International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:4, No:7, 2010

On Finite Hjelmslev Planes of parameters (p^{k-1}, p)

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***Abstract*—In this paper, we study on finite projective Hjelmslev planes $\mathbf{M}(\mathbf{Z}_q)$ coordinatized by Hjelmslev ring $\mathbf{Z}_q$ (where prime power $q = p^k$). We obtain finite hyperbolic Klingenberg planes from these planes under certain conditions. Also, we give a combinatorical result on $\mathbf{M}(\mathbf{Z}_q)$, related by deleting a line from lines in same neighbour.

***Keywords*—finite Klingenberg plane, finite hyperbolic Klingenberg plane.

I. Introduction

Projective Klingenberg and Hjelmslev planes (more briefly: PK-planes and PH-planes, resp.) are generalizations of ordinary projective planes. These structures were introduced by Klingenberg in [17], [18]. Finite PK-planes introduced by Drake and Lenz in [10] have been studied in detail by Bacon in [5]. As for finite hyperbolic Klingenberg planes (briefly, HK-planes), these structures introduced by Celik in [8].

In our previous papers [1], [9] we have studied a certain class (which we will denote by $\mathbf{M}(\mathcal{A})$) of Moufang-Klingenberg (briefly, MK) planes coordinatized by a local alternative ring $\mathcal{A} := \mathbf{A}(\varepsilon) = \mathbf{A} + \mathbf{A}\varepsilon$ of dual numbers (an alternative ring $\mathbf{A}$, $\varepsilon \notin \mathbf{A}$ and $\varepsilon^2 = 0$) introduced by Blunck in [7]. Besides, in the papers of [2], [3] we have dealed with finite Klingenberg planes of parameters (p^{2k-1}, p), coordinatized by a local ring $\mathbf{Z}_q + \mathbf{Z}_q\varepsilon$ which is not a H-ring.

In the present paper we are interested in finite PK-planes $\mathbf{M}(\mathbf{Z}_q)$ coordinatized by local ring $\mathbf{Z}_q$ (where q is a prime power), which is also an H-ring. So, we will show that the planes obtained by deleting m equivalence classes of lines (which are pairwise non-neighbour lines such that no three of them are concurrent) from the finite PK-plane $\mathbf{M}(\mathbf{Z}_q)$ are examples of finite hyperbolic Klingenberg planes, in the sense of [8]. Finally, we give a combinatorical result on $\mathbf{M}(\mathbf{Z}_q)$, related by deleting a line from lines in same neighbour.

II. Preliminaries

Let $\mathbf{M} = (\mathbf{P}, \mathbf{L}, \in, \sim)$ consist of an incidence structure $(\mathbf{P}, \mathbf{L}, \in)$(points, lines, incidence) and an equivalence relation '$\sim$' (neighbour relation) on $\mathbf{P}$ and on $\mathbf{L}$. Then $\mathbf{M}$ is called a *projective Klingenberg plane* (PK-plane), if it satisfies the following axioms:

(PK1) If P, Q are two non-neighbour points, then there is a unique line PQ through P and Q.

(PK2) If g, h are two non-neighbour lines, then there is a unique point $g \wedge h$ on both g and h.

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(PK3) There is a projective plane $\mathbf{M}^* = (\mathbf{P}^*, \mathbf{L}^*, \in)$ and incidence structure epimorphism $\Psi : \mathbf{M} \to \mathbf{M}^*$, such that the conditions

$$\Psi(P) = \Psi(Q) \Leftrightarrow P \sim Q,\ \Psi(g) = \Psi(h) \Leftrightarrow g \sim h$$

hold for all $P, Q \in \mathbf{P}$, $g, h \in \mathbf{L}$.

PK-plane $\mathbf{M}$ is called a *projective Hjelmslev plane* (PH-plane) If $\mathbf{M}$ furthermore provides the following axioms:

(PH1) If P, Q are two neighbour points, then there are at least two lines through P and Q.

(PH2) If g, h are two neighbour lines, then there are at least two points on both g and h.

A *Moufang-Klingenberg plane* (MK-plane) is a PK-plane $\mathbf{M}$ that generalizes a Moufang plane, and for which $\mathbf{M}^*$ is a Moufang plane (for the details see [4]).

A point $P \in \mathbf{P}$ is called *near* a line $g \in \mathbf{L}$ iff there exists a line h such that $P \in h$ for some line $h \sim g$.

Now we give the definition of an n-gon, which is meaningful when $n \geq 3$: An n-tuple of pairwise non-neighbour points is called an (ordered) *n-gon* if no three of its elements are on neighbour lines [9].

Let $\mathbf{M} = (\mathbf{P}, \mathbf{L}, \in, \|, \sim)$ consist of an incidence structure $(\mathbf{P}, \mathbf{L}, \in)$(points, lines, incidence), an equivalence relation '$\|$' on $\mathbf{L}$ (parallelism relation) and also an equivalence relation '$\sim$' (neighbour relation) on $\mathbf{P}$ and on $\mathbf{L}$. Then $\mathbf{M}$ is called a *hyperbolic Klingenberg plane* (HK-plane), if it satisfies the following axioms:

(HK1) If P, Q are two non-neighbour points, then there is a unique line PQ through P and Q.

(HK2) If $l \in \mathbf{L}$, then there is $P, Q \in l$ such that P is not neighbour to Q.

(HK3) There exists at least one 4-gon.

(HK4) For each point-line pair (P, l) where P is not near to l, there are at least two non-neighbour lines through P parallel to l.

(HK5) There is a hyperbolic plane $\mathbf{M}^* = (\mathbf{P}^*, \mathbf{L}^*, \in)$ and incidence structure epimorphism $\Psi : \mathbf{M} \to \mathbf{M}^*$, such that the conditions

$$\Psi(P) = \Psi(Q) \Leftrightarrow P \sim Q, \quad \forall P, Q \in \mathbf{P}$$
$$\Psi(g) = \Psi(h) \Leftrightarrow g \sim h, \forall g, h \in \mathbf{L}$$

and if there is no point on both g and h then $\Psi(g) \parallel \Psi(h)$.

An *alternative ring* (*field*) $\mathbf{R}$ is a not necessarily associative ring (field) that satisfies the alternative laws $a(ab) = a^2b$, $(ba)a = ba^2$, $\forall a, b \in \mathbf{R}$. An alternative ring $\mathbf{R}$ with identity element 1 is called *local* if the set $\mathbf{I}$ of its non-unit elements is an ideal.

We summarize some basic concepts about the coordinatization of MK-planes from [6].

Let $\mathbf{R}$ be a local alternative ring. Then $\mathbf{M}(\mathbf{R}) = (\mathbf{P}, \mathbf{L}, \in, \sim)$ is the incidence structure with neighbour relation defined as follows:

$$\begin{aligned}
\mathbf{P} &= \{(x,y,1) \mid x,y \in \mathbf{R}\} \\
&\quad \cup \{(1,y,z) : y \in \mathbf{R}, z \in \mathbf{I}\} \\
&\quad \cup \{(w,1,z) : w,z \in \mathbf{I}\}, \\
\mathbf{L} &= \{[m,1,p] : m,p \in \mathbf{R}\} \\
&\quad \cup \{[1,n,p] : p \in \mathbf{R}, n \in \mathbf{I}\} \\
&\quad \cup \{[q,n,1] : q,n \in \mathbf{I}\} \\
[m,1,p] &= \{(x, xm+p, 1) : x \in \mathbf{R}\} \\
&\quad \cup \{(1, zp+m, z) : z \in \mathbf{I}\}, \\
[1,n,p] &= \{(yn+p, y, 1) : y \in \mathbf{R}\} \\
&\quad \cup \{(zp+n, 1, z) : z \in \mathbf{I}\}, \\
[q,n,1] &= \{(1, y, yn+q) : y \in \mathbf{R}\} \\
&\quad \cup \{(w, 1, wq+n) : w \in \mathbf{I}\}
\end{aligned}$$

and

$$\begin{aligned}
P &= (x_1,x_2,x_3) \sim (y_1,y_2,y_3) = Q \\
&\Leftrightarrow \ x_i - y_i \in \mathbf{I} \ (i = 1,2,3)), \forall P,Q \in \mathbf{P}; \\
g &= [x_1,x_2,x_3] \sim [y_1,y_2,y_3] = h \\
&\Leftrightarrow \ x_i - y_i \in \mathbf{I} \ (i = 1,2,3)), \forall g,h \in \mathbf{L}.
\end{aligned}$$

For more detailed information about the coordinatization see [4] and [6].

Now it is time to give the following theorem from [4].

Theorem 2.1: $\mathbf{M}(\mathbf{R})$ is an MK-plane, and each MK-plane is isomorphic to some $\mathbf{M}(\mathbf{R})$.

Moreover, we can state the following, from [11, Theorem 4.1].

Theorem 2.2: If $\mathbf{R}$ is a (not necessarily commutative) local ring then $\mathbf{M}(\mathbf{R})$ is a PK-plane.

Drake and Lenz [10, Proposition 2.5] observed that the following corollary is true for PK-planes. This corollary is a generalization of results which are given for PH-planes by Kleinfeld [16, Theorem 1] and Lüneburg [19, Satz 2.11].

Corollary 2.3: Let $\mathbf{M}(\mathbf{R})$ be PK-plane. Then there are natural numbers t and r which are called the parametres of $\mathbf{M}(\mathbf{R})$ and they are uniquely determined by incidence structure of a finite PK-plane [10, Proposition 2.7], with

1) *every point (line) has t^2 neighbours;*
2) *given a point P and a line l with $P \in l$, there exist exactly t points on l which are neighbours to P and exactly t lines through P which are neighbours to l;*
3) Let r be order of the projective plane $\mathbf{M}^*$. If $t \neq 1$ we have $r \leq t$ (then $\mathbf{M}$ is called *proper*; we have $t = 1$ iff $\mathbf{M}$ is an ordinary projective plane)
4) *every point (line) is incident with $t\,(r+1)$ lines (points);*
5) $|\mathbf{P}| = |\mathbf{L}| = t^2\left(r^2+r+1\right)$.

Now consider ring $\mathbf{Z}_q$ where prime power $q = p^k$. We can state the elements of $\mathbf{Z}_q$ as $\mathbf{Z}_q = \mathbf{U}' \cup \mathbf{I}$ where $\mathbf{U}'$ is the set of units of $\mathbf{Z}_q$ and $\mathbf{I}$ is the set of non-units of $\mathbf{Z}_q$. Here it is clear that $\mathbf{I} = \left\{0p, 1p, 2p, ..., \left(p^{k-1}-1\right)p\right\} = p\mathbf{Z}_q$ for all p primes, and so $|\mathbf{I}| = p^{k-1}$. Since $\mathbf{Z}_q$ is a proper local ring and $\mathbf{Z}_q/\mathbf{I} = \mathbf{Z}_p$, Ψ induces an incidence structure epimorphism from finite PK-plane $\mathbf{M}(\mathbf{Z}_q)$ onto the Desarguesian projective plane (with order p) coordinatized by the field $\mathbf{Z}_p$ [11, page 169, above Theorem 4.1]. So, we have the following

Corollary 2.4: For finite PK-plane $\mathbf{M}(\mathbf{Z}_q)$, the parameters t and r in Corollary 2.3 are equal to p^{k-1} and p, respectively.

A local ring $\mathbf{R}$ is called a *Hjelmslev ring* (briefly, H-ring) if it satisfies the following two conditions:

(HR1) $\mathbf{I}$ consists of two-sided zero divisor.

(HR2) For $a, b \in \mathbf{I}$, one has $a \in b\mathbf{R}$ or $b \in a\mathbf{R}$, and also $a \in \mathbf{R}b$ or $b \in \mathbf{R}a$.

By the last definition, we can say that $\mathbf{Z}_q$ is an H-ring with maximal ideal $\mathbf{I}$ [11, example 4.8.b].

Now it is time to give the following theorem from [17].

Theorem 2.5: Let $\mathbf{R}$ be a local ring; $\mathbf{M}(\mathbf{R})$ be the Desarguesian PK-plane over $\mathbf{R}$. Then $\mathbf{R}$ is an H-ring if and only if $\mathbf{M}(\mathbf{R})$ is a PH-plane.

So, we can immediately say that $\mathbf{M}(\mathbf{Z}_q)$ is a PH-plane. Now we give the following theorem from [12].

Theorem 2.6: Let $\mathbf{R}$ be a finite H-ring with maximal ideal $\mathbf{I}$, and write $r = |\mathbf{R}/\mathbf{I}|$. Then $|\mathbf{R}| = r^n$ for some positive integer n, and $\mathbf{M}(\mathbf{R})$ is an n-uniform PH-plane.

By the last theorem, we can say that $\mathbf{M}(\mathbf{Z}_q)$ is a k-uniform PH-plane since $|\mathbf{Z}_q/\mathbf{I}| = p$ and $|\mathbf{Z}_q| = p^k$.

From now on we assume char $\mathbf{Z}_q \neq \mathbf{2}$ and also we restrict ourselves to finite PH-plane $\mathbf{M}(\mathbf{Z}_q) = (\mathbf{P}, \mathbf{L}, \in, \sim)$ coordinatized by H-ring $\mathbf{Z}_q$, with neighbour relation defined above.

III. Construction of Finite Hyperbolic Klingenberg Planes

It is well known that if a line is deleted from a projective plane then the remaining substructure forms an affine plane. Sandler [20] showed that if three non-concurrent lines are deleted from a projective plane then the remaining structure forms a hyperbolic plane in the sense of Graves [13]. Sandler's construction is extended by Kaya-Özcan [15], and it is obtained that result: if m lines no three of which are non-concurrent lines are deleted from a projective plane then the remaining structure forms a hyperbolic plane. Now we will adopt the method of [15] to obtain a finite hyperbolic Klingenberg plane from finite PK-plane $\mathbf{M}(\mathbf{Z}_q)$.

In $\mathbf{M}(\mathbf{Z}_q)$, let $l_1, l_2, l_3, \cdots, l_m$ be pairwise non-neighbour lines such that no three of them are concurrent. $\mathbf{M}(\mathbf{Z}_q)_m =$

$(\mathbf{P}_m, \mathbf{L}_m, \in, \sim)$ be substructure obtained from $\mathbf{M}(\mathbf{Z}_q)$ by deleting all lines l_i with the points which are near to l_i for $i = 1, 2, 3, \cdots, m$ and $m \geq 3$ is any natural number.

Some combinatorics properties of $\mathbf{M}(\mathbf{Z}_q)_m$ are given in [8, Section III and IV]. By [8, Lemma 3.1] we give some basic combinatorical properties which are valid in $\mathbf{M}(\mathbf{Z}_q)_m$. So, we have the following corollary.

Corollary 3.1: Following properties are valid in $\mathbf{M}(\mathbf{Z}_q)_m$ where $m \leq p + 2$:

1) *Two non-neighbour points of* $\mathbf{M}(\mathbf{Z}_q)_m$ *are on exactly one line.*
2) *Through each point of* $\mathbf{M}(\mathbf{Z}_q)_m$ *there pass exactly* $p^{k-1}(p+1)$ *lines and there pass exactly* $p+1$ *pairwise non-neighbour lines of* $\mathbf{M}(\mathbf{Z}_q)_m$.
3) *There are exactly* $\left(p^{k-1}\right)^2 \left(p^2 + p + 1 - m\right)$ *lines and there are exactly* $p^2 + p + 1 - m$ *pairwise non-neighbour lines in* $\mathbf{M}(\mathbf{Z}_q)_m$.
4) *There are exactly* $\left(p^{k-1}\right)^2 p^2 + \frac{\left(p^{k-1}\right)^2}{2}(m-1)(m-2p-2)$ *points and there are exactly* $p^2 + \frac{1}{2}(m-1)(m-2p-2)$ *pairwise non-neighbour points in* $\mathbf{M}(\mathbf{Z}_q)_m$.

Now we give the following definition from [8].

A point of $\mathbf{M}(\mathbf{Z}_q)$ is called *corner point* if it is an intersection point of any two non-neighbour lines in the set of deleted lines.

By this definition and its combinatoric properties [8, from Lemma 3.2 to Propositon 3.1] we can say that $\mathbf{M}(\mathbf{Z}_q)_m$ is an example of finite hyperbolic Klingenberg Planes. The following corollary is related to this [8, Proposition 3.1].

Corollary 3.2: Let n be the minimum number of pairwise non-neighbour corner points on a line of $\mathbf{M}(\mathbf{Z}_q)_m$. If $3 \leq m \leq p + n + \frac{1}{2}(1 - \sqrt{4p+5})$, then $\mathbf{M}(\mathbf{Z}_q)_m$ is a finite hyperbolic Klingenberg Plane.

Now we would like to give a combinatoric result on $\mathbf{M}(\mathbf{Z}_q)$, related to the number of points on the remaining lines in this neighbour when it is deleted a line with the points near to this line from lines in same neighbour, by using some incidence matrices of $\mathbf{M}(\mathbf{Z}_q)$.

Lines in same neighbour intersect at p^i points where $1 \leq i \leq k-1$ to neighbour of a certain point on the lines. The table I gives more detailed information about this. For example, in $\mathbf{M}(\mathbf{Z}_{3^3})$, lines in same neighbour intersect at 3 or 3^2 points to neighbour of a certain point on the lines. The number of the lines which intersect at 3 or 3^2 points are 18 or 2, respectively.

For finite PH-plane $\mathbf{M}(\mathbf{Z}_{p^k})$, we obtain that generalization: "the number of lines which intersect at p^i points where $1 \leq i \leq k-1$ to neighbour of a certain point on the lines is $(p-1)p^{2(k-1-i)}$".

If we extend this calculation to all points of a line, by considering $p+1$ points on a line, then we reach the number

TABLE I
INTERSECTION OF LINES IN THE SAME NEIGHBOUR

	p	p^2	p^3	$\cdots$	p^{k-1}
$\mathbf{M}(\mathbf{Z}_{2^2})$	$1(p-1)$	.	.	$\cdots$	.
$\mathbf{M}(\mathbf{Z}_{2^3})$	4	1	.	.	$\cdots$
$\mathbf{M}(\mathbf{Z}_{2^4})$	2.2^3	2.2	1	$\cdots$	.
$\mathbf{M}(\mathbf{Z}_{3^2})$	$2(p-1)$	.	.	$\cdots$	.
$\mathbf{M}(\mathbf{Z}_{3^3})$	3.6	2	.	$\cdots$	.
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\mathbf{M}(\mathbf{Z}_{p^k})$	$(p-1)p^{2(k-2)}$	$(p-1)p^{2(k-3)}$	$(p-1)p^{2(k-4)}$	$\cdots$	$p-1$

of lines in a neighbour, that is,

$$\begin{aligned} & (p+1)\sum_{i=1}^{k-1}(p-1)p^{2(k-1-i)} \\ = \; & (p+1)\,(p-1)p^{2(k-1)}\sum_{i=1}^{k-1}p^{-2i} \\ = \; & \left(p^2-1\right)p^{2(k-1)}\frac{p^{-2}}{1-p^{-2}} \\ = \; & \left(p^2-1\right)p^{2(k-1)}\frac{1}{(p^2-1)} \\ = \; & p^{2(k-1)} = \left(p^{k-1}\right)^2. \end{aligned}$$

So, as a result, we can say that lines in same neighbour intersect at points in same property to neighbour of any point. For lines in same neighbour, the number of lines which intersect at p points is

$$(p+1)\,(p-1)p^{2(k-2)} = \left(p^2-1\right)p^{2(k-2)},$$

the number of lines which intersect at p^2 points is

$$(p+1)\,(p-1)p^{2(k-3)} = \left(p^2-1\right)p^{2(k-3)},$$

the number of lines which intersect at p^3 points is

$$(p+1)\,(p-1)p^{2(k-4)} = \left(p^2-1\right)p^{2(k-4)},$$

the number of lines which intersect at p^{k-2} points is

$$(p+1)\,(p-1)p^{2(k-k+1)} = \left(p^2-1\right)p^2,$$

and finally the number of lines which intersect at p^{k-1} points is

$$(p+1)\,(p-1)p^{2(k-k)} = \left(p^2-1\right).$$

Now, we can give some results about the number of points on the remaining lines when any one (with points near to this line) of lines in same neighbour deletes from the neighbour. Of lines in same neighbour, it is deleted that

p points of $\left(p^2-1\right)p^{2(k-2)}$ lines,

p^2 points of $\left(p^2-1\right)p^{2(k-3)}$ lines,

p^3 points of $\left(p^2-1\right)p^{2(k-4)}$ lines,

$\cdots$

p^{k-2} points of $\left(p^2-1\right)p^2$ lines, and finally

$$p^{k-1} \text{ points of } p^2-1 \text{ lines.}$$

So, considering that any line of the plane contains $(p+1)\,p^{k-1}$ points, the number of points on remaining lines in same neighbour is that:

$$\left(p^2-1\right)p^{2(k-2)} \text{ lines contain } (p+1)\,p^{k-1}-p,$$
$$\left(p^2-1\right)p^{2(k-3)} \text{ lines contain } (p+1)\,p^{k-1}-p^2,$$
$$\left(p^2-1\right)p^{2(k-4)} \text{ lines contain } (p+1)\,p^{k-1}-p^3,$$
$$\ldots$$
$$\left(p^2-1\right)p^{2} \text{ lines contain } (p+1)\,p^{k-1}-p^{k-2},$$

and finally

$$p^2-1 \text{ lines contain } (p+1)\,p^{k-1}-p^{k-1}.$$

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