

# Simultaneous Term Structure Estimation of Hazard and Loss Given Default with a Statistical Model using Credit Rating and Financial Information

Tomohiro Ando, and Satoshi Yamashita

**Abstract**—The objective of this study is to propose a statistical modeling method which enables simultaneous term structure estimation of the risk-free interest rate, hazard and loss given default, incorporating the characteristics of the bond issuing company such as credit rating and financial information. A reduced form model is used for this purpose. Statistical techniques such as spline estimation and Bayesian information criterion are employed for parameter estimation and model selection. An empirical analysis is conducted using the information on the Japanese bond market data. Results of the empirical analysis confirm the usefulness of the proposed method.

**Keywords**—Empirical Bayes, Hazard term structure, Loss given default.

## I. INTRODUCTION

FACTORS such as the introduction of the second BIS regulation and diversification of direct financing activities have lead to increased recognition on the importance of credit risk quantification. There has been active development of mathematical models which estimate future default probability or loss given default. These credit risk measuring models can be categorized into two types; statistical models, such as the logit models, which are based on actual default data; and stochastic process models, such as the structural model and reduced form model, which are based on marketability data.

In recent studies, practical estimation accuracy for default probability is achieved using these models. However, based on a requirement to elaborate credit risk quantification, the necessity to estimate factors other than default probability is recognized. These are, for example, term structure of default probability (or hazard), loss given default, default correlations, and fluctuation of exposure. Improving the estimation accuracy of these parameters is effective not only for evaluating debt exposure but also for bond pricing and derivative development.

These parameters can be estimated simultaneously with some types of credit risk quantification models. For example,

the reduced form model [10,18], which is frequently used for bond pricing that relates to credit risk, assumes under the risk neutral measure that three factors (i.e., interest rate term structure, hazard term structure, loss given default) influence bond price, and estimates these parameters using various approaches to analyze the theoretical price of a bond. This model, however, is used to calculate default probability and loss given default is usually a given parameter.

When estimating the above credit risk related factors, diverse information such as the level, volatility, and term structure of the interest rate of corporate bonds as well as financial data of the corporation and rating information is used. However, parameter estimation is generally done independently; i.e., estimating interest rate term structure using price data of risk-free government bonds (or swap rate), or estimating hazard term structure using corporate bond data. Since diverse information is fragmented and estimation is done independently, the information is not necessarily leveraged effectively.

The objective of this study is to pull the divided information together and to propose a statistical modeling method which enables simultaneous term structure estimation of the risk-free interest rate, hazard and loss given default incorporating the characteristics of the bond issuing company (credit rating, financial information). The reduced form model was enhanced for this purpose. The benefit of using the reduced form model is that the model incorporates financial theory and can reflect / analyze the feature of actually observed data (i.e., formation of the expectation of market participants) at the same time. Furthermore, since the market includes credit risk premium in the process of corporate bond pricing, estimation results can be directly used for derivative pricing (for example, refer to [8]).

The rest of this paper is organized as follows. Section II discusses the basic concept of bond pricing and the reduced form model and introduces related researches. In Section III, a statistical modeling method with integrated information is proposed. In Section IV, effectiveness of the method is confirmed by implied estimation of interest rate term structure, hazard term structure, and loss given default using the data of the Japanese bond market. Section V discusses the estimation of hazard term structure and term structure of loss given

Tomohiro Ando is with the Graduate School of Business Administration, Keio University, 4-1-1 Hiyoshi, Kohoku-ku, Yokohama-shi, Kanagawa, 223-8526, Japan (Phone: +81-45-564-2039; e-mail: andoh@kbs.keio.ac.jp).

Satoshi Yamashita is with the Institute of Statistical Mathematics, 4-6-7 Minami-Azabu, Minato-ku, Tokyo 106-8569, Japan (e-mail: yamasita@ism.ac.jp).

default. Section VI gives conclusion.

## II. PRELIMINARIES

### A. An Overview of Related Researches

Before discussing models regarding credit risk, we briefly introduce related researches examining term structure. In general, estimation method for the term structure of interest rates can be categorized into two types, i.e., stochastic process model and statistical model, and both have long histories (for example, refer to [3]). The former model which attempts to estimate term structure of interest rates based on a stochastic process was started with a modeling of short term interest rates [27] and various models have been proposed including [4,6,16,37].

The advantage of stochastic process model is that the necessary computational intensity is low when an analytic solution is obtainable. However, since the stochastic process is easy to express mathematically, it should be used with caution in estimating the term structure of interest rates. Since these models assume a fairly simplified stochastic process in order to facilitate analysis, these simplifications prevent from expressing complex market fluctuations.

Statistical models, on the other hand, do not explicitly assume term structure but estimate term structure by directly fitting curves to the yield curve using asset price data. The advantages of using the statistical model is the fact that, while the model is based on financial theory, it can also reflect and analyze the formation of the expectations of market participants. One of the pioneer researchers on the term structure of interest rates is [25,26] who developed a method based on spline function. Others who proposed models include:[30,31,33,38].

Next we will discuss hazard term structure and Loss Given Default (LGD). As discussed in the next section, in order to evaluate discounted cash flow of bonds, it is necessary to focus on hazard term structure and LGD. As with the term structure of interest rates, there are two types of models to estimate hazard term structure; the statistical model which uses actual default data for analysis and the stochastic process model which uses market data. One of the pioneering studies using the statistical model is the research by [24] based on the survival model. For the stochastic process model, there are structural model [28] and reduced form model [10,18]. With regard to the structural model, it is defined that defaults occur when the variable representing business worth fall below a certain level and the ease in model estimation has lead to its use in financial applications. In the reduced form model, hazard itself is expressed using a stochastic process and its term structure is estimated using data observed in the market (e.g., bond price). A common example of the stochastic process used includes an average regression type stochastic process frequently applied in estimating the term structure of interest rates [9,19,20,23]. In the reduced form model, based on assumption of risk neutrality, the discounted future cash flow of the bond is formulated to balance with the current bond price and the hazard term

structure and LGD are impliedly derived from the bond price. LGD impliedly derived from the reduced form model is called the Market Implied LGD (i.e., LGD assumed by the bond market) and reflect the consensus of investors on LGD.

It is also important to note that there are many definitions for LGD. For example Workout LGD is based on the actual recovery value by creditors against a defaulting company, Market LGD is the hypothetical market value of a bond sold by the creditor after the default, Historical Implied LGD is based on past default and loss data, and Market Implied LGD which is based on the risk inherent in the bond indicated by the spread of the bond before default. For Workout LGD and Market LGD, while its estimation is technically difficult due to the lack of appropriate databases in Japan, there are adequate databases overseas and a number of results have been reported [1,2]. Nonetheless, as data related to Workout LGD is being accumulated and the default bond market is developing in Japan, models to estimate these LGDs are likely to be developed in the future.

Generally, it is considered difficult to simultaneously estimate the hazard term structure and Market Implied LGD solely from the bond price using the reduced form model framework and analysis is conducted by externally providing the hazard term structure or Market Implied LGD. Previous studies that have attempted simultaneous estimation include an estimation method using bond and share price data [17] and estimation method using bond and liability data [36].

### B. Preparation for the Proposed Model

All tables and figures you insert in your document are only to help you gauge the size of your paper, for the convenience of the referees, and to make it easy for you to distribute preprints. This chapter will discuss the relationship between future cashflow and bond pricing theory based on discounted cashflow. We consider an example based on a default risk free government coupon bond with a coupon amount of  $C$  and a face value of  $R$  (redeemed at maturity). Assume that the number of coupon payments between now and maturity is  $L$  (this includes the coupon payment on the maturity date,  $T_L$ ) and timing of payments is expressed as  $t = (t_1, \dots, t_L)'$ . As the current price of the government bond equals the sum of the discounted value of future cash flow, assuming  $r(t)$  is the term structure of the risk free instantaneous forward rate, the present value (PV) of a government bond with a maturity date  $T_L$  years from now can be expressed as:

$$PV(r(\cdot), t) = \sum_{i=1}^L C \cdot \exp\left\{-\int_0^{t_i} r(u) du\right\} + R \cdot \exp\left\{-\int_0^{T_L} r(u) du\right\}$$

In other words, the nominal cash flow is multiplied by the discount function  $d(t) = \exp\left\{-\int_0^t r(u) du\right\}$  to calculate the present value. The present value of the government bond is denoted as  $PV(r(\cdot), t)$  to emphasize its dependence on the timing of cashflow  $t$  and the risk free instantaneous forward rate  $r(t)$ .

In actual situations, it is necessary to take into consideration the impact of accrued or unearned interest as well as differences

in country specific practices such as the number of days in one year when calculating accrued interest.

Next, we examine the theoretical price of a corporate bond with a coupon amount of  $C$  and face value of  $R$  (redeemed at maturity). As with the previous example, the number of coupon payments between now and maturity is  $L$  (this includes the coupon payment on the maturity date,  $T_L$ ) and timing of payments is expressed as  $t = (t_1, \dots, t_L)'$ .

Generally, defaults occur at an unpredictable point in the future. Therefore, when pricing financial instruments with default risk, it is customary to use a method that expresses the future default date as a formulation of instantaneous rate of default (the fluctuation of default probability at dimensionless time). Assuming  $r$  is the random variable of the default date, the "likelihood" of default at each instant can be expressed as:

$$h(t) = \lim_{\Delta t \rightarrow 0} \frac{P(t < r \leq t + \Delta | t < r)}{\Delta}.$$

This  $h(t)$ , called the hazard term structure, is the standardization of "the conditional probability of a bond which have not defaulted up till time  $t$  and defaulting within the next interval  $t + \Delta$ " when  $\Delta$  is dimensionless time. The un-default probability of a bond with the above hazard term structure not defaulting up to date  $t$  can be expressed as:

$$P(r > t) = \exp \left\{ - \int_0^t h(u) du \right\}.$$

In the reduced form model [10,18],  $h(t)$  is expressed as a stochastic process and  $P(r > t)$  is evaluated as an expected value of the risk neutral measure. In this example, we assume the hazard term structure is given.

Furthermore, when the bond defaults before maturity, the discounted cash flow of the bond, given the hazard term structure  $h(t)$  and LGD,  $\delta$ , can be expressed as the following formula

$$\begin{aligned} PV(r(\cdot), h(\cdot), \delta, t) = & \sum_{i=1}^L C \cdot \exp \left[ - \int_0^{t_i} \{r(u) + h(u)\} du \right] \\ & + R \cdot \exp \left[ - \int_0^{t_L} \{r(u) + h(u)\} du \right] \\ & + R \cdot \delta \cdot \int_0^{t_L} \left[ h(t) \cdot \exp \left\{ - \int_0^t \{r(u) + h(u)\} du \right\} \right] dt \end{aligned} \quad (1)$$

assuming that  $\delta$  of the face value is recoverable upon default of the bond (generally,  $\delta$  is called LGD) and there is no coupon payments after default. The first term is the discounted present value of coupon payments; the second term is the discounted present value of the face value redemption payment (the case in which there was no default); and the third term is the discounted present value of the recovered amount (the case in which there was default). Compared with the government bond, for the discount function  $d(t)$  used to discount the nominal cash flow to calculate the present value, we use the discount function  $d(t) = \exp \left[ - \int_0^t \{r(u) + h(u)\} du \right]$  which

incorporates the LGD adjusted hazard term structure. In Appendix A, we provide a theoretical derivation of the formula

(1).

The above discussion shows that the discounted cash flow of government bonds can be derived if the risk free instantaneous forward rate  $r(t)$  is given and that the discounted cash flow of corporate bonds containing default risk can be derived if the hazard term structure  $h(t)$  and LGD  $\delta$  are available as well. However, in actual situations, this information is not directly observable and must be estimated using some type of market data and mathematical methods. The next section introduces a new method to estimate this information.

### III. METHODOLOGY

#### A. Formulation of the Model

This section will examine the proposed simultaneous estimation model for default term structure and LGD using the bond pricing model discussed in section II.B. Assume that the following information is available from transactions on a specific day; prices of government bonds  $P_i^g$ , prices of corporate bonds  $P_i^{cp}$ , and their respective accrued interests  $a_i^g, a_i^{cp}$ . The total number of government bond data is  $n_0$ . Each corporate bond is rated as 1, ..., J respectively and the total number of rating divisions is J. The total number of corporate bond data in each rating division is represented as  $n_1, \dots, n_j$ . The financial information for the issuer of each corporate bond is also available ( $x = (x_1, \dots, x_p)$ ). Parameters that must be estimated are term structure of interest rates  $r(t)$ , the hazard term structure  $h_j(t; x)$  by rating division which incorporates the company's financial information  $x$ , and LGD  $\delta_j(x)$  ( $j = 1, \dots, j$ ). The proposed model will estimate these parameters simultaneously.

It is extremely rare for the actual bond price to match the theoretical price discussed in the previous section. Therefore, in order to explain the formation of expectation of the market participants, the discounted present value of the government bond  $PV^g$  and of the corporate bond  $PV^{cp}$  are expressed by the following statistical models:

$$\begin{cases} \text{Government bond pricing model} \\ P_{i_0}^g + a_{i_0}^g = PV_{i_0}^g(r(\cdot), t_{i_0}) + \varepsilon_{i_0}^g, \\ \text{Corporate bond pricing model for the 1st credit rating} \\ P_{i_1}^{cp} + a_{i_1}^{cp} = PV_{i_1}^{cp}(r(\cdot), h_1(\cdot, x_{i_1}), \delta_1(x_{i_1}), t_{i_1}) + \varepsilon_{i_1}^1 \\ \cdot \\ \text{Corporate bond pricing model for the J-th credit rating} \\ P_{i_j}^{cp} + a_{i_j}^{cp} = PV_{i_j}^{cp}(r(\cdot), h_j(\cdot, x_{i_j}), \delta_j(x_{i_j}), t_{i_j}) + \varepsilon_{i_j}^J, \end{cases} \quad (2)$$

where noise  $\varepsilon_{i_0}^g, \varepsilon_{i_1}^1, \dots, \varepsilon_{i_j}^J$  are independent from each other and follow a normal distribution with mean 0 and variance  $\sigma_0^2, \sigma_1^2, \dots, \sigma_j^2$ . Since the timing of the cash flow  $t$  is different for each bond,  $t_{i_0}$  denotes the timing of the cash flow for government bond numbered  $i_0$ , and  $t_{i_j}$  denotes the timing of the cash flow for corporate bond numbered  $i_j$  in the rating

division  $j$ . The parameter  $x_{ij}$  in the bond pricing model (where rating is  $j$ ) denotes the financial information of the company issuing the bond.

In this paper, the following structure is assumed in order to flexibly estimate the term structure of interest rates  $r(t)$  and the hazard term structure  $h_j(t, x)$  ( $j = 1, \dots, J$ ) by rating which incorporates the financial information of the issuing company.

$$r(t, w_0) = \sum_{k=1}^{m_0} w_{0k} \phi_{0k}(t) = w'_0 \phi_0(t),$$

$$h_j(t, x, w_j, \beta_h) = \left[ \sum_{k=1}^{m_j} w_{jk} \phi_{jk}(t) \right] \cdot \exp \left\{ \sum_{\ell=1}^p \beta_{h\ell} x_{\ell} \right\} \quad (3)$$

$$= [w'_j \phi_j(t)] \cdot \exp \{ \beta'_h x \}, \quad (j = 1, \dots, J)$$

where  $\phi_j(t) = (\phi_{j1}(t), \dots, \phi_{jm_j}(t))'$  is a known basis function vector composed of basis functions and  $w_j = (w_{j1}, \dots, w_{jm_j})'$  and  $\beta_h = (\beta_{h1}, \dots, \beta_{hp})'$  are the unknown parameters to be estimated. This function plots the term structure (yield curve) as a smooth curve. The hazard term structure is also expressed as a smooth curve but its level will shift vertically depending on the financial ratio  $x$ . For the hazard term structure, the term structure is expressed as a linear sum of linear independent basis functions and the impact of financial information is expressed as its linear sum. As a result, this is the same as using the linear sum of basis functions for baseline hazard in the proportional hazard model [5].

For LGD by rating  $\delta_j(x)$  ( $j = 1, \dots, J$ ), it is assumed that the remaining years to maturity  $z$  will have an influence as well as the company's rating and financial information  $x$ . For LGD  $\delta_j(x)$  ( $j = 1, \dots, J$ ), logit transformation to limit LGD within a generally acceptable range ( $\delta \in [0.1]$ ) is applied to arrive at the following structure:

$$\delta_j(x, z, \beta_{\delta}, \gamma, \alpha) = \frac{\exp \left\{ \sum_{k=1}^{m_{\delta}} \gamma_{\delta k} \phi_{\delta k}(j) + \sum_{\ell=1}^p \beta_{\delta \ell} x_{\ell} + \alpha z \right\}}{1 + \exp \left\{ \sum_{k=1}^{m_{\delta}} \gamma_{\delta k} \phi_{\delta k}(j) + \sum_{\ell=1}^p \beta_{\delta \ell} x_{\ell} + \alpha z \right\}}$$

$$= \frac{\exp \{ \gamma' \phi_{\delta}(j) + \beta'_{\delta} x + \alpha z \}}{1 + \exp \{ \gamma' \phi_{\delta}(j) + \beta'_{\delta} x + \alpha z \}}, \quad (j = 1, \dots, J), \quad (4)$$

where  $\phi_{\delta}(j) = (\phi_{\delta 1}(j), \dots, \phi_{\delta m_{\delta}}(j))'$  is the given basis function vector composed of basis functions and  $\beta_{\delta} = (\beta_{\delta 1}, \dots, \beta_{\delta p})'$ ,  $\gamma = (\gamma_1, \dots, \gamma_{m_{\delta}})'$ , and  $\alpha$  are the parameters to be estimated.

The method to estimate the term structure of interest rates from cross-sectional data of coupon-bearing bonds using regression analysis based on basis function expansion has been widely established both in academic and practical applications since it was first proposed by [25, 26]. There have been various basis functions proposed including quadratic spline basis [25], cubic spline basis [26], Bernstein polynomial [31] and exponential spline [38].

We will use B-spline [13, 33] which has been the focus of attention due to its usefulness. Fig. 1 is an illustration of the cubic B-spline basis function. Each basis function  $\phi_j(t)$  is composed of piecewise polynomials which are smoothly connected (in a sense that second order derivatives are continuous) at (equally distributed) points  $t_j$  called knots. The

B-spline basis function can be constructed using [7]'s sequential algorithm as:

$$\phi_j(x, 0) = \begin{cases} 1, & t_j \leq x < t_{j+1}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\phi_j(x, p) = \frac{x - t_j}{t_{j+p} - t_j} \phi_j(x, p-1) + \frac{t_{j+p+1} - x}{t_{j+p+1} - t_{j+1}} \phi_{j+1}(x, p-1)$$

where  $\phi_j(x, p)$  is the order of an  $p$  order B-spline basis function. Since the zero order B-spline basis function is a constant between two knots, it is easy to construct a B-spline basis function of any order. Hereinafter,  $\phi_j(x, 3) = \phi_j(x)$ .

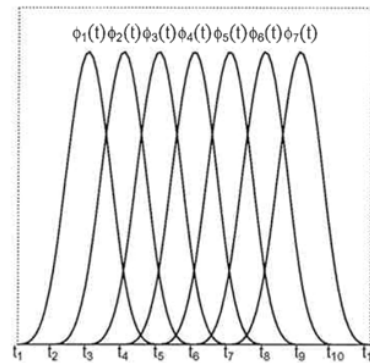


Fig. 1 An example of a cubic B-spline basis function

Under the premise of structures of formulas (3) and (4), the discounted present value of the government bond  $PV^g$  and discounted present value of the  $j$ -rated corporate bond  $PV^{cp}$  can be expressed by the following formulas (for both bonds, assume that the number of coupon payments between now and maturity is  $L$  (this includes the coupon payment on the maturity date,  $T_L$ )).

$$PV^g(w_0, t) = \sum_{i=1}^L C \cdot \exp \{ -w'_0 \psi_0(t_i) \} + R \cdot \exp \{ -w'_0 \psi_0(t_L) \}$$

$$PV^{cp}(x, z, w_0, w_j, \beta_h, \beta_{\delta}, \gamma, \alpha, t) \quad (5)$$

$$= \sum_{i=1}^L C \cdot \exp \{ -w'_0 \psi_0(t_i) - w'_j \psi_j(t_i) \exp(\beta'_h x) \}$$

$$+ R \cdot \exp \{ -w'_0 \psi_0(t_L) - w'_j \psi_j(t_L) \cdot \exp(\beta'_h x) \}$$

$$+ R \cdot \delta_j(x, z, \beta_{\delta}, \gamma, \alpha) \cdot \{ w'_j \bar{\psi}_j(t_L, w_0, w_j, \beta) \cdot \exp(\beta'_h x) \}.$$

In the above equations,  $t = (t_1, \dots, t_L)'$  is the timing of payments, and also  $\psi_j(t) = (\psi_{j1}(t), \dots, \psi_{jm_j}(t))'$  and  $\bar{\psi}_j(t, w_0, w_j, \beta) = (\bar{\psi}_{j1}(t, w_0, w_j, \beta), \dots, \bar{\psi}_{jm_j}(t, w_0, w_j, \beta))'$  are  $m_j$  dimensional vectors which are respectively composed of the following formulas.

$$\psi_{jk}(t) = \int_0^t \phi_{jk}(u) du,$$

$$\bar{\psi}_{jk}(t, w_0, w_j, \beta) = \int_0^t [\phi_{jk}(u) \cdot \exp \{ -w'_0 \psi_0(u) - w'_j \psi_j(u) \exp(\beta'_h x) \}] du.$$

The bond pricing model can be formulated as a normally distributed density function by using the statistical model (2) and the bond pricing model (5).

Government bond pricing model

$$f(y_{i_0}^g | t_{i_0}; w_0, \sigma_0^2) = \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp \left[ -\frac{\{y_{i_0}^g - \text{PV}_{i_0}^g(w_0, t_{i_0})\}^2}{2\sigma_0^2} \right],$$

Corporate bond pricing model for the 1st credit rating

$$f(y_{i_1}^{cp} | t_{i_1}, x_{i_1}, z_{i_1}; w_0, w_1, \beta_h, \beta_\delta, \gamma, \alpha, \sigma_0^2) = \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp \left[ -\frac{\{y_{i_1}^{cp} - \text{PV}_{i_1}^{cp}(x_{i_1}, z_{i_1}, w_0, w_1, \beta_h, \beta_\delta, \gamma, \alpha, t_{i_1})\}^2}{2\sigma_1^2} \right], \quad (6)$$

Corporate bond pricing model for the J-th credit rating

$$f(y_{i_j}^{cp} | t_{i_j}, x_{i_j}, z_{i_j}; w_0, w_j, \beta_h, \beta_\delta, \gamma, \alpha, \sigma_j^2) = \frac{1}{\sqrt{2\pi\sigma_j^2}} \exp \left[ -\frac{\{y_{i_j}^{cp} - \text{PV}_{i_j}^{cp}(x_{i_j}, z_{i_j}, w_0, w_j, \beta_h, \beta_\delta, \gamma, \alpha, t_{i_j})\}^2}{2\sigma_j^2} \right],$$

where  $u_{i_0}^g = p_{i_0}^g + a_{i_0}^g$ ,  $y_{i_j}^{cp} = p_{i_j}^{cp} + a_{i_j}^{cp}$  ( $j = 1, \dots, J$ ) is the bond transaction price which incorporates accrued interest and  $z_{i_j}$  contained in corporate bond pricing model (J-rated bond) is the remaining length of time until maturity. Consequently, the model estimates the parameters  $\theta = (w', \sigma', \beta_h', \beta_\delta', \gamma', \alpha)'$ ,  $w = (w_0', \dots, w_J')'$  and  $\sigma = (\sigma_0^2, \sigma_1^2, \dots, \sigma_J^2)'$  by entering the company's financial data  $x$ , the current government bond price including accrued interest  $p_{i_0}^g + a_{i_0}^g$ , the current corporate bond price including accrued interest  $p_{i_j}^{cp} + a_{i_j}^{cp}$ , their respective timing of payments, length to maturity  $z$ , and the rating for each corporate bond  $j$  ( $j = 1, \dots, R$ ). By entering the estimated parameters in formula (3) and (4), we arrive at the term structure of interest rates, hazard term structure by rating which incorporates the company's financial information  $x$ , and LGD. The next section will present a method to estimate parameter  $\theta$  contained in the model.

### B. Parameter Estimation

This section A possible method to estimate parameters in the bond pricing model based on the B-spline basis function (6) is the maximum likelihood method and the maximum likelihood estimate can be obtained by maximizing the log likelihood.

$$\ell(\theta) = \sum_{i_0=1}^{n_0} \log f(y_{i_0}^g | t_{i_0}; w_0, \sigma_0^2) + \sum_{j=1}^J \sum_{i_j=1}^{n_j} \log f(y_{i_j}^{cp} | t_{i_j}, x_{i_j}, z_{i_j}; w_0, w_j, \beta_h, \beta_\delta, \gamma, \alpha, \sigma_j^2) \quad (7)$$

However, depending on the design of the model, the maximum likelihood estimate can become unstable leading to large fluctuations in the term structure of interest rates and the hazard term structure by rating which incorporates the financial information of the issuing company. Consequently, this method is not readily acceptable in practice. The parameter estimation is also unstable when there is multi-collinearity or an outlier in the financial ratio. Furthermore, it is suggested that the impact of ratings on the market participants' perception on LGD are

similar among adjacent rating divisions.

Taking the above facts into consideration, the parameter  $\theta$  in the bond pricing model can be estimated by maximizing the penalized log-likelihood.

$$\ell_\lambda(\theta) = \ell(\theta) - \sum_{j=0}^J \frac{n\lambda_j}{2} w_j' K_j w_j - \frac{n\lambda_\gamma}{2} \gamma' K_\gamma \gamma - \frac{n\lambda_{\beta_h}}{2} \beta_h' \beta_h - \frac{n\lambda_{\beta_\delta}}{2} \beta_\delta' \beta_\delta. \quad (8)$$

where  $n = \sum_{j=0}^J n_j$  is the total number of transaction data for government bonds and corporate bonds and  $\lambda_j$  is the smoothing parameter.  $K_j$  is a  $m_j \times m_j$  order differential matrix which can be expressed as  $K_j = D_j' D_j$  using the following  $(m_j - k) \times m_j$  order matrix;

$$D_{jk} = \begin{pmatrix} (-1)^0 C_0 & \dots & (-1)^k C_k & 0 & \dots & 0 \\ 0 & (-1)^0 C_0 & \dots & (-1)^k C_k & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & (-1)^0 C_0 & \dots & (-1)^k C_k \end{pmatrix}$$

where  ${}_k C_i$  is a binomial coefficient. The penalty based on a differential matrix has a penalizing effect on the curvature of the B-spline curve (second derivatives) resulting in limiting extreme fluctuations in the term structures of interest rates and hazard. The similarity of LGD between adjacent ratings can be expressed by introducing a penalty for parameter  $\gamma$ . The second derivative penalty is usually used in the estimation of statistical models based on B-spline for penalized maximum likelihood method and studies on its theory can be found in papers including [12,14]

In maximizing the penalized log-likelihood, the information on the cumulative default rate by rating published by the rating companies is also used. For example, assuming that the actual cumulative default rate one year from now for rating  $j$  is given as  $p(1) = P(r \leq 1)$ , the parameter  $w_j$  in the baseline hazard function for rating  $j$  is restricted to satisfy the following equation.

$$\exp\{-w_j' \psi_j(1)\} = 1 - p(1) \quad (9)$$

It is important to note that  $w_j \in R^{m_j - \gamma_j}$  portion of the parameter  $w_j$  not related to  $p(1)$  is not restricted. The reason for the restriction is as follows:

Investors of corporate bonds are not interested in the level of hazard and LGD themselves but rather in the bond price that takes both of the two factors into account. Therefore, the information contained in the corporate bond transaction price can be considered to reflect the influence of the level of hazard and LGD simultaneously and therefore, transaction price alone is insufficient for simultaneously estimating the two factors. Generally with the reduced form model, the level of hazard or LGD is provided externally to enable estimation of the other. The restriction is placed, therefore, to evade the problem of inseparability between hazard and LGD in the reduced form model. For the above reason, the analysis assumes that the baseline hazard function level matches the cumulative default probability of the rating company.

The parameter  $\theta$  can be estimated by numerical optimization and the bond pricing model can be completed by entering the

estimated parameter in formula (6). The estimated interest rate / hazard term structure and LGD can be impliedly derived from the corporate bond market. That is to say, it is the estimated results in a risk neutral world and can be directly used in the pricing of derivatives.<sup>15</sup> However, it is important to note that the market risk premium is also contained and therefore, the level of hazard term structure in the actual world is lower than that of a risk neutral world by a margin of the risk premium.

The bond pricing model (6) based on penalized maximum likelihood method is dependent on the number of basis functions and the value of the smoothing parameter. Therefore, it is important to select these values appropriately. In particular, selecting the appropriate smoothing parameter enables more accurate estimation of the term structure of interest rates and the hazard term structure incorporating the financial information of the bond issuing company (i.e., the formation of market participants' expectation). The next section will cover the establishment of a model assessment criteria and the method for selecting the number of basis functions and the smoothing parameter.

### C. Selecting the Number of Basis Functions and the Smoothing Parameter

Models such as the penalized maximum likelihood method which do not use the maximum likelihood method for model estimation can be assessed using criteria such as the Bayesian information criterion [21], generalized information criterion [20], cross validation [34] and bootstrap method [11]. Since the bond pricing model (6) proposed in this paper is estimated by numerical optimization, any statistical resampling methods would require enormous hours of computation. To avoid this problem, a criterion expressed as analytical form, in particular the Bayesian information criterion (BIC), is used here.

In an empirical Bayes approach, a prior distribution of  $\pi(\theta|\zeta)$  is assumed for the parameter  $\theta$  in the model, where  $\zeta$  is a hyperparameter vector. Also, for this study,  $\zeta$  corresponds to  $m$  (the number of basis functions) and the smoothing parameter  $\lambda$ . Assuming that the prior probability of the model is equal, the model is selected to maximize the following marginal likelihood in the empirical Bayes approach.

$$\int \exp\{\ell(\theta)\} \pi(\theta|\zeta) d\theta$$

The empirical Bayes approach, proposed by [32] has been successfully applied in a wide range of areas since its introduction. However, Schwarz's [32] empirical Bayes approach is a criterion to assess models estimated by the maximum likelihood method. Therefore, there are many theoretical problems when it is used to assess models estimated by the penalized maximum likelihood method. In order to overcome this problem, [21] calculated the above integral equation using the Laplace approximation [35] in order to derive the BIC for assessing models based on the penalized maximum likelihood method.

From the standpoint of Bayes theory, using the penalized maximum likelihood method corresponds to the assumption

that the prior distribution of parameter  $\theta$  is the following normal distribution [21].

$$\begin{aligned} \pi(\theta|\zeta) = & \left( \frac{n\lambda_0}{2\pi} \right)^{(m_0-2)/2} |K_0|_+^{1/2} \exp \left\{ -\frac{n\lambda_0}{2} w'_0 K_0 w_0 \right\} \\ & \times \prod_{j=1}^J \left( \frac{n\lambda_j}{2\pi} \right)^{(m_j-r_j-2)/2} |K_j|_+^{1/2} \exp \left\{ -\frac{n\lambda_j}{2} w'_j K_j w_j \right\} \\ & \times \left( \frac{n\lambda_\gamma}{2\pi} \right)^{(m_\gamma-2)/2} |K_\gamma|_+^{1/2} \exp \left\{ -\frac{n\lambda_\gamma}{2} \gamma' K_\gamma \gamma \right\} \\ & \times \left( \frac{n\lambda_{\beta_s}}{2\pi} \right)^{p/2} \exp \left\{ -\frac{n\lambda_{\beta_s}}{2} \beta'_s \beta_s \right\} \times \left( \frac{n\lambda_{\beta_\gamma}}{2\pi} \right)^{p/2} \exp \left\{ -\frac{n\lambda_{\beta_\gamma}}{2} \beta'_\gamma \beta_\gamma \right\} \end{aligned}$$

where  $|K_0|_+$  is the product of an eigenvalue (other than zero) of the matrix  $K_0$  and  $|K_j|_+$  is the product of an eigenvalue (other than zero) of the matrix  $K_j$  related to the parameter segment unrestricted by the integer of the formula (9). By assuming that the prior distribution of the model is equal and applying the Laplace approximation [35] to the model's posterior distribution, BIC [21] based on Bayes theory can be derived from the following equation.

$$\begin{aligned} \text{BIC} = & -2\ell(\hat{\theta}) + n\lambda_0 \hat{w}'_0 K_0 \hat{w}_0 + n \sum_{j=1}^J \lambda_j \hat{w}'_j K_j \hat{w}_j + n\lambda_\gamma \hat{\gamma}' K_\gamma \hat{\gamma} + n\lambda_{\beta_s} \hat{\beta}'_s \hat{\beta}_s \quad (10) \\ & + n\lambda_{\beta_\gamma} \hat{\beta}'_\gamma \hat{\beta}_\gamma - \sum_{j=0}^J \log |K_j|_+ - \log |K_\gamma|_+ + \log |J(\hat{\theta})| - (m_0-2) \log \lambda_0 \\ & - \sum_{j=1}^J (m_j-r_j-2) \log \lambda_j - (m_\gamma-2) \log \lambda_\gamma - p \log (\lambda_{\beta_s} \lambda_{\beta_\gamma}) + \text{Const} \end{aligned}$$

where Const is a constant,  $J(\hat{\theta}) = -\frac{1}{n} \frac{\partial^2 \ell_\lambda(\hat{\theta})}{\partial \theta \partial \theta'}$ ,

common to all models. The optimal number of basis functions and the smoothing parameter can be selected by minimizing the BIC derived.

Although it is ideal to optimize the number of basis functions and the smoothing parameter, in practice, the computational intensity becomes significant as the number of rating divisions ( $J$ ) increases. In the modeling using the B-spline basis, it is possible to achieve results that fairly reflect the data structure by providing a sufficient number of basis functions and then optimizing only the smoothing parameter [12,13]. Consequently, one possible method to reduce the computational intensity is to provide a sufficient number of basis functions and then optimizing only the smoothing parameter. This method was used for empirical analysis in this paper.

## IV. EMPIRICAL ANALYSIS

In this section, we will conduct an implied estimation on the term structure of risk free interest rates, hazard term structure (characterized by rating and financial ratios) and LGD using data of the Japanese bond market as well as evaluate the validity of the proposed model.

### A. Data

For the Japanese bond market data, we used the "Statistical information on over the counter bond trading" published by Japan Securities Dealers Association. As a sample of the trading data for one day, we used the analyzed results of the data for 1417 issues traded on July 5, 2004. For rating data, we used Moody's rating information. Since it is necessary to

specify the total number of ratings ( $J$ ) for the analysis using the bond pricing model (6), the total number was set as 5. The groups are composed of the following ratings: Investment grade companies were categorized in to 4 groups: Sector 1 (containing Aaa, Aa1, Aa2 and Aa3; 276 issues), Sector 2 (containing A1 and A2; 242 issues), Sector 3 (containing A3 and Baa1; 229 issues) and Sector 4 (containing Baa2 and Baa3; 415 issues). Since most investors regard issues Ba1 and below as non-investment grade, these issues were categorized as Sector 5 (containing Ba1 and below; 199 issues). In addition, data of Japanese government bonds was available for 249 issues. Ideally, analysis should be based on a rating division that reflect notches (Aaa, Aa1, Aa2, Aa3, A1, ..., C). However, while it is possible to estimate the parameter, the reliability of the results is significantly reduced when the sample size for each division is small.

For the cumulative default rate  $p(t)$  in formula (9), the average cumulative default rate for the period 1983 to 2001 published by Moody's (Average Cumulative Default Rates by Alpha-Numeric Rating (2000)) was used. In this analysis, the average cumulative default rate within one year for the ratings Aa2, A1, A3, Baa2 and Ba2 were used respectively for sectors 1 to 5.

While there are potentially many financial ratios, the two ratios strongly related to ratings and LGD were used in the study. One is the cash flow sales ratio ( $x_1$ ) which is one of the ratios to measure profitability and efficiency and the other is the capital adequacy ratio ( $x_2$ ) which measures the possibility of a capital deficit. Data on ratings were obtained from Moody's website and the financial data was obtained from the "e-Aurora" database.

### B. Results of the Analysis

In the analysis, number of basis functions for the term structure of interest rates  $r(t, w_0)$ , hazard term structure by rating  $h_j(t, x)$  which incorporates the financial information of the bond issuer and the LGD by rating  $\delta_j(x, \beta_\delta, \gamma, \alpha)$  were determined respectively as  $m_0 = 8$ ,  $m_j = 4$  ( $j = 1, \dots, 5$ ), and  $m_\delta = 4$ . Under these conditions, the smoothing parameter was optimized by minimizing BIC. By calculating BIC using various smoothing parameters, the optimal smoothing parameter was determined as  $\lambda_0 = 0.01, \lambda_1 = 0.1, \lambda_2 = 0.1, \lambda_3 = 0.1, \lambda_4 = 0.01, \lambda_5 = 0.1, \lambda_\gamma = 0.01, \lambda_{\beta_\delta} = 0.01, \lambda_{\beta_\delta} = 0.01$ .

On the other hand, when estimating the model using the maximum likelihood method (this was achieved by setting all smoothing parameters as zero), the parameter could not be estimated due to the likelihood divergence. This also indicates that more stable model estimation is possible with the penalized maximum likelihood method.

The term structure of interest rates for government bonds and for corporate bonds (LGD adjusted) is shown in Figs. 3 and 4. The rates shown in Figs. 3 and 4 are instantaneous forward rates and zero coupon yields (spot rates). This yield curve is derived from the LGD adjusted discount function:

$$d_0(t) = \exp \left\{ \int_0^t r(u; \hat{w}_0) du \right\}$$

$$d_j(t) = \exp \left[ - \int_0^t \left\{ r(u; \hat{w}_0) + h_j(t; x, \hat{w}_j, \hat{\beta}_h) \right\} du \right],$$

In other words, it is derived from the relationship between the discount function  $d(t)$  and the instantaneous forward rate  $r(t)$  ( $r(t) = -d'(t)/d(t)$ ), as well as the relationship between the discount function  $d(t)$  and the zero coupon yield  $\eta(t)$  ( $\eta(t) = -\log d(t)/t$ ), where the effect of the financial ratios are eliminated by setting  $\exp\{\hat{\beta}_h'x\} = 1$  for each bond rating division.

As shown in Figs. 2 and 3, the level of the yield curve is the lowest for government bonds. The pattern of the estimated instantaneous forward rate for government bonds shows a bump around the years 7 and 8. However, as the spot rate around year 10 is low, it is suggested that the bump is an impact of the benchmark bond effect.<sup>22</sup> Furthermore, risk premium increases with lower rating and in particular, the instantaneous forward rate consistently increases for the low rated Sector 5 (Ba1 or below).

Figs. 4 and 5 show the estimated term structure of baseline hazard  $h_{0j}(t) = \sum_{k=1}^{m_j} \hat{w}_{jk} \phi_{jk}(t)$  and the term structure of the default probability derived from the baseline hazard. Looking at the baseline hazard term structure in Fig. 5, hazard term structures for sectors with high ratings (Sector 1 to 3) maintain low levels throughout the period. Nonetheless, the hazard shows a gradual upward trend which suggests that future uncertainty is taken into consideration. Furthermore, the hazard of the low rating Sector 5 follows a different pattern from those of the high rated sectors. It is on the rise from the period  $t=0$  to around year 4, then falls until year 6 then rises again.

Next, the term structure of default probability derived from the estimated baseline hazard was compared with the average cumulative default rate for the period from 1983 to 2001 published by Moody's [29]. The term structure of default probability for the sectors 1 to 5 were compared with the average cumulative default rates for the ratings used in the estimation, Aa2, A1, A3, Baa2 and Ba2 (Fig. 6). Looking at the data shown in Fig. 6, the term structure of default rates for sectors 1 to 4 are roughly the same with Moody's average cumulative default rates. However, the term structure of default rate for sector 5 is slightly lower than Moody's average cumulative default rate.

The coefficient  $\beta_h$  for the hazard function in formula (3) is also examined. In formula (3), when the coefficient weighting the financial ratio is positive, an increase in the value of the financial ratio results in a larger hazard. On the other hand, when the coefficient weighting the financial ratio is negative, an increase in the value of the financial ratio results in a smaller hazard. The estimated results were; cash flow sales ratio,  $\hat{\beta}_{h1} = -0.12527(0.0004)$  ; capital adequacy ratio,  $\hat{\beta}_{h2} = -0.0054(0.0003)$  (where the value in the parenthesis

is standard deviation). This indicates that companies that have smaller values for these ratios tend to have larger hazards.

Fig. 7 shows the boxplots of estimated LGD for each sector and Table I shows the average LGD and estimated error variance. By comparing the estimated error variance, the data indicates that the market holds a fairly common view towards bonds with high ratings. However, the error variance of default-free government bonds  $\sigma_0^2$  is less than half of that of below investment grade Sector 5  $\sigma_5^2$ . There are two explanations for this: One is that the estimations of the market fluctuate more for higher risk bonds, another is the fact that all below investment grade bonds were lumped in one sector (Sector 5).

The impact of rating on LGD in formula (4) was as follows;  $\hat{\gamma}'\phi_\delta(1) = 1.40, \hat{\gamma}'\phi_\delta(2) = 0.99$ ,  $\hat{\gamma}'\phi_\delta(3) = 0.53, \hat{\gamma}'\phi_\delta(4) = 0.41, \hat{\gamma}'\phi_\delta(5) = 0.27$ . LGD decreases with the decrease on ratings as indicated in Fig. 9. Hamilton et al. (2001) analyzed the actual LGD for bonds which defaulted between 1981 and 2000. Their results indicate that LGD is higher for companies with high ratings and they suggest ratings can be used to forecast LGD.

The coefficients  $\beta_\delta$  to weigh financial ratios were as follows; for the coefficient for the cash flow sales ratio,  $\hat{\beta}_{\delta_1} = 0.0386(0.0058)$ ; and for the coefficient for the capital adequacy ratio,  $\hat{\beta}_{\delta_2} = 0.0285(0.0054)$ . The value in the parenthesis is standard deviation. In formula (4), if the coefficient weighting the financial ratio is positive, an increase in the value of the financial ratio results in a larger LGD. On the other hand, if the coefficient weighting the financial ratio is negative, an increase in the value of the financial ratio results in a smaller LGD. The estimated results indicate that LGD is higher for companies with larger values for both the cash flow sales ratio and capital adequacy ratio. Furthermore, the coefficient for the years remaining to maturity  $z$  was estimated as  $\hat{\alpha} = -0.267(0.0074)$ . The value in the parenthesis is standard deviation. If we examine this coefficient using the same logic applied to coefficients to weigh financial ratios, the results suggest that LGD is lower for bonds with a longer period to maturity.

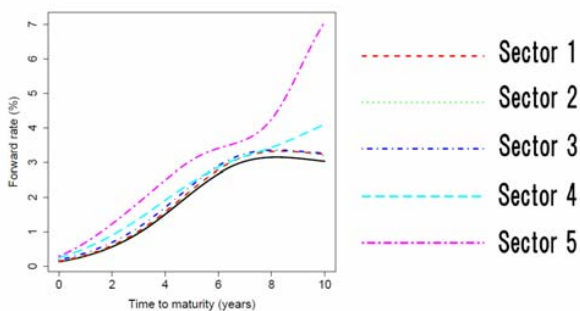


Fig. 2 Instantaneous forward rate of government bonds and LGD adjusted instantaneous forward rate for each sector

TABLE I  
ESTIMATED ERROR VARIANCE AND ESTIMATED AVERAGE LGD AND ITS STANDARD DEVIATION

	Error variance	1-LGD	<sup>a</sup> Standard deviation
Government bond	0.327	----	-----
Sector 1	0.379	0.808	0.063
Sector 2	0.411	0.743	0.043
Sector 3	0.413	0.551	0.164
Sector 4	0.687	0.539	0.149
Sector 5	0.856	0.413	0.133

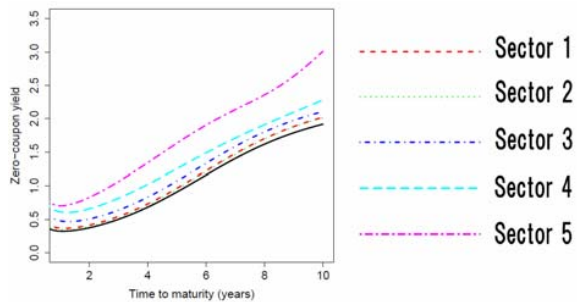


Fig. 3 Zero coupon yield of government bonds and LGD adjusted zero coupon yield for each sector

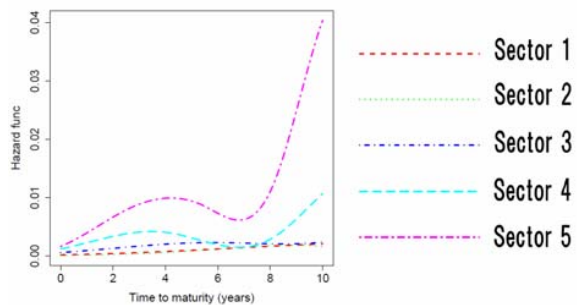


Fig. 4 Term structure of baseline hazard

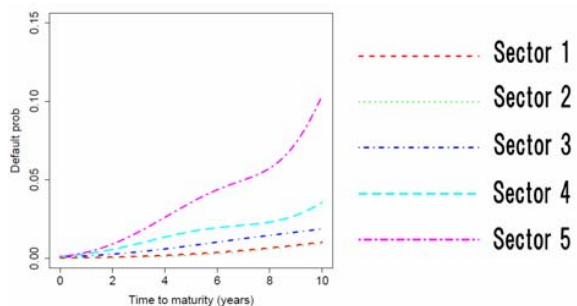


Fig. 5 Term structure of default probability derived from baseline hazard



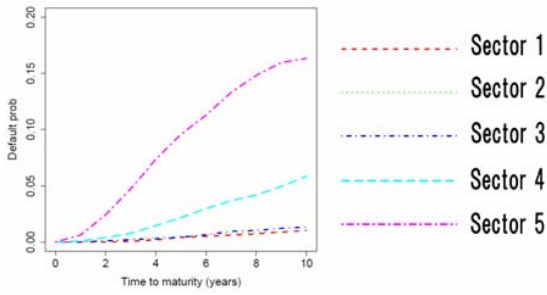


Fig. 6 Term structure of average cumulative default rate for ratings Aa2, A1, A3, Baa2 and Baa3

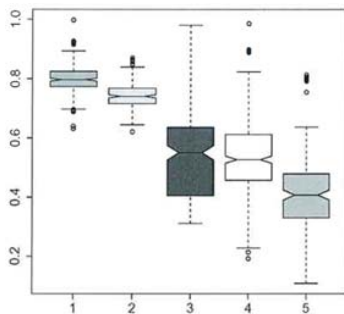


Fig. 7 Boxplots of LGD for Sector 1 (Aaa, Aa1, Aa2 and Aa3), Sector 2 (A1 and A2), Sector 3 (A3 and Baa1), Sector 4 (Baa2 and Baa3) and Sector 5 (Ba1 and below)

#### V. ESTIMATING THE TERM STRUCTURE OF LGD

In the previous sections, LGD was assumed to be constant throughout the period. However, we can also introduce LGD term structure into the estimations. LGD in Formula (4) can be expressed as:

$$\delta_j(t, x, \beta_\delta, \gamma, b) = \frac{\exp\left\{\sum_{k=1}^{m_j} \gamma_{jk} \phi_{\delta k}(j) + \sum_{\ell=1}^p \beta_{\delta \ell} x_\ell + \sum_{u=1}^{m_g} b_u \phi_{tu}(t)\right\}}{1 + \exp\left\{\sum_{k=1}^{m_j} \gamma_{jk} \phi_{\delta k}(j) + \sum_{\ell=1}^p \beta_{\delta \ell} x_\ell + \sum_{u=1}^{m_g} b_u \phi_{tu}(t)\right\}}$$

$$= \frac{\exp\{\gamma' \phi_\delta(j) + \beta'_\delta x + b' \phi_t(t)\}}{1 + \exp\{\gamma' \phi_\delta(j) + \beta'_\delta x + b' \phi_t(t)\}}, \quad (j = 1, \dots, J)$$

where  $\phi_\delta(t) = (\phi_{1j}(t), \dots, \phi_{m_g j}(t))'$  is a known basis function composed of basis functions and vector  $b = (b_1, \dots, b_m)$  is the parameter to be estimated.

If using LGD in formula (11), the third term in the discounted present value of the bond in formula (1) can be expressed as:

$$\sum_{k=1}^K P(t_k < r \leq t_k + \Delta) \cdot \left[ R \cdot \delta(t_k) \cdot \exp\left\{-\int_0^{t_k} r(u) du\right\} \right]$$

$$= R \cdot \left[ \sum_{k=1}^K \delta(t_k) \cdot P(t_k < r \leq t_k + \Delta | t_k < r) \cdot P(t_k < r) \cdot \exp\left\{-\int_0^{t_k} r(u) du\right\} \right]$$

$$= R \cdot \left[ \sum_{k=1}^K \delta(t_k) \cdot \frac{P(t_k < r \leq t_k + \Delta | t_k < r)}{\Delta} \cdot \exp\left\{-\int_0^{t_k} h(u) du\right\} \cdot \exp\left\{-\int_0^{t_k} r(u) du\right\} \cdot \Delta \right]$$

$$= R \cdot \int_0^{\infty} \left[ \delta(t) \cdot h(t) \cdot \exp\left\{-\int_0^t \{r(u) + h(u)\} du\right\} \right] dt \quad (\Delta \rightarrow 0)$$

Using this equation, the discounted present value of the bond in formula (5) can be modified and combined with formula (2)

to formulate a bond pricing model similar to formula (6).

By adding a second derivative penalty, which reduces extreme fluctuations in the term structure of LGD, to the penalized log-likelihood in formula (8), the parameter can be estimated through maximization. If we use the same argument as for BIC in section III.C, the assessment criterion for a model including the term structure of LGD can be derived.

Below are the results of the analysis on 1554 issues for February 24, 2005. As in section IV, the issues were grouped into sectors using the Moody's rating. Sector 1 (containing Aaa, Aa1, Aa2 and Aa3; 267 issues), Sector 2 (containing A1 and A2; 316 issues), Sector 3 (containing A3 and Baa1; 320 issues), Sector 4 (containing Baa2 and Baa3; 276 issues) and Sector 5 (containing Ba1 and below; 90 issues). In addition, data of Japanese government bonds was available for 285 issues. Cash flow sales ratio and capital adequacy ratio were used as financial ratios and the number of basis functions for LGD in formula (11) was set as  $m_{t_j} = 5$ .

Fig. 8 shows the estimated term structure of LGD. LGD decreases with the decrease in ratings and the increase in the time to maturity. However, we need to consider the fact that bond investors focus on bond price that simultaneously reflect the two factors, level of hazard and LGD. This means that although estimating the term structure of LGD is possible, its reliability is questionable. One possible solution is to introduce information on actual LGD but this was not possible due to difficulty in obtaining such data. This is an area for future study.

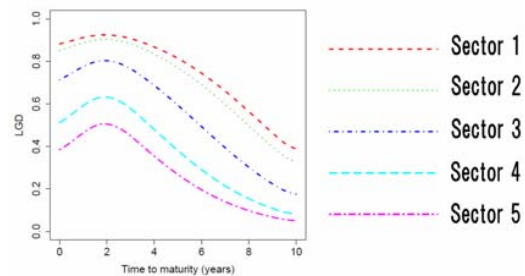


Fig. 8 Term structure of LGD of a bond issued by a company in a specific sector Term structure of LGD of a bond issued by the following companies in a specific sector. Sector 1: TOYOTA, Sector 2: SONY, Sector 3: NISSAN, Sector 4: FUJITSU and Sector 5: SOJITZ

#### VI. CONCLUSION

This paper proposed a statistical model that combined market information to simultaneously estimate term structure of interest rates, hazard term structure which incorporates the characteristics of the issuing company (rating, financial information) and LGD by rating. Generally, simultaneous estimation of these parameters was considered difficult but the method proposed in this paper has proved otherwise. An empirical analysis using data of the Japanese bond market indicated the link between falling ratings / weakening financial conditions and deteriorating structure of default probability /

LGD. This indicates that the market considers rating and financial data as important criteria.

Thus study can be carried forward in three areas: First, the paper assumed that the hazard term structure of the bond issuing company was dependent on its rating and financial information but possible areas for improvement include the use of share price information or the introduction of default correlation. Second, for LGD, risk factors specific to individual bonds such as senior / junior structure can be introduced into the model. Third, since hazard term structure and LGD by rating are estimated under risk neutral measure in this study, some adjustments would be necessary to estimate hazard term structure in the actual world. These areas are left to be discussed in future studies.

#### APPENDIX

The third term in  $PV(r(\cdot), h(\cdot), \delta, t)$  denotes the discounted present value of the future cash flow when the bond defaults by the redemption period  $T_L$  and the recovery is  $\delta$  of face value  $R$ . First divide the period from  $t = 0$  to the redemption period  $t_L$  into equal dimensionless time periods ( $\Delta_1 = (t_1, t_1 + \Delta]$ ,  $\Delta_2 = (t_2, t_2 + \Delta]$ , ...,  $\Delta_K = (t_K, t_K + \Delta]$ ) totaling  $K$  (assume;  $t_1 = 0$ ,  $t_L = t_K + \Delta$ ,  $t_k = t_{k-1} + \Delta$ ). Also assume  $\delta$  of face value  $R$  is recoverable when default occurs in  $\Delta_k$  and that the discounted present value of the future cash flow can be approximated as  $R \cdot \delta \cdot \exp\left\{-\int_0^{t_k} r(u) du\right\}$ . In this case, the sum of the nominal cash flow  $R \cdot \delta$  up till the redemption period  $t_L$ , incorporating the default occurrence probability  $P(t_k < r \leq t_k + \Delta)$  in each dimensionless time period  $\Delta_k$  is the discounted present value of the nominal cash flow that will occur in the future when the bond defaults by the redemption period  $t_L$ .

$$\begin{aligned} & \sum_{k=1}^K P(t_k < r \leq t_k + \Delta) \cdot \left[ R \cdot \delta \cdot \exp\left\{-\int_0^{t_k} r(u) du\right\} \right] \\ &= R \cdot \delta \cdot \left[ \sum_{k=1}^K P(t_k < r \leq t_k + \Delta | t_k < r) \cdot P(t_k < r) \cdot \exp\left\{-\int_0^{t_k} r(u) du\right\} \right] \\ &= R \cdot \delta \cdot \left[ \sum_{k=1}^K \frac{P(t_k < r \leq t_k + \Delta | t_k < r)}{\Delta} \cdot \exp\left\{-\int_0^{t_k} h(u) du\right\} \cdot \exp\left\{-\int_0^{t_k} r(u) du\right\} \cdot \Delta \right] \end{aligned}$$

The third term in  $PV(r(\cdot), h(\cdot), \delta, t)$  is derived by decreasing  $\Delta_k$  towards zero ( $K \rightarrow \infty$ ).

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**Tomohiro Ando** (BS'00-MS'02—Ph.D'04) is an Associate Professor of Graduate School of Business Administration, Keio University. He received B.S., M.S., and Ph.D. in Mathematics from Kyushu University in 2000, 2002 and 2004, respectively. His current research interests include the development of Bayesian statistical modelling method, marketing science and financial econometrics. He has research papers published in *Biometrika*, *Journal of Statistical inference*, *Annals of institute of statistical mathematics*, *Computational statistics and data analysis* and in many journals.

**Satoshi Yamashita** (BS'87-MS'89—Ph.D'97) is an Associate Professor of Institute of Statistical Mathematics, Japan. His current research interest is financial econometrics. He received B.S., M.S., and Ph.D. in Mathematics from Kyoto University in 2000, 2002 and 2004, respectively.