The inverse eigenvalue problem via orthogonal matrices

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Abstract—In this paper we study the inverse eigenvalue problem for symmetric special matrices and introduce sufficient conditions for obtaining nonnegative matrices. We get the HROU algorithm from [1] and introduce some extension of this algorithm. If we have some eigenvectors and associated eigenvalues of a matrix, then by this extension we can find the symmetric matrix that its eigenvalue and eigenvectors are given. At last we study the special cases and get some remarkable results.

Keywords—Householder matrix; Nonnegative matrix; Inverse eigenvalue problem.

I. INTRODUCTION

AT first we recall some lemmas and theorems from [1].

Lemma I.1. . *If $(\lambda_1, \lambda_2, \cdots \lambda_n)$ are eigenvalues of $n \times n$ matrix A and (λ_k, u) is a particular eigenpair, then for each arbitrary vector $v \in \Re^n$ the eigenvalues of matrix $A + uv^T$ agree with those of A, except that λ_k is replaced by $\lambda_k + v^T u$.*

Theorem I.2. *Let A be an $n \times n$ arbitrary matrix with eigenvalues $\lambda_1, \lambda_2, \cdots \lambda_n$. Let $X = [X_1|X_2|\cdots|X_r]$ be such that $rank(X) = r$ and $AX_i = \lambda_i X_i$, $i = 1, 2, \cdots, r$, $r \le n$. Let C be an $r \times n$ arbitrary matrix, then the matrix $A + XC$ has eigenvalues $\mu_1, \mu_2, \cdots \mu_r, \lambda_{r+1}, \lambda_{r+2}, \cdots, \lambda_n$, where $\mu_1, \mu_2, \cdots \mu_r$ are eigenvalues of the matrix $\Omega + CX$ with diagonal matrix $\Omega = \text{diag}\{\lambda_1, \lambda_2, \cdots \lambda_r\}$.*

Theorem I.3. *Let $\{(\lambda_i, u_i)\}_{i=1}^n$ be eigenpairs of an $n \times n$ matrix A, $u_i^T u_j = 0$ for any $i \ne j$, and let $v_k = \frac{(\mu - \lambda_k) u_k}{u_k^T u_k}$, then $\{(\lambda_i, u_i)\}_{i=1, i \ne k}^n$ and (μ, u_k) are eigenpairs of matrix $A + u_k v_k^T$.*

Theorem I.4. *Let $A = xx^T$, where $x \in \Re^n$, $x > 0$, and $Hx = \| x \|_2 e_1$, then*

1) all the columns of H constitute an orthogonal eigenvectors set of A,

2) $h_1 = x/ \| x \|_2^2$ and $x^T x h_1 h_1^T = A$.

Lemma I.5. *The matrices $h_1 h_1^T, h_2 h_2^T, \ldots, h_n h_n^T$ are linearly independent, if h_i is the i-th column of a Householder matrix H.*

Recall the following problem from [1] .
Problem 1. Find a real symmetric matrix $B \in \Re^{n \times n}$ with prescribed spectrum $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, and a positive eigenvector x.

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Algorithm 1. (HROU : Householder - based rank one updating)
1) Input x, $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$; initial $B = 0$;
2) Compute the Householder matrix, s.t. $Hx =\| x \|_2 e_1$;
3) for $i = 1, 2, \ldots, n$
4) $B \leftarrow B + \lambda_i h_i h_i^T$
5) end for
6) output B, λ_1, h_1.

In above algorithm for given spectrum $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and the positive eigenvector x we obtain the symmetric matrix $B = \sum_{i=1}^n \lambda_i h_i h_i^T$, where h_i is the ith column of Householder matrix $H = I - \frac{2}{v^T v} vv^T$ and $v = x- \| x \|_2 e_1$.
In fact by useing algorithm 1 matrix $B = \sum_{i=1}^n \lambda_i h_i h_i^T$ is answer of problem 1.

II. SPECIAL FORMS OF PROBLEM 1

We are noting that the problem (1) for each eigenvector x (not necessary positive) is solving with same method. In algorithm of HROU if assume input spectrum be $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $x = (1, 0, \ldots, 0, 1)^T$, then the matrix H has a special form as follows

$$H = I - \frac{2}{v^T v} vv^T = \begin{pmatrix} \beta & 0 & \cdots & 0 & \beta \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ \beta & 0 & \cdots & 0 & 1-\alpha \end{pmatrix} \quad (1)$$

where $\beta = \frac{1}{\sqrt{2}}$ and $\alpha = \frac{1}{\sqrt{2}(\sqrt{2}-1)}$.
Now we obtain $H_j = h_j h_j^T$ for $j = 1, 2, \ldots, n$:

$$H_1 = h_1 h_1^T = \begin{pmatrix} \beta^2 & 0 & \cdots & 0 & \beta^2 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ \beta^2 & 0 & \cdots & 0 & \beta^2 \end{pmatrix} \quad (2)$$

and for $j = 2, \ldots, n-1$, we have

$$H_j = h_j h_j^T = \begin{pmatrix} & & \overbrace{0}^{jth} & & \\ 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ j\{ \; 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \quad (3)$$

and also

$$H_n = h_n h_n^T = \begin{pmatrix} \beta^2 & 0 & \cdots & 0 & \beta(1-\alpha) \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ \beta(1-\alpha) & 0 & \cdots & 0 & (1-\alpha)^2 \end{pmatrix} \quad (4)$$

therefore by notice to construction of H_i the matrix B has following form

$$b_{11} = \beta^2(\lambda_1 + \lambda_n) \quad (5)$$

$$b_{1n} = \beta^2\lambda_1 + \beta(1-\alpha)\lambda_n \quad (6)$$

$$b_{nn} = \beta^2\lambda_1 + (1-\alpha)^2\lambda_n \quad (7)$$

the another elements of matrix B as follows :

$$b_{ii} = \lambda_i, \quad i = 2,\ldots,n-1 \quad (8)$$

$$b_{ij} = 0, \quad 2 \le i < j \le n \quad (9)$$

Thus the matrix B has bordered diagonal form , this means

$$B = \begin{pmatrix} * & 0 & \cdots & 0 & * \\ 0 & * & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & * & 0 \\ * & 0 & \cdots & 0 & * \end{pmatrix}. \quad (10)$$

Since $\beta(1-\alpha) = \frac{-1}{2}$ and $(1-\alpha)^2 = \frac{1}{2}$, $\beta^2 = \frac{1}{2}$, then we can write the relations (5), (6), (7) as follows

$$b_{11} = \frac{(\lambda_1 + \lambda_n)}{2} \quad (11)$$

$$b_{1n} = \frac{(\lambda_1 - \lambda_n)}{2} \quad (12)$$

$$b_{nn} = \frac{(\lambda_1 + \lambda_n)}{2}. \quad (13)$$

Example II.1.

Let $\Lambda = (3, 1, -2, -2)$, find the bordered diagonal matrix B in form (10) that Λ is its spectrum. By relations (11), (12), (13) the matrix B is obtained as follows

$$B = \begin{pmatrix} 0.5 & 0 & 0 & 2.5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 2.5 & 0 & 0 & 0.5 \end{pmatrix}.$$

Notice that we assume the vector $x = (1, 0, 0, 1)^T$ is eigenvector of B.

Let $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ be the spectrum and

$$x = (1, \ldots, \overbrace{1}^{kth}, 0, \ldots, 0, \overbrace{1}^{(n-l+1)th}, \ldots, 1)^T$$ be the eigenvector of matrix B , then B has the following form :

$$B = \begin{pmatrix} (A)_{k\times k} & 0 & \cdots & 0 & (C)_{k\times l} \\ 0 & * & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & * & 0 \\ (C^T)_{l\times k} & 0 & \cdots & 0 & (D)_{l\times l} \end{pmatrix} \quad (14)$$

Now we obtain the elements of matrix B from algorithm HROU. Let $F = \{1,\ldots,k,n-l+1,\ldots,n\}$ and $E = \{2,\ldots,k,n-l+1,\ldots,n\}$ then we have

$$b_{11} = \beta^2(\sum_{i\in F}\lambda_i) \quad (15)$$

$$b_{1j} = \beta^2\lambda_1 - \alpha\beta\sum_{i\in E}\lambda_i + \beta\lambda_j, \quad j \in E \quad (16)$$

$$b_{ii} = \beta^2\lambda_1 + \alpha^2\sum_{j\in E}\lambda_j + (1-2\alpha)\lambda_i, \quad i \in E \quad (17)$$

$$b_{ij} = \beta^2\lambda_1 + \alpha^2\sum_{k\in E}\lambda_k - \alpha(\lambda_i+\lambda_j), \quad i < j, \quad i,j \in E \quad (18)$$

$$b_{ii} = \lambda_i, \quad i = k+1,\ldots,n-l \quad (19)$$

$$b_{ij} = 0, \quad i < j, \quad i,j = k+1,\ldots,n-l \quad (20)$$

where

$$\alpha = \frac{1}{\sqrt{k+l}(\sqrt{k+l}-1)}, \quad \beta = \frac{1}{\sqrt{k+l}}. \quad (21)$$

Example II.2.

Let $\Lambda = (8, 5, 3, 2, -1, -3)$, the matrix B that Λ is its spectrum is in the following form.

$$B = \begin{pmatrix} 2.20000 & 3.35065 & 0 & 2.00901 & 0.66737 & -0.22705 \\ 3.35065 & 3.37467 & 0 & -0.53991 & 0.54549 & 1.26909 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 2.00901 & -0.53991 & 0 & 2.54549 & 1.63090 & 2.35450 \\ 0.66737 & 0.54549 & 0 & 1.63090 & 1.71631 & 3.43991 \\ -0.22705 & 1.26909 & 0 & 2.35450 & 3.43991 & 1.16352 \end{pmatrix}$$

Notice that we assume the vector $x = (1, 1, 0, 1, 1, 1)^T$ is eigenvector of B.

Now notice to the following lemma.

Lemma II.3. *For $l = k = 1, 2$ the matrix B be the symmetric matrix in following form*

$$B = \begin{pmatrix} (A)_{k\times k} & 0 & \cdots & 0 & (C)_{k\times l} \\ 0 & * & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & * & 0 \\ (C)_{l\times k} & 0 & \cdots & 0 & (A)_{l\times l} \end{pmatrix} \quad (22)$$

Proof. For $k = l = 1$ the relations (11),(13) hold since $b_{1n} = b_{nn}$.

For $k = l = 2$ assume the matrix B that obtain from HROU algorithm has form (15). By (22) for $k = l = 2$ we have

$$\alpha^2 = \beta^2 = (1-\alpha)^2 = \frac{1}{4} \quad (23)$$

from (18),(24) for $i = 2, n-1, n$ we have

$b_{ii} = \beta^2\lambda_1 + \alpha^2(\sum_{j=2}^{2}\lambda_j + \sum_{j=n-1}^{n}\lambda_j) + (1-2\alpha)\lambda_i = \beta^2\lambda_1 + \alpha^2(\sum_{j=2,j\neq i}^{2}\lambda_j + \sum_{j=n-1,j\neq i}^{n}\lambda_j) + (\alpha^2 - 2\alpha + 1)\lambda_i \equiv^{\alpha^2=\beta^2=(1-\alpha)^2} \beta^2(\lambda_1 + \sum_{j=2,j\neq i}^{2}\lambda_j + \sum_{j=n-1,j\neq i}^{n}\lambda_j + \lambda_i) = \beta^2(\lambda_1 + \lambda_2 + \lambda_{n-1} + \lambda_n) \equiv^{(16)} b_{11}.$

Then, to be attention to form of matrix B the main diagonal elements matrix A and D are equal. Also according to relations (17),(19) we have

$$b_{12} = \frac{1}{4}\lambda_1 - \frac{1}{4}(\lambda_2 + \lambda_{n-1} + \lambda_n) + \frac{1}{2}\lambda_2$$

$$= \frac{1}{4}(\lambda_1 + \lambda_2) - \frac{1}{4}(\lambda_{n-1} + \lambda_n) \qquad (24)$$

$$b_{n-1,n} = \frac{1}{4}\lambda_1 + \frac{1}{4}(\lambda_2 + \lambda_{n-1} + \lambda_n) - \frac{1}{2}(\lambda_{n-1} + \lambda_n)$$

$$= \frac{1}{4}(\lambda_1 + \lambda_2) - \frac{1}{4}(\lambda_{n-1} + \lambda_n). \qquad (25)$$

consequently from relations (25), (26) we deduce $b_{12} = b_{n-1,n}$. since the main diagonal elements matrix A and D are equal and B is symmetric matrix, from (15) we have form (23) in this case.

Example II.4.

Let $\Lambda = (4, 2, 1, 1, -2)$, the matrix B that Λ is its spectrum is in the following form.

$$B = \begin{pmatrix} 1.2500 & 1.7500 & 0 & 1.2500 & -0.2500 \\ 1.7500 & 1.2500 & 0 & -0.2500 & 1.2500 \\ 0 & 0 & 1 & 0 & 0 \\ 1.2500 & -0.2500 & 0 & 1.2500 & 1.7500 \\ -0.2500 & 1.2500 & 0 & 1.7500 & 1.2500 \end{pmatrix}$$

Notice that we assume the vector $x = (1, 1, 0, 1, 1)^T$ is eigenvector of B.

III. SUFFICIENT CONDITIONS FOR NONNEGATIVE

In this section we introduce some conditions for nonnegative solution of problem 1.
From relations (11)-(13) to fake into consideation that B is nonnegative matrix if in addition to satisfy in following condition

$$\lambda_1 + \lambda_2 + \ldots + \lambda_n \geq 0 \qquad (26)$$

$$|\lambda_1| \geq \lambda_j, \qquad 2 \leq j \leq n \qquad (27)$$

in another following conditions

$$\lambda_j \geq 0, \qquad j = 2, \ldots, n-1 \qquad (28)$$

because according to (28) we have

$$\lambda_1 > |\lambda_n| \Rightarrow -\lambda_1 < \lambda_n < \lambda_1 \Rightarrow \lambda_1 - \lambda_n > 0, \lambda_1 + \lambda_n > 0$$

Then from (11)-(13), we have

$$b_{11}, b_{1n}, b_{nn} > 0 \qquad (29)$$

then by (27)-(29) all of entries of B matrix in the form (10)be nonnegative.

Further more according to (16)-(20) moreover conditions (26),(27) for nonnegative matrix we have following conditions

$$b_{11} = \beta^2 (\sum_{i \in F} \lambda_i) \geq 0 \qquad (30)$$

$$min_{j \in E} b_{1j} = \beta^2 \lambda_1 - \alpha\beta \sum_{i \in E} \lambda_i + \beta\lambda_{min j \in E} \geq 0 \qquad (31)$$

$$min_{i \in E} b_{ii} = \beta^2 \lambda_1 + \alpha^2 \sum_{j \in E, j \neq i} \lambda_j + (\alpha - 1)^2 \lambda_{min j \in E} \geq 0 \qquad (32)$$

$$min_{i<j, i, j \in E} b_{ij} = \beta^2 \lambda_1 + \alpha^2 \sum_{k \in E} \lambda_k - \alpha(max_{i,j \in E}\lambda_i + \lambda_j) \geq 0 \qquad (33)$$

$$\lambda_{min k+1 \leq i \leq n-l} \geq 0. \qquad (34)$$

Example III.1.

Let $\Lambda = (4, 3, 1, -1, -2)$, the nonnegative matrix B that Λ is its spectrum is in the following form.

$$B = \begin{pmatrix} 0.3333 & 0 & 0 & 2.1220 & 1.5447 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 2.1220 & 0 & 0 & 0.0447 & 1.8333 \\ 1.5447 & 0 & 0 & 1.8333 & 0.6220 \end{pmatrix}$$

Notice that we assume the vector $x = (1, 0, 0, 1, 1)^T$ is eigenvector of B.

IV. THE EXTENSION OF PROBLEM 1.

In this section we intend to solve the following problem.
problem 2. Find the real symmetric matrix $B \in \Re^{n \times n}$, such that $(\lambda_1, \ldots, \lambda_n)$ be the its spectrum and x, y be two orthogonal eigenvectors.
Recall theorem (1), from this theorem we have next lemma.

Lemma IV.1. *Let $\{(\lambda_i, x_i)\}_{i=1}^n$ be the eigenpairs of $n \times n$ matrix A and x_1, x_2 be two its orthogonal eigenvectors.*

Let $X = (x_1, x_2)$, $C = \begin{pmatrix} \frac{\mu_1 - \lambda_1}{x_1^T x_1} x_1^T \\ \frac{\mu_2 - \lambda_2}{x_2^T x_2} x_2^T \end{pmatrix}_{2 \times n}$, then the matrix $A + XC$ has eigenvalues $\mu_1, \mu_2, \lambda_3, \ldots, \lambda_n$ and the its eigenvectors are the eigenvectors of matrix A.

Proof. According to theorem (1) the eigenvalues of matrix $A + XC$, are $\mu_1, \mu_2, \lambda_3, \ldots, \lambda_n$ where μ_1 and μ_2 are the eigenvalues of matrix $\Omega + CX$, where $\Omega = diag\{\lambda_1, \lambda_2\}$.

$$\Omega + CX = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} + \begin{pmatrix} \frac{\mu_1 - \lambda_1}{x_1^T x_1} x_1^T \\ \frac{\mu_2 - \lambda_2}{x_2^T x_2} x_2^T \end{pmatrix} \begin{pmatrix} x_1 & x_2 \end{pmatrix} =$$

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} + \begin{pmatrix} \mu_1 - \lambda_1 & 0 \\ 0 & \mu_2 - \lambda_2 \end{pmatrix} = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$$

For solving problem (2) we need to a good candidate $A = xx^T + yy^T$.

Theorem IV.2. *Let $x, y \in \Re^n$ are two orthogonal vectors and $A = xx^T + yy^T$. Moreover let $Hx = \|x\|_2 e_1$ and $Hy = \|y\|_2 e_2$, then*
1) the all of columns $(h_i, i = 1, 2, \ldots, n)$ of H^T constitute of an orthogonal eigenvectors set of A,
2) $h_1 = x/\|x\|_2, h_2 = y/\|y\|_2$ and $x^T x h_1 h_1^T + y^T y h_2 h_2^T = A$.

Proof. Since

$$HAH^T = H(xx^T + yy^T)H^T = Hx(Hx)^T + Hy(Hy)^T$$

$$= \|x\|_2^2 e_1 e_1^T + \|y\|_2^2 e_2 e_2^T,$$

H is orthogonal matrix, then we have $HH^T = I$. Furthermore according to above computations we have

$$HAH^T = \| x \|_2^2 e_1 e_1^T + \| y \|_2^2 e_2 e_2^T$$

$$\Rightarrow AH^T = H^T(\| x \|_2^2 e_1 e_1^T + \| y \|_2^2 e_2 e_2^T) \Rightarrow$$

$$A(h_1, h_2, \ldots, h_n) = (x^T x) H e_1 e_1^T + (y^T y) H e_2 e_2^T$$

$$\Rightarrow (Ah_1, Ah_2, \ldots, Ah_n) = (x^T x h_1, y^T y h_2, 0, \ldots, 0) \Rightarrow$$

$$Ah_1 = x^T x h_1, \quad Ah_2 = y^T y h_2, \quad Ah_i = 0, \quad \text{for } i = 3, \ldots, n,$$

Then all columns of H^T (h_i) are orthogonal eigenvectors of A.

Also, we have

$$Hx = \|x\|_2 e_1 \Rightarrow H^T H x = H^T \|x\|_2 e_1$$

$$\Rightarrow x = \|x\|_2 H^T e_1 \Rightarrow x = \|x\|_2 h_1 \Rightarrow h_1 = x/\|x\|_2,$$

$$Hy = \|y\|_2 e_2 \Rightarrow H^T H y = H^T \|y\|_2 e_2$$

$$\Rightarrow y = \|y\|_2 H^T e_2 \Rightarrow y = \|y\|_2 h_2 \Rightarrow h_2 = y/\|y\|_2.$$

Also we see that

$$x^T x h_1 h_1^T + y^T y h_2 h_2^T =$$

$$x^T x \frac{xx^T}{\|x\|_2^2} + y^T y \frac{yy^T}{\|y\|_2^2} = xx^T + yy^T = A,$$

and this complete the proof.

Now by above subjects we introduce the following algorithm for solving problem (2). From now we assume that h_i denote the i-th column of matrix H^T.

Algorithm 2.

1) Input two orthogonal vectors x, y, $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$; initial $B = 0$;

2) Compute the orthogonal matrix, s.t. $Hx = \|x\|_2 e_1$, $Hy = \|y\|_2 e_2$;

3) for $i = 1, 2, \ldots, n$

4) $B \leftarrow B + \lambda_i h_i h_i^T$

5) end for

6) output $B, \lambda_1, h_1, \lambda_2, h_2$.

Now we study the some properties of above algorithm. By lemma (4) we have $B = A + XC$, where $X = (h_1, h_2)$ and $C = \begin{pmatrix} (\lambda_1 - x^T x) h_1^T \\ (\lambda_2 - y^T y) h_2^T \end{pmatrix}$, then $B = A + (\lambda_1 - x^T x) h_1 h_1^T + (\lambda_2 - y^T y) h_2 h_2^T$.

In order that the eigenvalues of B are $(\lambda_1, \lambda_2, 0, \ldots, 0)$ (to take into consideration $A = xx^T + yy^T$ has eigenvalues $(x^T x, y^T y, 0, \ldots, 0)$).

Now according to then theorem(2) the other eigenvalues change to λ_i as following

$$B = A + (\lambda_1 - x^T x) h_1 h_1^T + (\lambda_2 - y^T y) h_2 h_2^T + \sum_{i=3}^{n} \lambda_i h_i h_i^T.$$

On the other hands by theorem (4) we have

$$B = A + \sum_{i=1}^{n} \lambda_i h_i h_i^T - (x^T x h_1 h_1^T + y^T y h_2 h_2^T) \Rightarrow$$

$$B = \sum_{i=1}^{n} \lambda_i h_i h_i^T.$$

The most important section of algorithm (2) is the computation of Householder matrix H. we express the construction of matrix H, this matrix obtain from product of two matrices H_1 and H_2 i.e $H = H_2 H_1$.

Let $H_1 = I - \frac{2}{v^T v} vv^T$ where $v = x - \| x \|_2 e_1$. To construct the matrix H_2, since $v = x - \| x \|_2 e_1$ we have :

$$z = H_1 y = (I - \frac{2}{v^T v} vv^T) y = y - \frac{2v}{v^T v} v^T y =$$

$$y - \frac{2v}{2(x^T x - \| x \|_2 x_1)} (\overbrace{x^T y}^{0} - \| x \|_2 \overbrace{e_1^T y}^{y_1}) \Rightarrow$$

$$z = H_1 y = y + \frac{\| x \|_2 v y_1}{\| x \|_2 (\| x \|_2 - x_1)}$$

Pay attention that the first element of z, namely z_1 is zero , because

$$z_1 = y_1 + \frac{(x_1 - \| x \|_2) y_1}{\| x \|_2 - x_1} = y_1 - y_1 = 0$$

Then the vector z separates to the

$$z = \begin{pmatrix} 0 \\ \hat{z} \end{pmatrix}$$

and $\hat{z}$ is vector with $n - 1$ elements. Now we construct the Householder matrix $\widehat{H_2}$ as following

$$\hat{v} = \hat{z} - \| \hat{z} \|_2 e_1$$

and

$$\widehat{H_2} = I - \frac{2}{\hat{v}^T \hat{v}} \hat{v} \hat{v}^T$$

Then the matrix H_2 has the following form

$$H_2 = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & \widehat{H_2} & \\ 0 & & & \end{pmatrix}$$

For satisfying $Hy = \| y \|_2 e_2$ must holds $\| z \|_2 = \| y \|_2$ and this is satisfy if $y_1 = 0$. Since $z_1 = 0$ and $z_i = y_i + \frac{x_i y_1}{\|x\|_2 - x_1}$ for $i = 2, \ldots, n$, then $z = y$.

Remark 1. In the line 1 of algorithm (2) moreover two vectors x and y must be orthogonal, the first element of y be necessary zero.

V. NUMERICAL EXAMPLES

Example V.1.

Find the matrix $B \in \Re^{n \times n}$, such that $(\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T$ be the its spectrum and $x = (1, 0, \ldots, 0, 1)^T$ and $y = (0, 1, 0, \ldots, 0, 1, 0)^T$ be its orthogonal eigenvectors.

Solution. We construct the Householder matrix H according to the last of previous section :

$$v = x - \| x \|_2 e_1 = (1 - \sqrt{2}, 0, \ldots, 0, 1)^T$$

and

$$H_1 = I - \frac{2}{v^T v} vv^T = \begin{pmatrix} \beta & 0 & \ldots & 0 & \beta \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \\ \beta & 0 & \ldots & 0 & 1-\alpha \end{pmatrix}$$

where

$$\alpha = \frac{1}{\sqrt{2}(\sqrt{2}-1)}, \qquad \beta = \frac{1}{\sqrt{2}}.$$

$\widehat{z} = (1, 0, \ldots, 0, 1, 0)^T$, then $\widehat{v} = \widehat{z} - \| \widehat{z} \|_2 \, e_1 = (1 - \sqrt{2}, 0, \ldots, 0, 1, 0)^T$

$$\widehat{H_2} = I - \frac{2}{\widehat{z}^T \widehat{z}} \widehat{z}\widehat{z}^T = \begin{pmatrix} \beta & 0 & \ldots & 0 & \beta & 0 \\ 0 & 1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \beta & 0 & \ldots & 0 & 1-\alpha & 0 \\ 0 & 0 & \ldots & 0 & 0 & 1 \end{pmatrix}$$

and

$$H_2 = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & \beta & 0 & \ldots & 0 & \beta & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \beta & 0 & \ldots & 0 & 1-\alpha & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & 1 \end{pmatrix}$$

$$H = H_2 H_1 = \begin{pmatrix} \beta & 0 & 0 & \ldots & 0 & 0 & \beta \\ 0 & \beta & 0 & \ldots & 0 & \beta & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 & 0 \\ 0 & \beta & 0 & \ldots & 0 & 1-\alpha & 0 \\ \beta & 0 & 0 & \ldots & 0 & 0 & 1-\alpha \end{pmatrix}$$

Now we obtain the matrices $h_i h_i^T$ for $i = 1, \ldots, n$:

$$h_1 h_1^T = \begin{pmatrix} \beta^2 & 0 & \ldots & 0 & \beta^2 \\ 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 \\ \beta^2 & 0 & \ldots & 0 & \beta^2 \end{pmatrix},$$

$$h_2 h_2^T = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & \beta^2 & 0 & \ldots & 0 & \beta^2 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & \beta^2 & 0 & \ldots & 0 & \beta^2 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 \end{pmatrix}$$

$$h_j h_j^T = \begin{pmatrix} 0 & \ldots & \overbrace{0}^{jth} & \ldots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \ldots & 1 & \ldots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & \ldots & 0 \end{pmatrix}, \qquad j = 3, \ldots, n-2$$

$$h_{n-1} h_{n-1}^T = \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 \\ 0 & \beta^2 & \ldots & \beta(1-\alpha) & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \beta(1-\alpha) & \ldots & (1-\alpha)^2 & 0 \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix},$$

$$h_n h_n^T = \begin{pmatrix} \beta^2 & 0 & \ldots & 0 & \beta(1-\alpha) \\ 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 \\ \beta(1-\alpha) & 0 & \ldots & 0 & (1-\alpha)^2 \end{pmatrix}$$

Then the element of matrix $B = \sum_{i=1}^{n} \lambda_i h_i h_i^T$ is the following form

$$b_{11} = \beta^2(\lambda_1 + \lambda_n) \tag{35}$$

$$b_{1n} = \beta^2 \lambda_1 + \beta(1-\alpha)\lambda_n \tag{36}$$

$$b_{nn} = \beta^2 \lambda_1 + (1-\alpha)^2 \lambda_n \tag{37}$$

$$b_{22} = \beta^2(\lambda_{n-1} + \lambda_2) \tag{38}$$

$$b_{2,n-1} = \beta^2 \lambda_2 + \beta(1-\alpha)\lambda_{n-1} \tag{39}$$

$$b_{n-1,n-1} = \beta^2 \lambda_2 + (1-\alpha)^2 \lambda_{n-1} \tag{40}$$

$$b_{ii} = \lambda_i, \qquad i = 3, \ldots, n-2 \tag{41}$$

$$b_{ij} = 0, \qquad i < j, \quad i, j = 3, \ldots, n-2 \tag{42}$$

where

$$\beta = \frac{1}{\sqrt{2}}, \qquad \alpha = \frac{1}{\sqrt{2}(\sqrt{2}-1)}$$

This means :

$$\begin{pmatrix} * & 0 & 0 & \ldots & 0 & 0 & * \\ 0 & * & 0 & \ldots & 0 & * & 0 \\ 0 & 0 & * & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & * & 0 & 0 \\ 0 & * & 0 & \ldots & 0 & * & 0 \\ * & 0 & 0 & \ldots & 0 & 0 & * \end{pmatrix} \tag{43}$$

Now notice that following example .

Example V.2.

Let $\Lambda = (8, 5, 4, 4, 3, 1, -1, -2)$, the matrix B that Λ is its spectrum is in the following form.

$$B = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 2 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 2 & 0 \\ 5 & 0 & 0 & 0 & & 0 & 0 & 3 \end{pmatrix}$$

Notice that we assume the vectors $x = (1, 0, 0, 0, 0, 0, 0, 1)^T$ and

$$y = (0, 1, 0, 0, 0, 0, 1, 0)^T$$

are orthogonal eigenvectors of B.

With changing the vectors x and y we can find the another form for matrix B

$$\begin{pmatrix} * & 0 & 0 & \ldots & 0 & 0 & * \\ 0 & (\ )_{k \times k} & 0 & \ldots & 0 & (\ )_{k \times l} & 0 \\ 0 & 0 & * & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & * & 0 & 0 \\ 0 & (\ )_{l \times k} & 0 & \ldots & 0 & (\ )_{l \times l} & 0 \\ * & 0 & 0 & \ldots & 0 & 0 & * \end{pmatrix} \quad (44)$$

where $x = (1, 0, \ldots, 0, 1)^T$ and

$$y = (0, \overbrace{1, \ldots, 1}^{k}, 0, \ldots, 0, \overbrace{1, \ldots, 1}^{l}, 0)^T$$

are two orthogonal eigenvectors of above matrix.

$$\begin{pmatrix} (\ )_{k \times k} & 0 & (\ )_{k \times l} \\ 0 & (\ ) & 0 \\ (\ )_{l \times k} & 0 & (\ )_{l \times l} \end{pmatrix} \quad (45)$$

where $x = (\overbrace{1, \ldots, 1}^{k}, 0, \ldots, 0, \overbrace{1, \ldots, 1}^{l})^T$ and $y = (\overbrace{0, \ldots, 0}^{k}, 1, \ldots, 1, \overbrace{0, \ldots, 0}^{l})^T$ are two orthogonal eigenvectors of above matrix.

$$\begin{pmatrix} (\ )_{k \times k} & 0 & 0 & \ldots & 0 & 0 & (\ )_{k \times l} \\ 0 & * & 0 & \ldots & 0 & * & 0 \\ 0 & 0 & * & & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & * & 0 & 0 \\ 0 & * & 0 & \ldots & 0 & * & 0 \\ (\ )_{l \times k} & 0 & 0 & \ldots & 0 & 0 & (\ )_{l \times l} \end{pmatrix} \quad (46)$$

where

$$x = (\overbrace{1, \ldots, 1}^{k}, 0, \ldots, 0, \overbrace{1, \ldots, 1}^{l})^T$$

and

$$y = (\overbrace{0, \ldots, 0}^{k}, 1, 0, \ldots, 0, 1, \overbrace{0, \ldots, 0}^{l})^T$$

are two orthogonal eigenvectors of above matrix.

VI. Extension

In section 5 we explain problem (1) for two orthogonal eigenvectors x and y. We can extend this problem for r vectors that $n \geq r > 2$ and and integer number. In this case the algorithm (3) reduce as following

Algorithm 3.

1) Input orthogonal vectors $x_1, \ldots, x_r$ such that the vector $x_i (i = 2, \ldots, r)$ has $i - 1$ zero in the first elements and matrix $B = 0$

2) The matrix Householder H such that $Hx_i = \|x_i\|_2 e_i$ for $i = 1, \ldots, r$.

3) for $i = 1, 2, \ldots, n$

4) $B \leftarrow B + \lambda_i h_i h_i^T$

5) end for

6) output B, λ_i, h_i for $i = 1, \ldots, r$.

Now we introduce the interesting example that give us matrix in form "X".

Example VI.1.

Find the matrix B such that the following vectors are its eigenvectors and $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T$ be its spectrum

$$x_1 = (1, 0, \ldots, 0, 1)^T,$$
$$x_2 = (0, 1, 0, \ldots, 0, 1, 0)^T, \ldots,$$
$$x_k = (0, \ldots, 0, \overbrace{1}^{kth}, 0, \ldots, 0, \overbrace{1}^{(n-k+1)th}, 0, \ldots, 0)^T$$

where $k = [n/2]$. The element of matrix B is the following form

$$b_{ii} = \beta^2 (\lambda_i + \lambda_{n-i+1}), \qquad i = 1, \ldots, [n/2] \quad (47)$$
$$b_{i,n-i+1} = \beta^2 \lambda_i + \beta(1 - \alpha)\lambda_{n-i+1}, \qquad i = 1, \ldots, [n/2] \quad (48)$$
$$b_{n-i+1,n-i+1} = \beta^2 \lambda_i + (1 - \alpha)^2 \lambda_{n-i+1}, \qquad i = 1, \ldots, [n/2] \quad (49)$$

where

$$\beta = \frac{1}{\sqrt{2}}, \qquad \alpha = \frac{1}{\sqrt{2}(\sqrt{2} - 1)}$$

.Also if n is odd

$$b_{[n/2]+1,[n/2]+1} = \lambda_{[n/2]+1} \quad (50)$$

Now notice that the following example .

Example VI.2.

Find the matrix in form "X" such that $(8, 5, 3, 2, -1, -3)$ is its spectrum.

solution.

$$B = \begin{pmatrix} 2.5 & 0 & 0 & 0 & 0 & 5.5 \\ 0 & 2 & 0 & 0 & 3 & 0 \\ 0 & 0 & 2.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 2.5 & 0 & 0 \\ 0 & 3 & 0 & 0 & 2 & 0 \\ 5.5 & 0 & 0 & 0 & 0 & 2.5 \end{pmatrix}$$

Notice that vectors

$$x_1 = (1, 0, 0, 0, 0, 1)^T,$$
$$x_2 = (0, 1, 0, 0, 1, 0)^T,$$
$$x_3 = (0, 0, 1, 1, 0, 0)^T$$

are orthogonal eigenvectors of B.

References

[1] Jonathan Axtell, Lixing Han, Daniel Hershkowitz, Michael Neumann, Nung-Sing Sze , *Optimition of the spectral radius of a product for nonnegative matrices*, Linear Algebra and its Applications 430 (2009) 1442-1451.

[2] J. Stoer and R. Bulirch, *Introduction to numerical analysis*, Springer Verlag 1991.

[3] Fuzhen Zhang. *Matrix Theory.* Springer-Verlage,1999.

[4] R.Behatia, *Matrix Analysis.* Springer-Verlage,1973.