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The Proof of Two Conjectures Related to Pell's Equation $x^2 - Dy^2 = \pm 4$

Armend Sh. Shabani

Abstract—Let $D \neq 1$ be a positive non-square integer. In this paper are given the proofs for two conjectures related to Pell's equation $x^2 - Dy^2 = \pm 4$, proposed by *A*. Tekcan.

Keywords— Pell's equation, solutions of Pell's equation.

I. Introduction

Let $D \neq 1$ be any positive non-square integer and *N* be any fixed integer. The equation

$$x^2 - Dy^2 = \pm N \tag{1}$$

is known as Pell's equation. It is named mistakenly after John Pell (1611-1685) who was a mathematician who in fact did not contribute for solving it (see [2]).

For $N = 1$, the Pell's equation $x^2 - Dy^2 = \pm 1$ is known as classical Pell's Equation and it has infinitely many solutions (x_n, y_n) for $n \geq 1$. There are different methods for finding the first non-trivial solution (x_1, y_1) called the fundamental solution from which all others are easily computed (see [3], amd [8]).

There are many papers in which are considered different types of Pell's equation (see [4], [5], [6], [7]).

In these notes we will be focused on paper [1] in which A. Tekcan considered the equations:

$$x^2 - Dy^2 = \pm 4 \tag{2}$$

and among other results he obtained the following:

Theorem 1.1. If (x_1, y_1) is the fundamental solution of the Pell's equation $x^2 - Dy^2 = 4$ then

$$x_n = \frac{x_1 x_{n-1} + D y_1 y_{n-1}}{2};\ y_n = \frac{y_1 x_{n-1} + x_1 y_{n-1}}{2} \tag{3}$$

for $n \geq 2$.

Theorem 1.2. If (x_1, y_1) is the fundamental solution of the Pell's equation $x^2 - Dy^2 = -4$ then:

$$\begin{cases} x_{2n+1} = \dfrac{(x_1^2 + Dy_1^2)x_{2n-1} + 2Dx_1 y_1 y_{2n-1}}{4} \\ y_{2n+1} = \dfrac{2x_1 y_1 x_{2n-1} + (x_1^2 + Dy_1^2) y_{2n-1}}{4} \end{cases}, \tag{4}$$

for $n \geq 1$.

Also the following conjectures were proposed:

Conjecture 1.3. If (x_1, y_1) is the fundamental solution of the Pell's equation $x^2 - Dy^2 = 4$ then (x_n, y_n) satisfy the following recurrence relations

$$\begin{cases} x_n = (x_1 - 1)(x_{n-1} + x_{n-2}) - x_{n-3} \\ y_n = (x_1 - 1)(y_{n-1} + y_{n-2}) - y_{n-3} \end{cases} \tag{5}$$

for $n \geq 4$.

Conjecture 1.4. If (x_1, y_1) is the fundamental solution of the Pell's equation $x^2 - Dy^2 = -4$ then (x_{2n+1}, y_{2n+1}) satisfy the following recurrence relations

$$\begin{cases} x_{2n+1} = (x_1^2 + 1)(x_{2n-1} + x_{2n-3}) - x_{2n-5} \\ y_{2n+1} = (x_1^2 + 1)(y_{2n-1} + y_{2n-3}) - y_{2n-5} \end{cases} \tag{6}$$

for $n \geq 3$.

II. Main Results. Proof of Conjectures

We will prove above mentioned conjectures using the method of mathematical induction.

Proof of conjecture 1.3

First, we show that relations (5) are true for $n = 4$, so we have to show:

$$\begin{cases} x_4 = (x_1 - 1)(x_3 + x_2) - x_1 \\ y_4 = (x_1 - 1)(y_3 + y_2) - y_1 \end{cases}. \tag{7}$$

Using (3) we obtain the following:

$$\begin{cases} x_2 = \dfrac{x_1^2 + Dy_1^2}{2} = \dfrac{x_1^2 + x_1^2 - 4}{2} = x_1^2 - 2 \\ y_2 = x_1 y_1 \end{cases}. \tag{8}$$

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Armend Sh. Shabani is with Department of Mathematics, University of Prishtina, Avenue "Mother Theresa", 5 Prishtine, 10000, Kosova-UNMIK (e-mail: armend_shabani@hotmail.com)

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Then by (3) and (8) we get:

$$\begin{cases} x_3 = \dfrac{x_1x_2 + Dy_1y_2}{2} = \dfrac{x_1(x_1^2-2) + Dy_1^2x_1}{2} \\ \quad = x_1(x_1^2-3) \\ y_3 = \dfrac{y_1x_2 + x_1y_2}{2} = \dfrac{y_1(x_1^2-2) + x_1^2y_1}{2} \\ \quad = y_1(x_1^2-1) \end{cases}. \tag{9}$$

Next by (3) and (9) we find x_4 and y_4:

$$x_4 = \frac{x_1x_3 + Dy_1y_3}{2} = \frac{x_1^2(x_1^2-3) + Dy_1^2(x_1^2-1)}{2}$$

$$= \frac{x_1^2(x_1^2-3) + (x_1^2-4)(x_1^2-1)}{2} = x_1^4 - 4x_1^2 + 2$$

$$y_4 = \frac{y_1x_3 + x_1y_3}{2} = \frac{y_1x_1(x_1^2-3) + x_1y_1(x_1^2-1)}{2}$$

$$= x_1y_1(x_1^2-2).$$

So we obtained:

$$\begin{cases} x_4 = x_1^4 - 4x_1^2 + 2 \\ y_4 = x_1y_1(x_1^2-2) \end{cases}. \tag{10}$$

Now, replacing (8) and (9) in (7) one obtains:

$$x_4 = (x_1-1)(x_1(x_1^2-3) + x_1^2 - 2) - x_1$$

$$= (x_1-1)(x_1^3 + x_1^2 - 3x_1 - 2) - x_1 = x_1^4 - 4x_1^2 + 2$$

and

$$y_4 = (x_1-1)(y_1(x_1^2-1) + x_1y_1) - y_1$$

$$= y_1((x_1-1)(x_1^2 - 1 + x_1) - 1) = x_1y_1(x_1^2-2)$$

which are the same formulas as in (10).

It means for $n = 4$, the recurrence relations (5) hold.

Next, we assume that (5) holds for *n* and we show that it holds for $n+1$.

Indeed, by (3) and by hypothesis we have:

$$x_{n+1} = \frac{x_1((x_1-1)(x_{n-1} + x_{n-2}) - x_{n-3})}{2} +$$

$$+ \frac{Dy_1((x_1-1)(y_{n-1} + y_{n-2}) - y_{n-3})}{2}$$

$$= (x_1-1)\left(\frac{x_1(x_{n-1} + x_{n-2}) + Dy_1(y_{n-1} + y_{n-2})}{2}\right) - \frac{x_1x_{n-3} + Dy_1y_{n-3}}{2}$$

$$= (x_1-1)\left(\frac{x_1x_{n-1} + Dy_1y_{n-1}}{2} + \frac{x_1x_{n-2} + Dy_1y_{n-2}}{2}\right) - x_{n-2}$$

$$= (x_1-1)(x_n + x_{n-1}) - x_{n-2},$$

$$y_{n+1} = \frac{y_1((x_1-1)(x_{n-1} + x_{n-2}) - x_{n-3})}{2} +$$

$$+ \frac{x_1((x_1-1)(y_{n-1} + y_{n-2}) - y_{n-3})}{2}$$

$$= (x_1-1)\left(\frac{y_1(x_{n-1} + x_{n-2}) + x_1(y_{n-1} + y_{n-2})}{2}\right) - \frac{y_1x_{n-3} + x_1y_{n-3}}{2}$$

$$= (x_1-1)\left(\frac{y_1x_{n-1} + x_1y_{n-1}}{2} + \frac{y_1x_{n-2} + x_1y_{n-2}}{2}\right) - y_{n-2}$$

$$= (x_1-1)(y_n + y_{n-1}) - y_{n-2},$$

completing the proof.

Proof of conjecture 1.4

First, we show that relations (6) are true for $n = 3$, so we show that:

$$\begin{cases} x_7 = (x_1^2+1)(x_5 + x_3) - x_1 \\ y_7 = (x_1^2+1)(y_5 + y_3) - y_1 \end{cases}. \tag{11}$$

Using (4) we obtain the following:

$$x_3 = \frac{(x_1^2 + Dy_1^2)x_1 + 2Dx_1y_1^2}{4} = \frac{x_1(x_1^2 + 3Dy_1^2)}{4}.$$

Since (x_1, y_1) is the fundamental solution of $x^2 - Dy^2 = -4$ then $x_1^2 + 3Dy_1^2 = 4(Dy_1^2 - 1)$ and since $Dy_1^2 = x_1^2 + 4$ we obtain $x_3 = x_1(x_1^2+3)$. Then by (4) we have

$$y_3 = \frac{2x_1y_1x_1 + (x_1^2 + Dy_1^2)y_1}{4} = \frac{y_1(3x_1^2 + Dy_1^2)}{4}$$

$$= \frac{y_1(3x_1^2 + x_1^2 + 4)}{4} = y_1(x_1^2+1).$$

So we have:

$$\begin{cases} x_3 = x_1(x_1^2+3) \\ y_3 = y_1(x_1^2+1) \end{cases}. \tag{12}$$

Then, using (4) and (12) we obtain:

$$x_5 = \frac{(x_1^2 + Dy_1^2)x_1(x_1^2+3) + 2Dx_1y_1^2(x_1^2+1)}{4}$$

$$= \frac{(2x_1^2+4)x_1(x_1^2+3) + 2x_1(x_1^2+4)(x_1^2+1)}{4}$$

$$= \frac{x_1}{2}((x_1^2+2)(x_1^2+3) + (x_1^2+4)(x_1^2+1))$$

$$= x_1(x_1^4 + 5x_1^2 + 5)$$

and

$$y_5 = \frac{2x_1y_1x_3 + (x_1^2 + Dy_1^2)y_3}{4}$$

$$= \frac{2x_1y_1x_1(x_1^2+3) + 2(x_1^2+2)y_1(x_1^2+1)}{4}$$

$$= \frac{y_1}{2}(x_1^2(x_1^2+3) + (x_1^2+2)(x_1^2+1)) = y_1(x_1^4+3x_1^2+1).$$

So we have.

$$\begin{cases} x_5 = x_1(x_1^4+5x_1^2+5) \\ y_5 = y_1(x_1^4+3x_1^2+1) \end{cases}. \quad (13)$$

Finally using (13) we obtain formulas for x_7 and y_7 depending on the fundamental solution (x_1, y_1).

$$x_7 = \frac{(x_1^2+Dy_1^2)x_5 + 2Dx_1y_1y_5}{4}$$

$$= \frac{2(x_1^2+2)x_1(x_1^4+5x_1^2+5) + 2x_1Dy_1^2(x_1^4+3x_1^2+1)}{4}$$

$$= \frac{(x_1^2+2)x_1(x_1^4+5x_1^2+5) + x_1^2(x_1^2+4)(x_1^4+3x_1^2+1)}{2}$$

$$= \frac{x_1}{2}((x_1^2+2)(x_1^4+5x_1^2+5) + (x_1^2+4)(x_1^4+3x_1^2+1))$$

$$= x_1(x_1^6+7x_1^4+14x_1^2+7)$$

and

$$y_7 = \frac{2x_1y_1x_5 + (x_1^2+Dy_1^2)y_5}{4}$$

$$= \frac{2x_1y_1x_1(x_1^4+5x_1^2+5) + 2(x_1^2+2)y_1(x_1^4+3x_1^2+1)}{4}$$

$$= \frac{y_1}{2}(x_1^2(x_1^4+5x_1^2+5) + (x_1^2+2)(x_1^4+3x_1^2+1))$$

$$= y_1(x_1^6+5x_1^4+6x_1^2+1).$$

So

$$\begin{cases} x_7 = x_1(x_1^6+7x_1^4+14x_1^2+7) \\ y_7 = y_1(x_1^6+5x_1^4+6x_1^2+1) \end{cases}. \quad (14)$$

Now replacing (12) and (13) in (11) we obtain:

$$x_7 = (x_1^2+1)(x_5+x_3) - x_1$$

$$= (x_1^2+1)(x_1(x_1^4+5x_1^2+5) + x_1(x_1^2+3)) - x_1$$

$$= x_1((x_1^2+1)(x_1^4+6x_1^2+8) - 1) = x_1(x_1^6+7x_1^4+14x_1^2+7)$$

and

$$y_7 = (x_1^2+1)(y_5+y_3) - y_1$$

$$= (x_1^2+1)(y_1(x_1^4+3x_1^2+1) + y_1(x_1^2+1)) - y_1$$

$$= y_1((x_1^2+1)(x_1^4+4x_1^2+2) - 1) = y_1(x_1^6+5x_1^4+6x_1^2+1)$$

which proves that for $n = 3$ the recurrence relations (6) are true.

Now we assume that (x_{2n+1}, y_{2n+1}) satisfies (6) and we prove that also (x_{2n+3}, y_{2n+3}) satisfies (6).

Indeed by (4) and hypothesis we obtain.

$$x_{2n+3} = \frac{(x_1^2+Dy_1^2)((x_1^2+1)(x_{2n-1}+x_{2n-3}) - x_{2n-5})}{4} +$$

$$+ \frac{2Dx_1y_1((x_1^2+1)(y_{2n-1}+y_{2n-3}) - y_{2n-5})}{4}$$

$$= (x_1^2+1)\frac{((x_1^2+Dy_1^2)(x_{2n-1}+x_{2n-3}) + 2Dx_1y_1(y_{2n-1}+y_{2n-3}))}{4} -$$

$$- \frac{(x_1^2+Dy_1^2)x_{2n-5} + 2Dx_1y_1y_{2n-5}}{4}$$

$$= (x_1^2+1)\left(\frac{(x_1^2+Dy_1^2)x_{2n-1} + 2Dx_1y_1y_{2n-1}}{4} + \right.$$

$$\left. + \frac{(x_1^2+Dy_1^2)x_{2n-3} + 2Dx_1y_1y_{2n-3}}{4}\right) - x_{2n-3}$$

$$= (x_1^2+1)(x_{2n+1}+x_{2n-1}) - x_{2n-3}$$

and

$$y_{2n+3} = \frac{2x_1y_1((x_1^2+1)(x_{2n-1}+x_{2n-3}) - x_{2n-5})}{4} +$$

$$+ \frac{(x_1^2+Dy_1^2)((x_1^2+1)(y_{2n-1}+y_{2n-3}) - y_{2n-5})}{4}$$

$$= (x_1^2+1)\left(\frac{2x_1y_1(x_{2n-1}+x_{2n-3})}{4} + \frac{(x_1^2+Dy_1^2)(y_{2n-1}+y_{2n-3})}{4}\right) -$$

$$- \frac{2x_1y_1x_{2n-5} + (x_1^2+Dy_1^2)y_{2n-5}}{4}$$

$$= (x_1^2+1)\left(\frac{2x_1y_1x_{2n-1} + (x_1^2+Dy_1^2)y_{2n-1}}{4}\right.$$

$$\left. + \frac{2x_1y_1x_{2n-3} + (x_1^2+Dy_1^2)y_{2n-3}}{4}\right) - y_{2n-3}$$

$$= (x_1^2+1)(y_{2n+1}+y_{2n-1}) - y_{2n-3},$$

completing the proof.

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