral type operators, Gen. Math. 14 (4) (2006), 29-36.
[7] D. K. Verma, V. Gupta and P. N. Agrawal, Some approximation properties of Baskakov-Durrmeyer-Stancu operators, Appl. Math. Comput., 218 (11) (2012), 6549-6556.
[8] X. M. Zeng, On the rate of convergence of the generalized Szász type operators for functions of bounded variation, J. Math. Anal. Appl., 226 (1998), 309-325.
[9] X. M. Zeng, W. Tao, Rate of convergence of the integral type Lupas operators, J. Kyungpook Math., 43 (2003), 593-604.
[10] X. M. Zeng, X. Cheng, Pointwise approximation by the modified Szász-Mirakyan operators, J. Comput Anal. Appl., 9 (4) (2007), 421-430.