

# Computing SAGB-Gröbner Basis of Ideals of Invariant Rings by Using Gaussian Elimination

Sajjad Rahmany, Abdolali Basiri

*Abstract*—The link between Gröbner basis and linear algebra was described by Lazard [4,5] where he realized the Gröbner basis computation could be archived by applying Gaussian elimination over Macaulay's matrix .

In this paper, we indicate how same technique may be used to SAGBI- Gröbner basis computations in invariant rings.

*Keywords*— Gröbner basis, SAGBI- Gröbner basis, reduction, Invariant ring, permutation groups.

## I. INTRODUCTION

**T**HE concept of SAGBI- Gröbner bases( a generalisation of Gröbner bases to ideals of sub algebras of polynomial ring) has been developed by Miller [9,10]. In fact, it is a method to compute bases of ideals of sub algebras in a similar way to computing Gröbner bases for ideals [1,2]. SAGBI-Gröbner bases and Gröbner bases have analogous reduction properties. The main difference is that SAGBI- Gröbner bases need not be finitely generated. Therefore, we restrict our study to partial SAGBI- Gröbner bases up to given degree  $D$ .

The main goal of this note is to establishes the relation between linear algebra and SAGBI- Gröbner bases (SG- bases) and present an algorithm for computing SG-basis(up to degree  $D$ ) for ideals of invariant rings of permutation groups. For this, we first describe link between SG- bases and linear algebra and then provide an algorithm like Lazard's algorithm for construction of SG- basis. The advantage of our method lies in this fact that it be compute SG-bases (up to degree  $D$ ) by applying Gaussian elimination on special matrix.

The paper is organized as follows. Section 2 has been divided into two parts:subsection (2.a), we review the necessary mathematical notations and in (2.b) we will give some basic definitions of invariants rings. In section 3, we recall the definition of SG-basis. Also we will present basic properties of SG-basis in invariant rings. In Section 4, we concentrate on our main goal. We will establishes the relation between linear algebra and SG- basis for ideals in invariant rings. In Section 5, We will give an algorithm for computing SG-basis.

S. Rahmany : Member of the department of Mathematics, Damghan University of Basic Science , Damghan, Iran, e-mail: (see <http://www.dubs.ac.ir/contact.html>).

A. Basiri : Member of the department of Mathematics, Damghan University of Basic Science , Damghan, Iran, e-mail: (see <http://www.dubs.ac.ir/contact.html>).

Manuscript received October 31, 2009.

## II. INVARIANT RINGS

### A. Standard notations

In this paper, we suppose that  $\mathbb{K}$  is a field of characteristic zero,  $R = \mathbb{K}[x_1, \dots, x_n]$  is the ring of polynomials and monomial order  $\prec$  has been fixed. For a polynomial  $f \in R$ , denote the leading monomial, leading term, and leading coefficient of  $f$  with respect to  $\prec$  by  $LM(f)$ ,  $LT(f)$ , and  $LC(f)$  respectively. We use the notation  $T(f)$  for the set of terms of  $f$ . We denote by  $T$ , the set of all terms of  $x_1, \dots, x_n$ . By extension, for any set  $B$  of polynomials, define  $LM(B) = \{LM(p) \mid p \in B\}$  and  $LT(B) = \{LT(p) \mid p \in B\}$ .

### B. Invariants rings

In this subsection, we will give some basic definitions of invariants rings and describe the main properties of them. In the rest of this paper we assume that  $G$  be a subgroup of  $\hat{S}_n$  where

$$\hat{S}_n = \{\Pi \cdot \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \mid \Pi \text{ is a permutation matrix}\}.$$

Also, we use the notation  $X$ , for column vector of the variables  $x_1, \dots, x_n$ . In other words,

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

**Definition 2.1:** Let  $A = (a_{ij}) \in G$  and  $f \in \mathbb{K}[x_1, \dots, x_n]$ . We define  $f(A.X) \in \mathbb{K}[x_1, \dots, x_n]$  by following:

$$f(A.X) = f(a_{11}x_1 + \dots + a_{1n}x_n, \dots, a_{n1}x_1 + \dots + a_{nn}x_n).$$

A polynomial  $f \in R$  is called *invariant polynomial* if  $f(A.X) = f(X)$  for all  $A \in G$ . The *invariant ring*  $R^G$  of  $G$  is the set of all invariant polynomials.

**Example 2.1:** Consider the cyclic matrix group  $G$  generated by matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Clearly  $f = x_1^2 + x_2^2$  is invariant while  $g = x_1x_2$  is not invariant, because  $g(A.X) \neq g(X)$ .

It is immediately clear than  $R^G$  is not finite dimensional as a vector space  $\mathbb{K}$ . But we have a decomposition of  $R^G$  into its homogeneous components, which are finite dimensional. This decomposition is similar to decomposition of  $R$ .

Let  $R_d$  denote the vector space of all homogeneous polynomials of degree  $d$ , then we have

$$R = \bigoplus_{d \geq 0} R_d$$

The monomials of degree  $d$  are a vector space basis of  $R_d$ . Now, observe that the action  $G$  preserves the homogeneous components. Hence we get a decomposition of the invariant ring

$$R^G = \bigoplus_{d \geq 0} R_d^G.$$

A method for calculate a vector space basis of  $R^G$  is Reynolds operator, which is defined as follows

**Definition 2.2:** Let  $G$  be a finite group. The *Reynolds operator* of  $G$  is the map  $\mathfrak{R} : R \rightarrow R^G$  defined by the formula

$$\mathfrak{R}(f) = \frac{1}{|G|} \sum_{\sigma \in G} f(\sigma.X)$$

for  $f \in R$ .

Following properties of the Reynolds operator is easily verified.

**Proposition 2.1:** ([3]) Let  $\mathfrak{R}$  be the Reynolds operator of the finite group  $G$ .

- (a)  $\mathfrak{R}$  is  $K$ -linear in  $f$ .
- (b) If  $f \in R$ , then  $\mathfrak{R}(f) \in R^G$ .
- (c) If  $f \in R^G$ , then  $\mathfrak{R}(f) = f$ .

It is easy to prove that, for any monomial  $m$  the Reynolds operator gives us a homogeneous invariant  $\mathfrak{R}(m)$ . Such invariants are called *orbit sums*.

The set orbit sums is a vector space basis of  $R^G$ , so any invariant can be uniquely written as a linear combination of orbit sums. Now, we give a special representation of invariant polynomials which is used in the next section. For this, we require the following terminology.

**Definition 2.3:** A monomial in  $LM(R^G)$  is called an *initial*.

Using 2.1 and definition 2.3 we can simply derive the following lemma.

**lemma 2.1:** Every  $f \in R^G$  can be written uniquely as  $f = \sum_{\alpha} c_{\alpha} \mathfrak{R}(m_{\alpha}^*)$ , where  $c_{\alpha} \in \mathbb{K}$  and  $m_{\alpha}^*$  are initial monomials.

In rest of this paper, we suppose that all representations of invariant polynomials are in the above form.

### III. SG-BASIS IN INVARIANT RINGS

In this section, we recall the definition of SG-basis which is an analogs of Gröbner basis for ideals in  $k$ -sub algebras. Also, we will present basic properties of SG-basis in invariant rings.

The following symbol will be needed throughout the paper. Let  $f_1, \dots, f_n$  be invariant polynomials and  $I, I^G$  represent the ideal generates by  $f_1, \dots, f_n$  in  $R$  and  $R^G$  respectively. For the sake of simplicity, we assume that  $I$  is homogeneous. The extension to the non-homogeneous case raise no difficulty.

**Definition 3.1:** A subset  $F \subseteq I^G$  is *SG-basis* for  $I^G$  if  $LT(F)$  generates the initial ideal  $\langle LT(I^G) \rangle$  as an ideal over algebra  $\langle LT(R^G) \rangle$ . It is a partial SG-basis up to degree  $D$  of  $I^G$  if  $LT(F)$  generates  $\langle LT(I^G) \rangle$  up to the degree  $D$ .

Recall that in ordinary Gröbner basis theory every ideal is assured to have a finite Gröbner bases but SG-basis need not be finite. We continue by describing an appropriate reduction for the current context.

**Definition 3.2:** Let  $f, g, p \in R^G$  with  $f, p \neq 0$  and let  $P$  be a subset of  $R^G$ . Then we say

- i)  $f$  *SG-reduces* to  $g$  modulo  $p$  (written  $f \xrightarrow{p}_{SG} g$ ), if  $\exists t \in T(f), \exists s \in LM(R^G)$  such that  $s.LT(p) = t$  and  $g = f - (\frac{a}{Lc(p).Lc(\mathfrak{R}(s))}).\mathfrak{R}(s).p$  where  $a$  is the coefficient of  $t$  in  $f$  and  $\mathfrak{R}$  is Reynolds operator of  $G$ .
- ii)  $f$  *SG-reduces* to  $g$  modulo  $P$  (written  $f \xrightarrow{P}_{SG} g$ ), if  $f$  SG-reduces to  $g$  modulo  $p$  for some  $p \in P$ .

Finally, the definition of *SG-reducible*, *SG-normalform* are straightforward.

Basic properties of SG-basis presented in [9,10,6]. We will review some of the standard fact on SG-bases. The proofs of the following proposition and its corollary proceed in the standard way.

**Proposition 3.1:** The following are equivalent for a subset  $F$  of an ideal  $I^G \subseteq R^G$ :

- a)  $F$  is an SG-basis for  $I^G$ .
- b) For every  $h \in I^G$ , every SG-normalform of  $h$  modulo  $F$  is 0.

**Corollary 3.1:** A SG-basis for  $I^G$  generates  $I^G$  as an ideal of  $R^G$ .

**Corollary 3.2:** Suppose that  $F$  is an SG-basis for  $I \subseteq R^G$ . Then  $f \in R^G$  belongs to  $I \iff f \xrightarrow{F}_{SG} 0$ .

It is easy to show that the proposition above continues to hold if we restrict our discussion to SG-basis up to degree  $D$ . Hence, if a SG-basis up to degree  $D$  of  $I^G$  has already been computed, then this is enough to test for membership in  $I^G$  for any polynomial  $f$  with  $deg(f) \leq D$ .

### IV. LINEAR ALGEBRA AND SG-BASIS

The link between Gröbner basis and linear algebra was described by Lazard[4,5] where he realized the Gröbner basis computation could be archived by applying Gaussian elimination over Macaulay's matrix .

In this section, we indicate how same technique may be used to SG-Gröbner basis computations. Also , we will establishes the relation between linear algebra and SG-Gröbner basis . For this, we assume  $I^G$  be an ideal generated by a finite set of invariants polynomials  $f_1, \dots, f_n$  in  $R^G$  and  $I_d^G$  denote the set of polynomials in  $I^G$  which are of degree less or equal than  $d$ , namely

$$I_d^G = \{f \in I^G | deg(f) \leq d\}.$$

It is easy to see that the ideal  $I^G$  itself is a subspace of the  $\mathbb{K}$ -vector space  $R^G$ , and so is  $I_d^G$  for each  $d \in \mathbb{N}$ .

Following proposition give a link between linear base of  $I_d^G$  and SG-Gröbner basis .

**Proposition 4.1:** Let  $F = \{g_1, \dots, g_l\} \subseteq I^G$ . Suppose  $d \in \mathbb{N}$  was fixed and for  $1 \leq i \leq l$  set

$$B_i = \{\mathfrak{R}(m)g_i \mid \deg((LT(\mathfrak{R}(m)g_i))) \leq d, LT(g_j) \nmid LT(\mathfrak{R}(m)g_i) \text{ for all } j < i\}.$$

where  $\mathfrak{R}$  is Reynolds operator of  $G$ . Then following conditions are equivalent

- (i)  $F$  is a SG-Gröbner basis up to degree  $d$  of  $I^G$  w.r.t a total degree order.
- (ii)  $B = \bigcup_{i=1}^l B_i$  is a basis of the  $\mathbb{K}$ -vector space  $I_d^G$ .

**Proof 4.1:** let  $F$  be a SG-Gröbner basis up to degree  $d$  of  $I^G$ . We have  $B \subseteq I_d^G$  by choice of the term order. It is clear that the head terms of elements of  $B_i$  are pairwise different for fixed  $1 \leq i \leq l$ .

If there were  $\mathfrak{R}(m_1)g_i, \mathfrak{R}(m_2)g_j$  with  $i < j$  and  $LT(\mathfrak{R}(m_1)g_i) = LT(\mathfrak{R}(m_2)g_j)$  then we would have  $LT(g_i) \mid LT(\mathfrak{R}(m_2)g_j)$  contrary to the construction of  $B_j$ . To prove the linear independence of  $B$ , let

$$p = \sum_{q \in B} \lambda_q \cdot q \quad (\lambda_q \in \mathbb{K})$$

where not all  $\lambda_q$  equal zero. Then  $\max\{LT(q) \mid \lambda_q \neq 0\} = LT(h)$  for exactly one  $h \in B$ , and we see that  $LT(h)$  is a term  $p$ . So  $p \neq 0$ . It remain to show that  $B$  is a generating system of  $I_d^G$ . Let  $f \in I_d^G$ . Then  $f \xrightarrow{*F}_{SG} 0$ . Among all possible reduction chains, consider the one where each reduction step  $f_k \xrightarrow{g_i}_{SG} f_{k+1}$  is a top reduction and has the property  $LT(g_j) \nmid LT(f_k)$  for all  $j < i$ . So,  $f_{k+1} = f_k - \mathfrak{R}(m_i)g_i$  such that  $\mathfrak{R}(m_i)g_i \in B_i$ .

Finally, we can find a representation of  $f$  as sum of orbit sums multiples of elements of  $B$ .

Conversely, let  $B$  generate  $I_d^G$ . Then for  $f \in I_d^G$  we have  $f = \sum_{q \in B} \lambda_q \cdot q$

According to the above observation, there is a  $q \in B$  such that  $LT(q) = LT(f)$ . It is means that there exist a  $g_i \in F$  ( $q = \mathfrak{R}(m)g_i$ ) such that  $LT(g_i) \mid LT(f)$ .

Let us mention one important consequence of the above proposition. In rest of this section, we assume  $I^G$  be an ideal generated by homogeneous polynomials  $f_1, \dots, f_m$  with  $\deg(f_i) = d_i$  and  $d_1 \leq \dots \leq d_m$ . Also, let  $I_d^G$  denote the set of homogeneous polynomials in  $I^G$  which are of degree  $d$ .

The characterization of SG-Gröbner basis of the last proposition may be used to make the following.

**Corollary 4.1:** A set  $F = \{f_1, \dots, f_k\}$  of homogeneous polynomials is a SG-Gröbner basis for degree  $D$  of  $I^G$  if and only if

$$\{\mathfrak{R}(m)f_i \mid i \in \{1, \dots, k\}, \deg((LT(\mathfrak{R}(m)f_i))) = d, LT(f_j) \nmid LT(\mathfrak{R}(m)f_i) \text{ for all } j < i\}$$

a linear basis for  $I_d^G$ . According to the above corollary, compute a SG-Gröbner basis in degree  $d$  for ideal  $I^G$  is equivalent to find the linear basis for  $I_d^G$ . Then, our goal (i.e. compute a SG-Gröbner basis in degree  $d$ ) becomes to compute a linear basis for  $I_d^G$ .

#### V. MACAULAY'S MATRIX INVARIANT AND LAZARD'S ALGORITHM

In this section, we will propose new method for computing SG-Gröbner basis in degree  $d$  for ideals in invariant rings of

finite groups. The advantage of this method lies in the fact that it be achieve by applying Gaussian elimination on a special matrix. Now, we provide the following definition which is an adaptation of Macaulay's matrix [7,8] in invariant rings;

**Definition 5.1:** The *Macaulay's matrix invariant*  $f_1, \dots, f_m$  of degree  $d$  is matrix which rows are all coefficients multiples  $\mathfrak{R}(m).f_i$  where  $m$  is an initial monomial of degree  $d - d_i$  and columns indexed by initial monomials of degree  $d$  (stored by  $\prec$ ).

We use the symbol  $M_{d,m}$  to denote Macaulay's matrix invariant.

$$M_{d,m} = \begin{matrix} \mathfrak{R}(\tilde{m}_1).f_1 \\ \vdots \\ \mathfrak{R}(m_i).f_j \\ \vdots \\ \mathfrak{R}(m_t).f_m \end{matrix} \begin{pmatrix} \mathfrak{R}(\tilde{m}_1) & \mathfrak{R}(\tilde{m}_2) & \dots & \mathfrak{R}(\tilde{m}_k) \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

It is easy to see that, Macaulay's matrix invariant is a representation of vector space  $I_d^G$  by an array of coefficients and also the following facts are straightforward

- (1) The leading terms of a row is the leading term the corresponding polynomial.
- (2) The result of applying a row operation on  $M_{d,m}$  gives a matrix whose rows generate the same ideal.

In fact, above representation is used to describe connection between SG-Gröbner basis and linear basis of an ideal. To find this relation, we will state the following definition and lemma. The proof of the lemma proceed in the standard way.

**Definition 5.2:** We denote by  $\tilde{M}_{d,m}$  the result of Gaussian elimination applied to the matrix  $M_{d,m}$  using a sequence of the elementary rows operations.

**lemma 5.1:** The set of the all polynomials correspond with rows of  $\tilde{M}_{d,m}$  such that leading monomials of these not appear as leading monomials of polynomials correspond with rows  $M_{d,m}$  is a SG-Gröbner basis of degree  $d$  for ideal  $I^G$ .

Now, suppose  $Row(\tilde{M}_{d,m})$  be the set of polynomials corresponding with all rows of  $\tilde{M}_{d,m}$ . By using above lemma, we can introduce a new algorithm for computing SG-Gröbner basis up to degree  $D$  of homogeneous ideals which is similar to lazard's algorithm.

#### Algorithm 5.1: Algorithm For computing SG-basis

**Input:** homogeneous polynomials invariants  $(f_1, \dots, f_m)$  with degrees  $d_1 \leq \dots \leq d_m$ ; a maximal degree  $D$

**output:** The elements of degree at most  $D$  of SG-bases of  $(f_1, \dots, f_m)$ .

$G := \emptyset$

**for**  $d$  **from**  $d_1$  **to**  $D$  **do**

**Compute**  $\tilde{M}_{d,m}$  by Gaussian elimination from  $M_{d,m}$ .

**Set**  $L_d := \{p \in Row(\tilde{M}_{d,m}) \mid LT(p) \notin LT(M_{d,m})\}$

$G := G \cup L_d$

**return**  $G$

## VI. CONCLUSION

A first implementation of above algorithm has been made in maple 12 computer algebra system and have been successfully tried on a number of examples. The advantage of this algorithm lies in this fact that it is very easy to implement and well suited to complexity analysis.

## REFERENCES

- [1] Buchberger B., *Ein Algorithmus zum Auffinden der Basiselemente des Restklassenringes nach einem nulldimensionalen Polynomideal*, Innsbruck, 1965
- [2] Buchberger B., *Ein algorithmisches Kriterium für die Lösbarkeit eines algebraischen Gleichungssystems*, Aequationes Math., 4, pages 374–383, 1970.
- [3] David A. Cox and John B. Little and Don O’Shea, *Ideals, Varieties, and Algorithms : An introduction to computational algebraic geometry and commutative algebra*, Undergraduate Texts in Mathematics. Springer Verlag, New York, 3rd ed.2007.
- [4] D.Lazard, *Gröbner bases, Gaussian elimination and resolution of systems of algebraic equations*, Computer algebra (London, 1983), Lecture Notes in Comput. Sci.162.
- [5] D.Lazard, *Solving systems of algebraic equations*, ACM SIGSAM Bulletin, 35, pages 11–37, 2001.
- [6] Faugère, J-C. and Rahmany, S., *Solving systems of polynomial equations with symmetries using SAGBI-Gröbner bases*, ISSAC 2009.
- [7] F.S. Macaulay *On Some formulae in elimination*, proceedings of the London Mathematical Society, 33, page3–27, 1902.
- [8] F.S. Macaulay, *The Algebraic Theory of Modular Systems*, Cambridge Mathematical Librar, Cambridge University Press, 1916.
- [9] J.L.Miller, *Analogues of Gröbner bases in polynomial rings over a ring*, Journal of Symbolic Computation, 21(2), 139–153, 1996.
- [10] J.L.Miller, *Effective algorithm for intrinsically computing SAGBI-Gröbner bases in polynomial ring over a field*, Groebner bases and application (Linz),421–433, 1998.