

# Improvement of Gregory's formula using Particle Swarm Optimization

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**Abstract**—Consider the Gregory integration (G) formula

$$\int_0^n f(x) dx = \sum_{k=0}^n f(k) + \sum_{k \geq 0} \left( \frac{a_k}{k!} \right) (\Delta_{h_k}^k f(0) + \Delta_{-h_k}^k f(n)) \quad (1)$$

with end corrections where  $\Delta_{h_k}$  is the forward difference operator with step size  $h_k$ . In this study we prove that (1) can be optimized by minimizing some of the coefficient  $a_k$  in the remainder term by particle swarm optimization. Experimental tests prove that (1) can be rendered a powerful formula for library use.

**Keywords**—Numerical integration, Gregory Formula, Particle Swarm optimization.

## I. INTRODUCTION

NOTE that for  $h_k=1$  (1) reduces to the classical Gregory integration formula. To justify our formula, we shall use the umbral methods developed by Rota and his school [1]-[2] instead of classical generating function technique.

When  $f(x)$  is replaced by  $1, x, x^2, x^3 \dots$  we find.

$$\begin{aligned} a_0 &= -\frac{1}{2} \\ a_1(h_1) &= \frac{1}{12h_1} \\ a_2(h_1, h_2) &= \frac{-h_1}{24h_2} \\ a_3(h_1, \dots, h_3) &= \left( \frac{1}{720h_3^3} \right) (-10h_1^2 + 30h_2h_1 - 1) \\ a_4(h_1, \dots, h_4) &= \left( \frac{-1}{480h_4^4} \right) \left( \frac{5}{3}h_1^3 - 10h_3h_1^2 + 30h_3h_2h_1 - h_3 \right) \\ a_5(h_1, \dots, h_5) &= \left( \frac{1}{60480h_5^5} \right) (-42h_1^4 + 630h_2^3h_1 + 1050h_3^2h_2^2 - 3150h_3^2h_2h_1 \\ &\quad - 420h_4h_1^3 - 2520h_4h_3^3h_1 - 2940h_4h_2^2h_1 + 7560h_4h_3h_2h_1 + 105h_5^2 - 252h_4h_3 + 2) \end{aligned}$$

So, at the order 5 we can write:

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$$\begin{aligned} \int_0^n f(x) dx &\approx \sum_{k=0}^n f(k) + a_0[f(0) + f(n)] + a_1(h_1)[f(h_1) - f(0) + f(n-h_1) - f(n)] \\ &\quad + a_2(h_1, h_2)[f(2h_2) - 2f(h_2) + f(0) + f(n-2h_2) - 2f(n-h_2) + f(n)] \\ &\quad + a_3(h_1, h_2, h_3)[\Delta_{h_3}^3 f(0) + \Delta_{-h_3}^3 f(n)] \\ &\quad + a_4(h_1, h_2, h_3, h_4)[\Delta_{h_4}^4 f(0) + \Delta_{-h_4}^4 f(n)] \\ &\quad + a_5(h_1, h_2, h_3, h_4, h_5)[\Delta_{h_5}^5 f(0) + \Delta_{-h_5}^5 f(n)] \end{aligned} \quad (2)$$

This formula has a sense so  $n \geq 2$ . In the contrary case an appropriate variable change will permit us to do the integral without no difficulty.

Our goal is to optimize the remainder. For do it, we try to determine  $h_1, h_2, h_3, h_4$  and  $h_5$  that returns  $a_3, a_4$  and  $a_5$  as small as possible.  $a_3, a_4$  and  $a_5$  is a system non linear of 3 equations to 5 unknowns  $h_1, h_2, h_3, h_4$  and  $h_5$ ; we take  $h_4, h_5 (=1, \text{ in this study})$  as parameters and let's solve this system by PSO.

## II. PARTICLE SWARM OPTIMIZATION

### Overview and strategy of particle swarm optimization

Recently, a new stochastic algorithm has appeared, namely 'particle swarm optimization' PSO. The term 'particle' means any natural agent that describes the 'swarm' behavior. The PSO model is a particle simulation concept, and was first proposed by Eberhart and Kennedy [3]. Based upon a mathematical description of the social behavior of swarms, it has been shown that this algorithm can be efficiently generated to find good solutions to a certain number of complicated situations such as, for instance, the static optimization problems, the topological optimization and others [4]-[5]-[6]-[7]. Since then, several variants of the PSO have been developed [8]-[9]-[10]-[11]-[12]-[13]-[14]. It has been shown that the question of convergence of the PSO algorithm is implicitly guaranteed if the parameters are adequately selected [15]-[16]-[17]. Several kinds of problems solving start with computer simulations in order to find and analyze the solutions which do not exist analytically or specifically have been proven to be theoretically intractable.

The particle swarm treatment supposes a population of individuals designed as real valued vectors - particles, and some iterative sequences of their domain of adaptation must be established. It is assumed that these individuals have a social behavior, which implies that the ability of social conditions, for instance, the interaction with the neighborhood, is an important process in successfully finding good solutions to a given problem.

The strategy of the PSO algorithm is summarized as follows: We assume that each agent (particle)  $i$  can be represented in a  $N$  dimension space by its current position  $x_i = (x_{i1}, x_{i2}, \dots, x_{iN})$  and its corresponding velocity. Also a memory of its personal (previous) best position is represented by,  $p = (p_{i1}, p_{i2}, \dots, p_{iN})$  called (pbest), the subscript  $i$  range from 1 to  $s$ , where  $s$  indicates the size of the swarm. Commonly, each particle localizes its best value so far (pbest) and its position and consequently identifies its best value in the group (swarm), called also (sbest) among the set of values (pbest).

The velocity and position are updated as.

$$v_{ij}^{k+1} = w_j v_{ij}^k + c_1 r_1^k [(pbest)_{ij}^k - x_{ij}^k] + c_2 r_2^k [(sbest)_{ij}^k - x_{ij}^k] \quad (3)$$

$$x_{ij}^{k+1} = v_{ij}^{k+1} + x_{ij}^k \quad (4)$$

where are the position and the velocity vector of particle  $i$

respectively at iteration  $k + 1$ ,  $c_1$  et  $c_2$  are acceleration coefficients for each term exclusively situated in the range of

2--4,  $w_{ij}$  is the inertia weight with its value that ranges from

0.9 to 1.2, whereas  $r_1$ ,  $r_2$  are uniform random numbers between zero and one. For more details, the double subscript in the relations (2) and (3) means that the first subscript is for the particle  $i$  and the second one is for the dimension  $j$ . The

role of a suitable choice of the inertia weight  $w_{ij}$  is important in the success of the PSO. In the general case, it can be initially set equal to its maximum value, and progressively we decrease it if the better solution is not reached. Too often, in the relation (1),  $w_{ij}$  is replaced by  $w_{ij} / \sigma$ , where  $\sigma$  denotes the constriction factor that controls the velocity of the particles. This algorithm is successively accomplished with the following steps [18]-[19]:

1. Set the values of the dimension space  $N$  and the size  $s$  of the swarm ( $s$  can be taken randomly).
2. Initialize the iteration number  $k$  (in the general case is set equal to zero).
3. Evaluate for each agent, the velocity vector using its memory and equation (3), where pbest and sbest can be modified.
4. Each agent must be updated by applying its velocity vector and its previous position using equation (4).
5. Repeat the above step (3, 4 and 5) until a convergence criterion is reached.

The practical part of using PSO procedure will be examined in the following section, where we'll optimize  $a_3$ ,  $a_4$  and  $a_5$  using PSO.

The PSO algorithm is applied, with parameter setting (Table I).

### III. SIMULATION STUDY

To test the performance of this algorithm we took various

TABLE I  
PARTICLE SWARM PARAMETER SETTING

Parameter	Setting
Population Size	20
Number of Iterations	500
$c_1$ and $c_2$	0.5
Inertial Weight	1.2 $\rightarrow$ 0.4
Desired Accuracy	$10^{-5}$

functions and we looked for an approximation with Gregory (G) and the new formula (GP) (Table II).

### IV. CONCLUSION

A Considering results we note that:

- the middle value turns around 0.3
- some well-known functions badly to integrate by the methods classic ( $1/(1+x, \dots)$ ) give good results with this formula .
- in general the GP formula improves the order of precision in a considerable way in relation to Gregory, without using as many assessments.

TABLE II  
COMPARISON

Function	Interval	Formula	$h_1$	$h_2$	$h_3$	R. Error.
$1/(1+x)$	[0, 1]	G	1	1	1	0.063
		GP	0.3	0.3	0.3	$-2.4732 \times 10^{-4}$
$e^x$	[0, 2]	G	1	1	1	0.0102
		GP	0.3	0.3	0.3	$-7.9663 \times 10^{-5}$
$x/(1-x)$	[0, 1]	G	1	1	1	0
		GP	0.3	0.3	0.3	0
$x/(1+x^2)$	[0, 2]	G	1	1	1	-0.0652
		GP	0.3	0.3	0.3	$7.1755 \times 10^{-4}$

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