

# Strong Law of Large Numbers for \*-Mixing Sequence

Bainian Li, Kongsheng Zhang

**Abstract**—Strong law of large numbers and complete convergence for sequences of \*-mixing random variables are investigated. In particular, Teicher's strong law of large numbers for independent random variables are generalized to the case of \*-mixing random sequences and extended to independent and identically distributed Marcinkiewicz Law of large numbers for \*-mixing.

**Keywords**—\*-mixing sequences; strong law of large numbers; martingale differences; Lacunary System

## I. INTRODUCTION

LET  $(\Omega, F, P)$  be a probability space and let  $\{X_n, n = 1, 2, \dots\}$  be a sequence of real-valued random variables defined on  $(\Omega, F, P)$ . For each positive integer  $n$ , let  $F_n$  be the smallest  $\sigma$ -algebra with respect to which  $X_n$  is measurable and for  $n \leq m$ , let  $F_n^m$  be the smallest  $\sigma$ -algebra with respect to which  $X_n, \dots, X_m$  are jointly measurable.

**Definition 1.1.** Let  $\{X_n, n \geq 1\}$  be a sequence of random variable,  $X_n$  is named \*-mixing if there exists a positive integer  $N$  and function  $f$  such that  $f \downarrow 0$  and for all  $n \geq N, m \geq 1, A \in F_1^m, B \in F_{m+n}^\infty$ ,

$$|P(AB) - P(A)P(B)| \leq f(n)P(A)P(B). \quad (1)$$

Evidently, inequality (1) is equivalent to the condition for all  $B \in F_{m+n}^\infty$ ,

$$|P(B|F_1^m) - P(B)| \leq f(n)P(B) \text{ a.s.} \quad (2)$$

It follows that  $X_n$  is integrable, and

$$|E(X_{m+n}|F_1^m) - E(X_{m+n})| \leq f(n)E|X_{m+n}| \quad (3)$$

The following strong law for \*-mixing sequences can be found in Blum[1].

**Theorem A.** Let  $\{X_n, n \geq 1\}$  be a \*-mixing sequence such that  $EX_n = 0$ , and  $EX_n^2 < \infty, n \geq 1$ , and  $\sum_{i=1}^\infty EX_i^2/i^2 < \infty$ , then

$$\sum_{i=1}^n X_i/n \rightarrow 0, \text{ a.s.} \quad (4)$$

In this paper we shall further generalize Theorem A.

## II. MAIN RESULTS

**Theorem 2.1.** Let  $\{Y_n, n \geq 1\}$  be a nonnegative \*-mixing sequence such that  $EY_i = \mu_i \leq K < \infty$ , for all  $i$ , and  $\sum_{i=1}^\infty EY_i^2/i^2 < \infty$ , then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n (Y_i - \mu_i) = 0, \text{ a.s.} \quad (5)$$

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To complete the proof, we need the following lemma

**Lemma 2.2.** ([2]) Suppose  $\{Y_n, n \geq 1\}$  is a \*-mixing sequence,  $EY_i < \infty, i \geq 1$ , for any  $\sigma$ -field  $B \in F_1^m, m \geq M$ , then

$$|E(Y_{m+n}|B) - E(Y_{m+n})| \leq f(m)E|Y_{m+n}|. \quad (6)$$

**Proof of Theorem 2.1.** Let  $X_i = Y_i - \mu_i$ , then  $EX_i = 0, E|X_i| \leq 2K$ , it suffices to prove

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n X_i = 0, \text{ a.s.} \quad (7)$$

By lemma 2.2, for any  $\varepsilon > 0$ , there exists  $M' > 0$  for all  $n \geq 2$ , we get

$$|E(X_{nM'+R}|F_{M'+R}^{(n-1)M'+R})| \leq f(M')E|X_{nM'+R}| \leq 2K\varepsilon, \quad (8)$$

where  $R$  is a nonnegative integer, and  $0 \leq R \leq M' - 1$ . For  $R = 0, 1, \dots, M' - 1$ , we prove

$$\sum_{n=2}^N X_{nM'+R}/N \rightarrow 0 (N \rightarrow \infty). \quad (9)$$

Let  $H_0 = F_0 = (\Omega, \Phi)$ ,  $H_n = F_{M'+R}^{nM'+R}, n \geq 2$ .

Clearly  $H_n \uparrow$ , for fixed  $R$ , let

$$Z_n = X_{nM'+R} - E(X_{nM'+R}|H_{n-1}), n \geq 2.$$

Obviously,  $\{Z_n, H_n, n \geq 2\}$  is a martingale difference. By virtue of condition expectation, we obtain

$$\begin{aligned} E\{(E(X_{nM'+R}|H_{n-1}))^2\} \\ &= E\{E(X_{nM'+R}|H_{n-1}) \cdot E(X_{nM'+R}|H_{n-1})\} \\ &= E\{E(X_{nM'+R} \cdot E(X_{nM'+R}|H_{n-1})|H_{n-1})\} \\ &= E\{X_{nM'+R}E(X_{nM'+R}|H_{n-1})\}. \end{aligned}$$

Hence,

$$\begin{aligned} EZ_n^2 &= EX_{nM'+R}^2 - 2E(X_{nM'+R}E(X_{nM'+R}|H_{n-1})) \\ &\quad + E\{(E(X_{nM'+R}|H_{n-1}))^2\} \\ &= EX_{nM'+R}^2 - E\{(E(X_{nM'+R}|H_{n-1}))^2\} \\ &\leq EX_{nM'+R}^2, \end{aligned}$$

it follows that

$$\begin{aligned} \sum_{n=2}^\infty EZ_n^2/n^2 &\leq \sum_{n=2}^\infty EX_{nM'+R}^2/n^2 \\ &= \sum_{n=2}^\infty EX_{nM'+R}^2/(nM' + R)^2 \left(\frac{nM' + R}{n}\right)^2 \\ &\leq 4(M')^2 \sum_{i=1}^\infty EX_i^2/i^2, \end{aligned} \quad (10)$$

since  $EX_i^2 \leq EY_i^2 + K^2$ .

Combined with (10) and  $\sum_{i=1}^{\infty} EY_i^2/i^2 < \infty$ , we deduce that  $\sum_{n=2}^{\infty} EZ_n^2/n^2 < \infty$ .

By using condition expectation again, one concludes

$$\sum_{n=2}^{\infty} E(Z_n^2|H_{n-1})/n^2 < \infty. \quad (11)$$

From (11) and Theorem 8.1(see Chow[3]), we have

$$\sum_{n=2}^N Z_n/N \rightarrow 0 \text{ a.s.} \quad (12)$$

Since (8) implies

$$|E(X_{nM'+R}|H_{n-1})| \leq 2K\varepsilon, n \geq 2,$$

hence

$$|\sum_{n=2}^N E(X_{nM'+R}|H_{n-1})|/N \leq 2K\varepsilon. \quad (13)$$

Combined with (12) and (13), this yields (9). For  $R = 0, 1, \dots, M' - 1$ , one has

$$(X_{2M'} + X_{3M'} + \dots + X_{NM'})/N \rightarrow 0, (N \rightarrow \infty) \text{ a.s.}$$

$$(X_{2M'+1} + X_{3M'+1} + \dots + X_{NM'+1})/N \rightarrow 0, (N \rightarrow \infty) \text{ a.s.}$$

$$\frac{1}{N}(X_{2M'+(M'-1)} + X_{3M'+(M'-1)} + \dots + X_{NM'+(M'-1)}) \rightarrow 0, (N \rightarrow \infty) \text{ a.s.}$$

From the above results, one has

$$\sum_{i=2M'}^{(N+1)M'-1} X_i/N \rightarrow 0, (N \rightarrow \infty) \text{ a.s.}, \quad (14)$$

which deduces (7) and completes the proof.

**Theorem 2.3.** ([3]) Let  $\{X_n, n \geq 1\}$  be a \*-mixing sequence such that  $EX_n = 0, EX_n^2 < \infty, n \geq 1$ . Suppose that  $\sum_{n=1}^{\infty} a_n^{-2} EX_n^2 < \infty$  and  $\sup_n a_n^{-1} \sum_{i=1}^n E|X_i| < \infty$ , where  $\{a_n\}$  is a sequence of positive constants increasing to  $\infty$ . Then

$$a_n^{-1} \sum_{i=1}^n X_i \rightarrow 0, \text{ a.s.} \quad (15)$$

**Proof.** Given  $\varepsilon > 0$ , choose  $n_0 \geq N$  so large that  $f(n_0) < \varepsilon$ .

From Lemma 2.2 we deduce that for all positive integers  $i$  and  $j$ ,

$$\begin{aligned} & |E(X_{in_0+j}|X_{n_0+j}, X_{2n_0+j}, \dots, X_{(i-1)n_0+j})| \\ &= |E[E(X_{in_0+j}|X_1, X_2, \dots, X_{(i-1)n_0+j}) \\ & \quad |X_{n_0+j}, X_{2n_0+j}, \dots, X_{(i-1)n_0+j})]| \\ &\leq f(n_0)E|X_{in_0+j}| \text{ a.s.} \end{aligned}$$

If  $n \geq n_0$ , choose nonnegative integers  $q$  and  $r$  such that  $0 \leq r \leq n_0 - 1$  and  $n = qn_0 + r$ . Then

$$\begin{aligned} a_n^{-1} \sum_{i=1}^n X_i &= a_n^{-1} \sum_{i=1}^{n_0} X_i + a_n^{-1} \sum_{i=1}^{q-1} \sum_{j=1}^{n_0-1} X_{in_0+j} \\ &\quad + a_n^{-1} \sum_{j=1}^r X_{qn_0+j} \end{aligned}$$

$$= I_1 + I_2, \quad (16)$$

where  $I_1 = a_n^{-1} \sum_{i=1}^{n_0} X_i, I_2 = a_n^{-1} \sum_{i=1}^{q-1} \sum_{j=1}^{n_0-1} X_{in_0+j} + a_n^{-1} \sum_{j=1}^r X_{qn_0+j}$ .

Obviously,  $I_1 \rightarrow 0, \text{ a.s. } (n \rightarrow \infty), I_2$  is dominated by

$$\begin{aligned} I_2 &= \sum_{j=1}^{q-1} a_n^{-1} \sum_{i=1}^{q-1} |E(X_{in_0+j}|X_{n_0+j}, X_{2n_0+j}, \dots, X_{(i-1)n_0+j})| \\ &\quad + \sum_{j=1}^r a_n^{-1} |X_{qn_0+j} - E(X_{qn_0+j}|X_{n_0+j}, X_{2n_0+j}, \dots, \\ &\quad X_{(q-1)n_0+j})| + f(n_0)a_n^{-1} \sum_{i=n_0+1}^n E|X_i|. \end{aligned}$$

Based on the fact  $\sum_{n=1}^{\infty} a_n^{-2} EX_n^2 < \infty$  and Theorem 2.18([3]), we see that the first two terms here converge a.s. to zero. The second term is convergent to zero since  $r$  is fixed, and by  $\sup_n a_n^{-1} \sum_{i=1}^n E|X_i| < \infty$ , the last term also converges a.s. to zero. We deduce that for all  $\varepsilon > 0$ ,

$$\limsup_n |b_n^{-1} \sum_{i=1}^n X_i| < \varepsilon (\sup_n b_n^{-1} \sum_{i=1}^n |X_i|) \text{ a.s.},$$

which completes the proof.

**Lemma 2.4.** Let  $\{X_n, n \geq 1\}$  be a sequence of \*-mixing random variables satisfying  $\sum_{n=1}^{\infty} f(n) < \infty, p \geq 2$ . Assume that  $EX_n = 0$  and  $E|X_n|^p < \infty$  for each  $n \geq 1$ . Then there exists a constant  $C$  depending only on  $p$  and  $f$  such that

$$E \left( \max_{1 \leq j \leq n} \left| \sum_{i=a+1}^{a+j} X_i \right|^p \right) \leq C \left[ \sum_{i=a+1}^{a+j} E|X_i|^p + \left( \sum_{i=a+1}^{a+j} EX_i^2 \right)^{p/2} \right],$$

for every  $a \geq 0$  and  $n \geq 1$ . In particular, we have

$$E \left( \max_{1 \leq j \leq n} \left| \sum_{i=1}^j X_i \right|^p \right) \leq C \left[ \sum_{i=1}^n E|X_i|^p + \left( \sum_{i=1}^n EX_i^2 \right)^{p/2} \right],$$

for every  $n \geq 1$ .

**Proof.**

$$\begin{aligned} E \left( \sum_{i=a+1}^{a+j} X_i \right)^2 &= \sum_{i=a+1}^{a+j} EX_i^2 + 2 \sum_{a+1 \leq i < j \leq a+n} E(X_i X_j) \\ &\leq \sum_{i=a+1}^{a+j} EX_i^2 + 2 \cdot \sum_{a+1 \leq i < j \leq a+n} f(j-i)E|X_i|E|X_j| \\ &\leq \sum_{i=a+1}^{a+j} EX_i^2 + 2 \cdot \sum_{a+1 \leq i < j \leq a+n} f(j-i)E(X_i^2)^{1/2}E(X_j^2)^{1/2} \\ &\leq \sum_{i=a+1}^{a+j} EX_i^2 + \end{aligned}$$

$$\begin{aligned}
& \sum_{k=1}^{n-1} \sum_{i=a+1}^{a+n-k} f(k)(EX_i^2 + EX_{k+i}^2) \\
& \leq \left(1 + 2 \sum_{k=1}^{\infty} f(k)\right) \sum_{i=a+1}^{a+j} EX_i^2 \\
& = C_1 \sum_{i=a+1}^{a+j} EX_i^2
\end{aligned}$$

It is well known that  $\ast$ -mixing is also  $\varphi$ -mixing. Therefore, by [4, Lemma 2.2]) we can immediately complete the proof of Lemma 2.4.

**Lemma 2.5.** Let  $\{X_n, n \geq 1\}$  be a zero-mean  $\ast$ -mixing and  $\sum_{k=1}^{\infty} f(k) < \infty$ , for some  $p \geq 2$ ,  $\sup_i E|X_i|^p < \infty$ . Then there exists constant  $C > 0$  depending only on  $p$  for any real-valued sequence  $\{a_{ni}\}$ , such that

$$E\left|\sum_{i=1}^n a_{ni}X_i\right|^p \leq C\left(\sum_{i=1}^n a_{ni}^2\right)^{p/2}.$$

**Proof.** Let  $a_{ni} = 0, i > n$ , since  $\sum_{k=1}^{\infty} f(k) < \infty$ ,  $\sup_i E|X_i|^p < \infty$ . By Lemma 2.4, we have

$$\begin{aligned}
E\left|\sum_{i=1}^n a_{ni}X_i\right|^p & \leq C\left[\sum_{i=1}^n E|a_{ni}X_i|^p + \left(\sum_{i=1}^n E a_{ni}^2 X_i^2\right)^{p/2}\right] \\
& \leq C\left[\sum_{i=1}^n |a_{ni}|^p + \left(\sum_{i=1}^n a_{ni}^2\right)^{p/2}\right].
\end{aligned}$$

Since  $p \geq 2$ , it follows that

$$\left(\sum_{i=1}^n |a_{ni}|^p\right)^{1/p} \leq \left(\sum_{i=1}^n a_{ni}^2\right)^{1/2}$$

$\Leftrightarrow \left(\sum_{i=1}^n |a_{ni}|^p\right) \leq \left(\sum_{i=1}^n a_{ni}^2\right)^{p/2}$ , which proves the statement.

**Remark 1.**

(1) Lemma 2.5 implies that  $\ast$ -mixing is a Lacunary System.

(2) If  $a_{ni} = 1$ , we have

$$E\left|\sum_{i=1}^n X_i\right|^p \leq cn^{p/2}.$$

### III. LARGER DEVIATIONS FOR $\ast$ -MIXING

**Theorem 3.1.** Let  $\{X_n, n \geq 1\}$  be a zero-mean  $\ast$ -mixing,  $\sum_{k=1}^{\infty} f(k) < \infty$ , for some  $p > 2$ ,  $E|X_i|^p < \infty$ . If there exists  $1/2 < r \leq 1$ ,  $\theta = 2r - 1$  and positive constant  $K$  such that  $\sum_{i=1}^n a_{ni}^2 \leq Kn^\theta, (i = 1, 2, \dots, n)$ , then

$$n^{-r} \sum_{i=1}^n a_{ni}X_i \longrightarrow 0, \quad a.s. \quad (17)$$

**Proof.** Denote  $S_n = \sum_{i=1}^n a_{ni}X_i$ , by Markov's inequality, we have

$$P\{S_n \geq n^r x\} \leq \frac{E|S_n|^p}{x^p n^{pr}}.$$

From lemma 2.5, we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} P\{|S_n| \geq n^r x\} & \leq \sum_{n=1}^{\infty} \frac{E|S_n|^p}{x^p n^{pr}} \\
& \leq \sum_{n=1}^{\infty} \frac{C(\sum_{i=1}^n a_{ni}^2)^{p/2}}{x^p n^{pr}} \\
& \leq \sum_{n=1}^{\infty} \frac{CK}{x^p n^{p/2}} < \infty.
\end{aligned}$$

Inequality (17) follows from Borel-Cantelli lemma.

**Remark 2.** Marcinkiewicz Law of large numbers of independent and identically distributed variables has been extended to the case of  $\ast$ -mixing.

**Theorem 3.2.** Let  $\{X_n, n \geq 1\}$  be a zero-mean  $\ast$ -mixing,  $\sum_{k=1}^{\infty} f(k) < \infty$ , for some  $p > 2$ ,  $E|X_i|^p < \infty$ . If there exists  $1/2 < r \leq 1$ ,  $\theta = 1 - 2/p$  and positive constant  $K$  such that  $\sum_{i=1}^n a_{ni}^2 \leq Kn^\theta, (i = 1, 2, \dots, n)$ , then

$$\frac{\sum_{i=1}^n a_{ni}X_i}{\sqrt{n \log n}} \longrightarrow 0, \quad a.s. \quad (18)$$

**Proof.** By Markov's inequality and lemma 2.5, we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} P\left\{\left|\sum_{i=1}^n a_{ni}X_i / \sqrt{n \log n}\right| \geq x\right\} & \leq \sum_{n=1}^{\infty} \frac{E|S_n|^p}{x^p n^{p/2} (\log n)^{p/2}} \\
& \leq \sum_{n=1}^{\infty} \frac{C(\sum_{i=1}^n a_{ni}^2)^{p/2}}{x^p n^{p/2} (\log n)^{p/2}} \\
& \leq \sum_{n=1}^{\infty} \frac{CK n^{p/2-1}}{x^p n^{p/2} (\log n)^{p/2}} \\
& = \sum_{n=1}^{\infty} \frac{CK}{x^p n (\log n)^{p/2}} \\
& < \infty.
\end{aligned}$$

Therefore, inequality (18) follows from Borel-Cantelli lemma.

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