The Effect of Slow Variation of Base Flow Profile on the Stability of Slightly Curved Mixing Layers

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Abstract—The effect of small non-parallelism of the base flow on the stability of slightly curved mixing layers is analyzed in the present paper. Assuming that the instability wavelength is much smaller than the length scale of the variation of the base flow we derive an amplitude evolution equation using the method of multiple scales. The proposed asymptotic model provides connection between parallel flow approximations and takes into account slow longitudinal variation of the base flow.

Keywords—shallow water, parallel flow assumption, weakly nonlinear analysis, method of multiple scales

I. INTRODUCTION

L INEAR stability of shallow mixing layers is investigated in [1]-[5]. Experimental study of shallow mixing layers is conducted in [6]-[8]. It is shown in [1]-[8] that bottom friction plays an important role in the development of instability. In particular, bottom friction (1) has a stabilizing influence on the flow; (2) reduces the growth of a mixing layer; and (3) prevents the development of three-dimensional instabilities because of the limited water depth.

The effect of stream curvature on the stability of mixing layers in deep water is analyzed in [9]-[11]. It is shown in [9] that curvature has a stabilizing effect on the growth rate of perturbations in a stably curved mixing layer. However, in the case of unstably curved mixing layer curvature increases the growth rate of the most unstable mode.

Theoretical analyses of linear stability of shallow mixing layers in [1]-[5] are based on a parallel flow assumption. In other words, the base flow profile is not allowed to evolve downstream (experiments in [5],[6] show that the base flow profile is slightly changing downstream). Asymptotic schemes have been developed in the past in order to take into account small non-parallelism of the base flow (see, for example, [12]). The basic assumption in [12] is that the instability wavelength λ is much smaller than the length scale L of the longitudinal variations of the base flow. The method described in [12] is used in [13] in order to analyze non-parallel effects in shallow water flows.

Slow longitudinal variation of the base flow is taken into account in the present paper. Method of multiple scales

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similar to [12] and [13] is applied to slightly curved shallow mixing layers. An evolution equation for the perturbation amplitude is derived. The coefficients of the equation depend on the two components of the base flow.

II. ASYMPTOTIC SCHEME

Consider the system of shallow water equations of the form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} + \frac{c_f}{2h} u \sqrt{u^2 + v^2} = 0, \tag{2}$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} + \frac{c_f}{2h} v \sqrt{u^2 + v^2} - \frac{1}{R} u^2 = 0, \tag{3}$$

where u and v are the depth-averaged velocity components in the x and y-directions, respectively, p is the pressure, c_f is the friction coefficient, h is water depth and R is the radius of curvature of the flow. It is assumed that the radius of curvature is much larger than the width of the mixing layer so that $1/R \ll 1$.

Introducing the stream function by the relations

$$u = \frac{\partial \psi}{\partial y} \text{ and } v = -\frac{\partial \psi}{\partial x}$$

and eliminating the pressure we rewrite (1)-(3) in the form

$$\begin{aligned} &(\Delta \psi)_t + \psi_y (\Delta \psi)_x - \psi_x (\Delta \psi)_y + \frac{2}{R} \psi_x \psi_y \\ &+ \frac{c_f}{2h} \Delta \psi \sqrt{\psi_x^2 + \psi_y^2} + \frac{c_f}{2h\sqrt{\psi_x^2 + \psi_y^2}} (\psi_y^2 \psi_{yy} \\ &+ 2\psi_x \psi_y \psi_{xy} + \psi_x^2 \psi_{xx}) = 0. \end{aligned} \tag{4}$$

Following [12] we introduce a slow longitudinal coordinate, $X = \varepsilon x$, where $\varepsilon \sim \lambda / L \ll 1$. Here λ is the wavelength of a perturbation and L is the length scale that characterizes the longitudinal evolution of the base flow.

The base flow stream function $\psi_0 (y, X)$ should satisfy (4). The velocity components of the base flow are denoted by $U(y, X)$ and $\varepsilon V(y, X)$, respectively. Both functions

$$U(y, X) = \frac{\partial \psi_0}{\partial y} \text{ and } V(y, X) = -\frac{\partial \psi_0}{\partial x}$$

are assumed to be of order unity.

The stream function $\psi(x, y, t)$ is assumed to be of the form

$$\psi(x, y, t) = \psi_0(y, X) + \psi'(x, y, t), \tag{5}$$

where $\psi'(x, y, t)$ is the fluctuating part.

Substituting (5) into (4), linearizing the resulting equation in the neighborhood of the base flow, keeping only terms of the first order in ε and dropping the primes we obtain

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) + U\frac{\partial}{\partial x}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) - \frac{\partial \psi}{\partial x}\frac{\partial^2 U}{\partial y^2}$$
$$+ \frac{c_f}{2h}\left[U\left(\frac{\partial^2 \psi}{\partial x^2} + 2\frac{\partial^2 \psi}{\partial y^2}\right) + 2\frac{\partial U}{\partial y}\frac{\partial \psi}{\partial y}\right]$$
$$+ \varepsilon\begin{bmatrix}\dfrac{\partial^2 U}{\partial y\partial X}\dfrac{\partial \psi}{\partial y} + V\dfrac{\partial}{\partial y}\left(\dfrac{\partial^2 \psi}{\partial x^2} + \dfrac{\partial^2 \psi}{\partial y^2}\right) \\[2mm] + \dfrac{c_f}{2h}\left(2\dfrac{\partial U}{\partial X}\dfrac{\partial \psi}{\partial x} - 2V\dfrac{\partial^2 \psi}{\partial y\partial X} + \dfrac{U_y}{U}V\dfrac{\partial \psi}{\partial x}\right) \\[2mm] + \dfrac{2}{R}\dfrac{\partial U}{\partial X}\dfrac{\partial \psi}{\partial y}\end{bmatrix} = 0 \tag{6}$$

Following the WKBJ approximation (see, for example, [12]) the perturbation stream function $\psi(x, y, t)$ is decomposed into a slow varying amplitude function $\varphi(y, X)$ and a fast varying phase function $\theta(X)/\varepsilon$ in accordance with the formula

$$\psi(x, y, t) = \varphi(y, X)\exp[i(\theta(X)/\varepsilon - \omega t)]. \tag{7}$$

The function $\varphi(y, X)$ is sought in the form

$$\varphi(y, X) = \varphi_1(y, X) + \varepsilon\varphi_2(y, X) + ... \tag{8}$$

Substituting (7) and (8) into (6) and collecting the terms containing the same powers of ε we obtain the set of equations for the functions $\varphi_1(y, X), \varphi_2(y, X),...$

At the leading order the following equation is obtained:

$$L(\varphi_1) = 0, \tag{9}$$

where

$$L(\varphi_1) = \varphi_1'' - k^2\varphi_1 - \frac{U''}{U - \omega/k}\varphi_1 + \frac{2}{R}U\varphi_1'$$
$$- \frac{c_f i}{2h(kU - \omega)}(-Uk^2\varphi_1 + 2U\varphi_1'' + 2U'\varphi_1').$$

Here $\theta_x = k$ and prime now denotes the derivative with respect to y.

Note that under the parallel flow assumption $U = U(y)$ and equation (9) together with zero boundary conditions at $\pm \infty$ represents the eigenvalue problem. The solution of the corresponding eigenvalue problem can be used in order to determine the conditions of the linear stability of the base flow $U = U(y)$. Assuming that the base flow is slowly evolving downstream, that is, $U = U(y, X)$ we formally obtain the same problem as for the linear stability case. The only difference is that now the argument X appear in (8) as a parameter. Thus, $\theta_x = k(X, \omega)$ satisfies the local dispersion relation

$$D(k, \omega, X) = 0. \tag{9}$$

In this case $\varphi_1(y, X)$ is the corresponding eigenfuction which can be written in the form

$$\varphi_1(y, X) = A(X)\Phi(y, X), \tag{10}$$

where $A(X)$ is an unknown amplitude and $\Phi(y, X)$ is a normalized eigenfuction (satisfying, for example, the condition $\Phi(0, X) = 1$). Using (10) the following equation at order ε is obtained:

$$L(\varphi_2) = F, \tag{11}$$

where

$$F = \frac{i}{kU - \omega}\frac{dA}{dX}(2\omega k\Phi - 3Uk^2\Phi + U\Phi'' - \Phi U'' - 2U/R\Phi'$$
$$+ c_f iUK\Phi/h) + \frac{i}{kU - \omega}A\{2\omega k\frac{\partial \Phi}{\partial X} + \omega\Phi\frac{dk}{dX} - 3Uk^2\frac{\partial \Phi}{\partial x}$$
$$- 3Uk\frac{dk}{dX}\Phi + U\frac{\partial \Phi''}{\partial X} - U''\frac{\partial \Phi}{\partial X} + \frac{\partial U'}{\partial X}\Phi' - 2\frac{U}{R}\frac{\partial \Phi'}{\partial X}$$
$$+ V(\Phi''' - k^2\Phi) + \frac{c_f}{2h}[2iUk\frac{\partial \Phi}{\partial X} + 2ik\frac{\partial U}{\partial X}\Phi + i\frac{dk}{dX}U\Phi$$
$$- 2ikV\Phi']\}. \tag{12}$$

In accordance with the Fredholm's alternative equation (11) has a solution if and only if the function F in (12) is orthogonal to the corresponding adjoint eigenfunction $\tilde{\Phi}(y, X)$ of the adjoint operator $\tilde{L}$:

$$\int_{-\infty}^{+\infty} F\tilde{\Phi}\,dy = 0. \tag{13}$$

Applying solvability condition (13) we obtain an amplitude evolution equation of the form

$$M(X)\frac{dA}{dX} + N(X)A = 0, \tag{14}$$

where

$$M(X) = i\int_{-\infty}^{+\infty} \frac{1}{kU-\omega}\begin{bmatrix} 2\omega k\Phi - 3Uk^2\Phi + U\Phi'' - \Phi U'' \\ -2\frac{U}{R}\Phi' + \frac{c_f}{h}iUk\Phi \end{bmatrix}\tilde{\Phi}\,dy,$$

$$N(X) = i\int_{-\infty}^{+\infty}\begin{Bmatrix} 2\omega k\frac{\partial\Phi}{\partial x} + \omega\Phi\frac{dk}{dX} - 3Uk^2\frac{\partial\Phi}{\partial X} \\ -3Uk\frac{dk}{dX}\Phi + U\frac{\partial\Phi''}{\partial X} - U''\frac{\partial\Phi}{\partial X} \\ +\frac{\partial U'}{\partial X}\Phi' - 2\frac{U}{R}\frac{\partial\Phi'}{\partial X} + V(\Phi''' - k^2\Phi) \\ +\frac{c_f}{2h}[2iUk\frac{\partial\Phi}{\partial x} + 2ik\frac{\partial U}{\partial X}\Phi \\ +i\frac{dk}{dX}U\Phi - 2ikV\Phi'] \end{Bmatrix}\tilde{\Phi}$$

$$/(kU-\omega)\,dy.$$

Thus, the asymptotic scheme described in the paper resulted in the following approximation for the fluctuating part of the stream function:

$$\psi(x,y,t) \sim A(X)\Phi(y,X)$$
$$\times \exp\left[i\left(\frac{1}{\varepsilon}\int_0^X k(X)\,dX - \omega t\right)\right] \tag{15}$$

Formula (15) takes into account slow longitudinal variation of the base flow.

III. DISCUSSION

A few important conclusions can be drawn from the asymptotic analysis presented in the previous section (see [14]). First, it follows from (15) that each multiplier on the right-hand side of (15) contains information related to both amplitude and phase of the perturbation. Second, the selection of the perturbed quantities plays an important role in the calculation of the growth rate and phase speed of the perturbation. Third, the growth rate and the phase speed of the perturbation depend not only on the perturbed quantity (velocity component or pressure), but also on the location of the downstream station where the quantities are calculated. Hence, a meaningful comparison of the weakly nonlinear model (14) can be made only if a particular quantity of interest Q is selected (for example, longitudinal velocity component or pressure). In this case (see [14]) a local wave number k_L can be defined by the formula

$$k_L(x,y) = -i\frac{\partial}{\partial x}\ln Q(x,y) \tag{16}$$

where $k_L = k_{Lr} + ik_{Li}$. The values of k_{Lr} and k_{Li} are interpreted as the local phase speed and local spatial growth rate, respectively. Thus, in order to compare weakly nonlinear model (14) with experimental data the following steps should be performed: (1) select a flow quantity Q; (2) measure the quantity Q at some point (x,y); compute the right-hand side of (16) at the same point (x,y). In summary, weakly nonlinear model (14) can be validated if detailed experimental data or numerical results of the solution of nonlinear shallow water equations are available.

IV. CONCLUSION

An amplitude evolution equation for the stability of slightly curved mixing layers is derived in the present paper under the assumption of small longitudinal variation of the base flow. It is assumed that a typical wavelength of a perturbation is much smaller than the wavelength associated with the longitudinal variation of the base flow. The method of multiple scales is used in the analysis. The perturbation stream function is decomposed into a slowly varying amplitude function and a fast varying phase function. Validity of the proposed model is discussed.

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