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Dissipation of Higher Mode using Numerical Integration Algorithm in Dynamic Analysis

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Abstract**—**In general dynamic analyses, lower mode response is of interest, however the higher modes of spatially discretized equations generally do not represent the real behavior and not affects to global response much. Some implicit algorithms, therefore, are introduced to filter out the high-frequency modes using intended numerical error. The objective of this study is to introduce the P-method and PC α-method to compare that with dissipation method and Newmark method through the stability analysis and numerical example. PC α-method gives more accuracy than other methods because it based on the α-method inherits the superior properties of the implicit α-method. In finite element analysis, the PC α-method is more useful than other methods because it is the explicit scheme and it achieves the second order accuracy and numerical damping simultaneously.

Keywords**—**Dynamic, α-Method, P-Method, PC α-Method, Newmark method.

I. Introduction

IN many dynamic application only lower mode response is of interest. The higher modes of spatially discretized equations generally do not represent the real behavior. In addition, the presence of numerical errors will lead to the spurious growth of high mode response.

Some implicit algorithms, therefore, are introduced to filter out the high-frequency modes such as the Wilson method, Houbolt's method, and the dissipation method. However, the Wilson method and Houbolt's methods damp out the lower frequency modes too strongly [1]. The dissipation method developed by H. M. Hilber, et al [2] gives the desired numerical damping effect and second order accuracy, even though dissipation method is unconditionally stable. Although the general Newmark method gives numerical damping, it has first order numerical damping, i.e., lower modes are too much damped out. To archive second order accuracy and numerical damping simultaneously, I. Miranda, R.M. Ferencz and T.J.R. Hughes [3] developed the predictor-corrector-method (PC-method) based on the implicit dissipation method and S.Y. Chang [4] also developed the P-method which is based on Newmark explicit method.

The objective of this study is to introduce the P-method and PC α-method to compare that with dissipation method and Newmark method through the stability analysis and numerical example.

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II. α -Method

The equation of motion for a linear system can be expressed as

$$M\ddot{\upsilon}+C\dot{\upsilon}+K\upsilon=F \tag{1}$$

where M is the mass matrix, K is the stiffness matrix, F is the vector of external force, is the displacement vector. Approximate solution of Eq.(1) is obtained as following

$$\begin{aligned} &M\ddot{\upsilon}_{n+1}+(1+\alpha)C\dot{\upsilon}_{n+1}-\alpha C\dot{\upsilon}_n \\ &\quad+(1+\alpha)K\upsilon_{n+1}-\alpha K\upsilon_n=F_{n+1+\alpha} \end{aligned} \tag{2a}$$

$$\upsilon_{n+1}=\upsilon_n+\Delta t\,\upsilon_n+\Delta t^2\left[\left(\frac{1}{2}-\beta\right)\ddot{\upsilon}_n+\beta\ddot{\upsilon}_{n+1}\right] \tag{2b}$$

$$\dot{\upsilon}_{n+1}=\dot{\upsilon}_n+\Delta t\left[(1-\gamma)\ddot{\upsilon}_n+\gamma\ddot{\upsilon}_{n+1}\right] \tag{2c}$$

If $\alpha=0$, this algorithm is reduced to the Newmark method. In this case, with $\gamma=0.5$, the algorithms possess no numerical dissipation. For homogeneous case, Eq.(2) can be expressed in the recursive form such as

$$X_{n+1}=A\,X_n \tag{3a}$$

where

$$X_n=\left\{\begin{array}{ccc}\upsilon_n & \Delta t\,\dot{\upsilon}_n & \Delta t^2\,\ddot{\upsilon}_n\end{array}\right\}^T \tag{3b}$$

and A is called the amplification matrix. Stability and accuracy of an algorithm depends on the eigenvalues of the amplification matrix and it is defined as Eq.(4) in case of undamped system

$$A=\frac{1}{D}\begin{bmatrix} 1+\alpha\beta\Omega^2 & 1 & \frac{1}{2}-\beta \\ -\gamma\Omega^2 & 1-(1+\alpha)(\gamma-\beta)\Omega^2 & 1-\gamma-(1+\alpha)\left(\frac{1}{2}\gamma-\beta\right)\Omega^2 \\ -\Omega^2 & -(1+\alpha)\Omega^2 & -(1+\alpha)\left(\frac{1}{2}\gamma-\beta\right)\Omega^2 \end{bmatrix} \tag{4a}$$

where,

$$\left.\begin{aligned} D &= 1+(1+\alpha)\beta\Omega^2 \\ \Omega &= \omega\,\Delta t \\ \omega &= \left(\frac{K}{M}\right)^{\frac{1}{2}} \end{aligned}\right\} \quad (4b)$$

The characteristic equation of A is

$$\det(A-\lambda I) = \lambda^3 - 2A_1\lambda + A_2\lambda - A_3 = 0 \quad (5)$$

In general, $A_3 \neq 0$, therefore, the amplification matrix has three non-zero eigenvalues. A consequence of convergence is that if Ω is in between 0 and $\Omega_{critical}$, then the characteristic equation has two complex conjugate roots $\lambda_{1,2}$, principal roots, and a so-called spurious root λ_3, which satisfy $|\lambda_3| < |\lambda_{1,2}| \leq 1$. Under these circumstance the principal roots become

$$\lambda_{1,2} = A \pm iB \quad (6)$$

If β and γ are taken as

$$\beta = \frac{1}{4}(1-\alpha)^2,\ \gamma = \frac{1}{2} - \alpha \quad (7)$$

then second-order accuracy and unconditionally stable condition are attained and the invariants of the amplification matrix become

$$\left.\begin{aligned} A_1 &= 1 - \frac{\Omega^2}{2D} + \frac{A_3}{2}, \quad A_2 = 1 + 2A_3 \\ A_3 &= \frac{\alpha(1+\alpha)^2\Omega^2}{4D} \end{aligned}\right\} \quad (8)$$

when Ω goes to zero, the complex conjugate roots become one and the real root becomes zero. On the other hand, in the limit Ω goes to infinity, the roots are real and the tendency is described in Fig. 1. It is revealed that the numerical damping cannot be increased by a below -1/3 through the numerical experiment. Thus the practical range of the value is

$$-\frac{1}{3} \leq \alpha \leq 0.$$

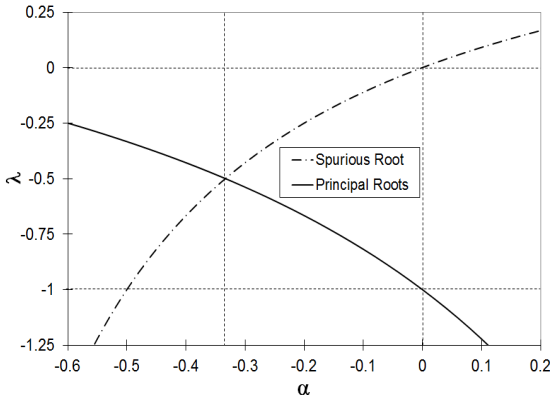


Fig. 1 Eigenvalues of the amplification matrix in the limit $\Delta t/T \to \infty$ vs α

To compute numerical damping and numerical natural frequency, one can assume the response of the system as

$$\begin{aligned} \upsilon_n &= C_1\lambda_1^n + C_2\lambda_2^n = C_1 e^{n(A+iB)} + C2e^{n(A-iB)} \\ &= e^{nA}\left[G_1\cos(nB) + G_2\sin(nB)\right] \end{aligned} \quad (9)$$

The analytical solution for free vibration response is expressed as

$$\upsilon(n\Delta t) = e^{-\xi\omega n\Delta t}\left[\hat{G}_1\cos(\omega_D\Delta t) + \hat{G}_2\sin(\omega_D\Delta t)\right] \quad (10)$$

If the viscous damping of system is ignored, the numerical damping and frequency can be derived as Eq.(11) through comparison Eq.(9) with Eq.(10).

$$\bar{\xi} = -\frac{\ln(A^2+B^2)}{2\bar{\omega}\Delta t}, \qquad \bar{\omega} = \frac{1}{\Delta t}\tan^{-1}\left(\frac{B}{A}\right) \quad (11)$$

Spectral radius is an important measure of stability and dissipation. Fig. 2 illustrates the spectra vs. $\Delta t/T$ for the Trapezoidal rule, Newmark method, and α-method. Trapezoidal rule is unconditionally stable, but it doesn't have the numerical dissipation. The results of trapezoidal with α-dissipation and Newmark method have some numerical dissipation, however, it is not an effective damping mechanism. Only α-method has the desirable numerical dissipation. It becomes obviously in Fig. 3. Newmark method has not second-order damping, i.e., lower frequencies are damped out too strongly. It also shows that higher frequencies are not damped out effectively and lower frequencies are damped strongly in trapezoidal rule with α-dissipation.

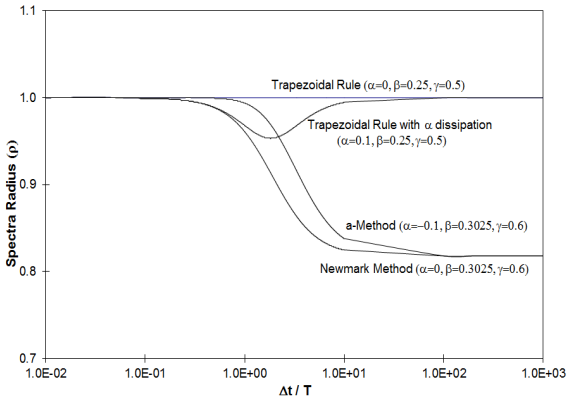


Fig. 2 Spectral Radius for Newmark and α-method

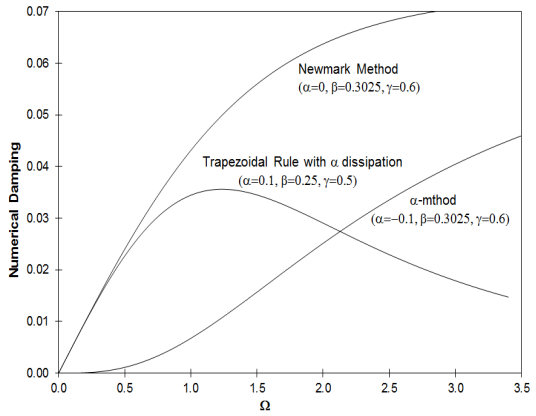


Fig. 3 Numerical damping ratio vs Ω for Newmark and α-method

III. Predictor-Corrector α-Method (PC α-Method)

Predictor-Corrector algorithm naturally arises from Eq.(2).
Predictor phase:

$$\tilde{\upsilon}_{n+1} = \upsilon_n + \Delta t\dot{\upsilon}_n + \frac{1}{2}(1-2\beta)\Delta t^2\ddot{\upsilon}_n$$
$$\dot{\tilde{\upsilon}}_{n+1} = \dot{\upsilon}_n + (1-\gamma)\Delta t\ddot{\upsilon}_n \tag{12}$$

Not likely α-method, time discrete equation of motion is taken at the predictor phase instead of t_{n+1} except acceleration term.

Time discrete equation of motion:

$$M\ddot{\upsilon}_{n+1} + (1+\alpha)C\dot{\tilde{\upsilon}}_{n+1} - \alpha C\dot{\upsilon}_n + (1+\alpha)K\tilde{\upsilon}_{n+1} - \alpha K\upsilon_n = F_{n+1+\alpha} \tag{13}$$

Corrector phase:

$$\upsilon_{n+1} = \tilde{\upsilon}_{n+1} + \beta\Delta t^2\ddot{\upsilon}_{n+1}$$
$$\dot{\upsilon}_{n+1} = \dot{\tilde{\upsilon}}_{n+1} + \gamma\Delta t\ddot{\upsilon}_{n+1} \tag{14}$$

For homogeneous case, Eq.(12) and (14) can be changed in recursive form such as Eq.(3a). The amplification matrix of PC α-method is derived as

$$\begin{bmatrix} 1-\beta\Omega^2 & 1-\beta\{2\xi\Omega+(1+\alpha)\Omega^2\} & \frac{1}{2}-\beta\left\{1+\left((1+\alpha)(1-\gamma)2\xi\Omega+(1+\alpha)\left(\frac{1}{2}-\beta\right)\Omega^2\right)\right\} \\ -\gamma\Omega^2 & 1-\gamma\{2\xi\Omega+(1+\alpha)\Omega^2\} & 1-\gamma\left\{1+\left((1+\alpha)(1-\gamma)2\xi\Omega+(1+\alpha)\left(\frac{1}{2}-\beta\right)\Omega^2\right)\right\} \\ -\Omega^2 & -\{2\xi\Omega+(1+\alpha)\Omega^2\} & -(1+\alpha)(1-\gamma)2\xi\Omega-(1+\alpha)\left(\frac{1}{2}-\beta\right)\Omega^2 \end{bmatrix} \tag{15}$$

The characteristic equation of the amplification matrix A becomes

$$\det(A-\lambda I) = \lambda^3 - 2A_1\lambda + A_2\lambda - A_3 = 0 \tag{16}$$

where,

$$A_1 = 1-\frac{1}{2}\left[2\xi\Omega(1+\alpha(1-\gamma)) + \Omega^2\left((1+\alpha)\left(\gamma+\frac{1}{2}\right)-\beta\alpha\right)\right]$$
$$A_2 = 1-\left[2\xi\Omega(1+2\alpha(1-\gamma)) + \Omega^2\left(\gamma-\frac{1}{2}+2\alpha(\gamma-\beta)\right)\right] \tag{17}$$
$$A_3 = -\left[2\xi\Omega\alpha(1-\gamma) + \Omega^2\alpha\left(\gamma-\beta-\frac{1}{2}\right)\right]$$

From Eq. (16) and (17), stability analysis can be carried if β and γ is defined as Eq. (7), the PC α-method becomes one parameter method and it has second-order accuracy in the range of

$$\alpha \in \left[-\frac{1}{3} \quad 0\right]$$

If γ does not satisfy the Eq.(7), the algorithm becomes first-order accurate.

Fig. 4 shows the stability limit of predictor -corrector method. Moreover, Fig. 5 indicates this method has the desired numerical damping effect, that is, second order damping mechanism.

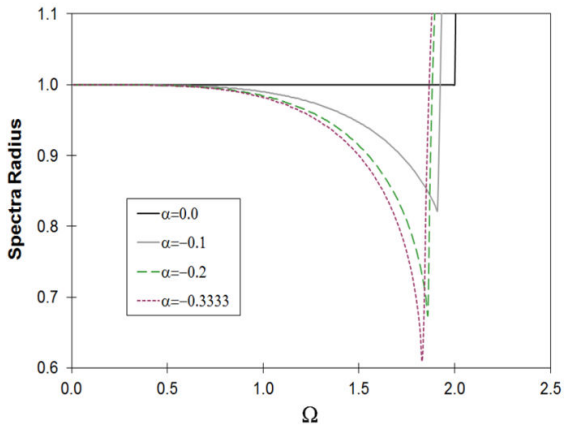


Fig. 4 Spectral Radius of the amplification matrix for PC α-method

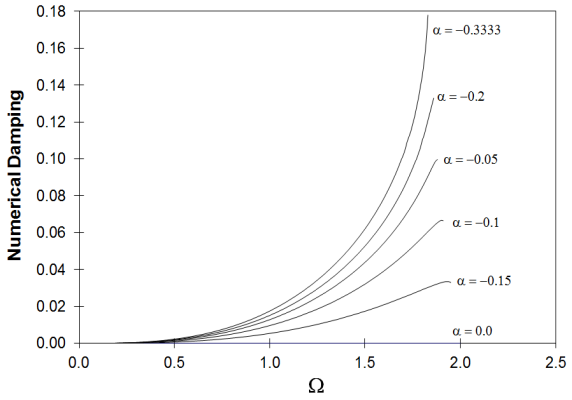


Fig. 5 Numerical damping for PC α-method

IV. P-Method

If α is equal to zero in Eq.(2a), the α method becomes Newmark method. In the case of β equal to 0, the algorithm is energy conserving if γ = 0.5, whereas numerical dissipation presents if γ > 0.5. It is necessary to assume β=0 to develop P-method since it is supposed to be explicit.

To make α-function dissipation method, γ is set to be 0.5 and the amplification matrix becomes

$$\begin{bmatrix} 1 & 1 & 1 \\ -\frac{1}{2}\Omega^2 & 1-\frac{1}{2}(1+\alpha)\Omega^2 & \frac{1}{2}-\frac{1}{4}(1+\alpha)\Omega^2 \\ -\Omega^2 & -(1+\alpha)\Omega^2 & -\frac{1}{2}(1+\alpha)\Omega^2 \end{bmatrix} \tag{18}$$

The characteristic equation and eigen values of the amplification matrix can be expressed as Eq.(19).

$$\lambda\left(\lambda^2 - A_1\lambda + A_2\right) = 0 \tag{19a}$$

where

$$A_1 = 2-(1+\alpha)\Omega^2, \quad A_2 = 1-\alpha\Omega^2, \quad A_3 = 0$$
$$\lambda_{1,2} = A \pm iB \tag{19b}$$

and

$$A = 1-\frac{1}{2}(1+\alpha)\Omega^2, \quad B = \Omega\sqrt{1-\frac{1}{4}(1+\alpha)^2\Omega^2} \tag{19c}$$

The algorithm is stable if and only if $\max\left\{\left|\lambda_i\right|\right\} \le 1$ and the roots of the amplification matrix should be complex conjugate to make the solution realistic (sinusoidal response). Therefore, α value is bounded as

$$0 \le \alpha \le \frac{2}{\Omega} - 1 \tag{20}$$

In general, numerical damping and frequency are function of Ω and increasing function with increasing positive slopes for an integration method having frequency-proportional damping. Thus, $\bar{\xi}$ and $\bar{\Omega}$ can be expressed as

$$\bar{\xi} = F(\Omega), \quad \bar{\Omega} = G(\Omega) \tag{21}$$

By substituting Eq.(21) into Eq.(11) and A, B are defined in Eq.(19b), one can obtain

$$\alpha = \left(\frac{1}{\Omega^2}\right)\left\{1-\exp\left[-\frac{2\Omega F(\Omega)}{1-G(\Omega)}\right]\right\} \tag{22}$$

For simplicity, F(Ω) and G(Ω) are assumed to be an increasing polynomial function of Ω.

$$F(\Omega) = p\Omega^q, \quad G(\Omega) = r\Omega^s \tag{23}$$

where p, q, r, and s are positive constants. In addition, q and s must be greater than one to become increasing function with increasing positive slope.

The effects of p, q, r and s are investigated through parametric studies and are shown in Fig. 6. Comparing Fig. 6(a) and 6(b) with Fig. 6(c) and 6(d), one can conclude that the variations of the constants p and q have larger effect on numerical damping than those of r and s.

Fig. 6(a) indicates that the curve moves upward with the increase of p value and all the curves can archive the desired numerical damping. Fig. 6(b) shows that the algorithm has a desired numerical dissipation when q is greater than three. Since the amount of the numerical damping is not affected by the coefficient of G(Ω), it is convenient to assume G(Ω) as zero and the q is set to 3. Thus Eq.(22) is reduced to

$$\alpha = \frac{1}{\Omega^2}\left\{1-\exp\left[-2p\Omega^4\right]\right\} \tag{24}$$

Using Taylor's expansion, one can obtain a simple polynomial function of Eq. (24) such as

$$\alpha = \sum_{n=1}^{\infty}(-1)^{n+1}\left[\frac{(2p)^n}{n!}\right]\Omega^{4n-2} = \sum_{n=1}^{\infty}(-1)^{n+1}\,p_n\left[\frac{k}{m}\Delta t^2\right]^{2n-1} \tag{25a}$$

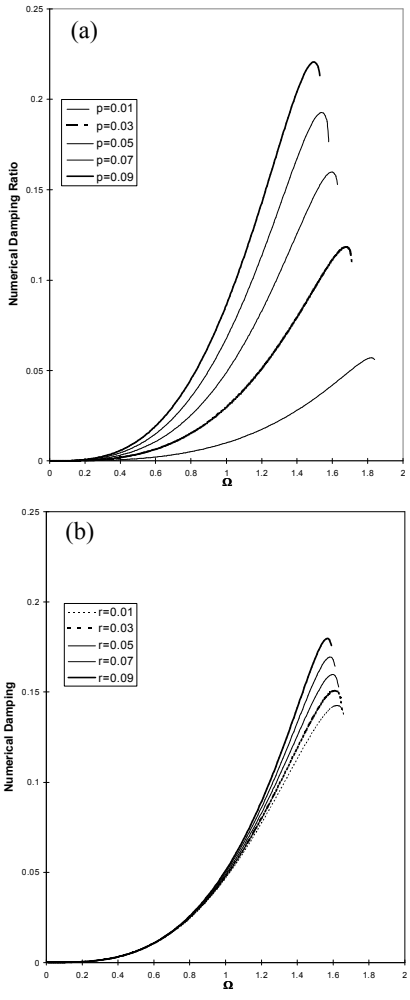
where

$$p_n = \frac{(2p)^n}{n!} \tag{25b}$$

If only the first term of Eq.(25) is taken, it becomes

$$\alpha = p_1 \frac{k}{m} \Delta t^2 \tag{26}$$

The spectra radii and the numerical damping ratio of the P-method have been plotted in Fig. 7 and 8. Explicit Newmark method has a linear numerical damping which damp out low frequency too. The results of P-method indicate it has the desired numerical damping. The damping curve with almost coincides with the curve of the PC α-method with α = -1/3 when Ω is less than 1.4, which has maximum numerical dissipation. Therefore, P-method has the larger numerical damping effect than that of PC α-method.



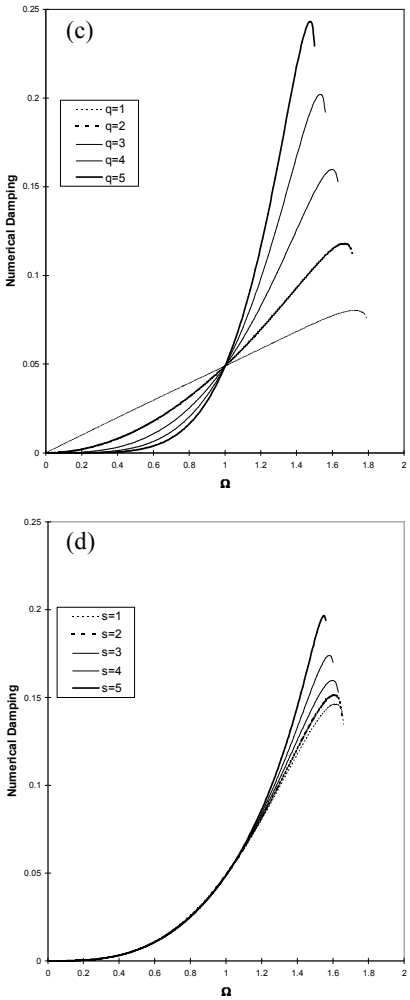


Fig. 6 Parametric effects on variation of numerical damping vs. Ω of (a) q=3, r=0.05and s=3, (b) p=0.05, q=3 and s=3, (c) p=0.05, r=0.05and s=3, (d) p=0.05, q=3 and r=0.05, respectively

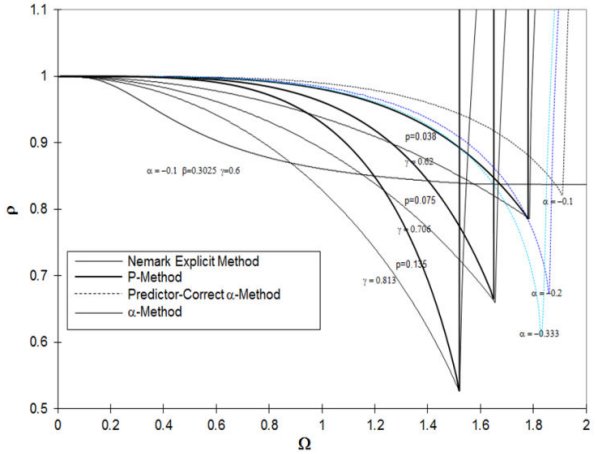


Fig. 7 Spectra Radius for Newmark explicit, PC α-method and P-method

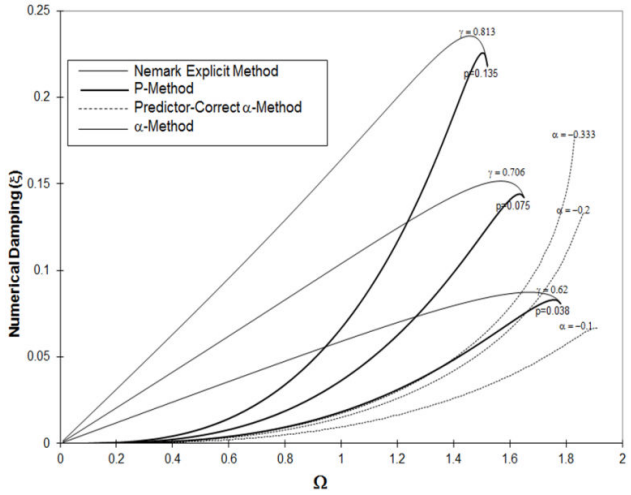


Fig. 8 Numerical damping for ρ-method, Newmark, and PC α-method

V. Comparison of Algorithms

The spectra radius, numerical damping, and period error for the presented algorithms are compared in this section. The trapezoidal method, Newmark method, and α-method when parameters are set to Eq.(7) achieve the unconditionally stable, and Newmark explicit, P-method, and PC α-Method are conditionally stable as Fig. 9.

The PC α-method shows lager stability limit than those of Newmark Explicit method and P-method, however, P-method indicates larger numerical dissipation ability. Fig. 10 shows that Newmark explicit method, Newmark implicit method and Trapezoidal Rule damp out the lower frequency too much, on the other hand, α-method, PC α-method and P-method have a second order numerical damping mechanism and it also shows that the effect of numerical damping in α-method are not effective than that of PC α-method and P-method. It seems that α-method needs large.

From Fig. 11 one can observe that PC α-method is more accurate than the other method in the range of $0 \le \Omega \le 1.5$. The numerical error of the P-method is closer to that of Newmark explicit method by increasing of Ω. It is obvious because P-method is derived from Newmark explicit method.

The α-method has the second order numerical error before Ω=0.6 and it is changed to the first order function. However, the numerical error of the P-method and PC α-method also is the second order function as they have the second order accuracy and numerical dissipation. That is, the numerical error of the P-method and the PC α-method rapidly grow up by the increasing of discretized time step. However, period error of α-method does not grow up rapidly because the error function becomes linear after Ω=0.6.

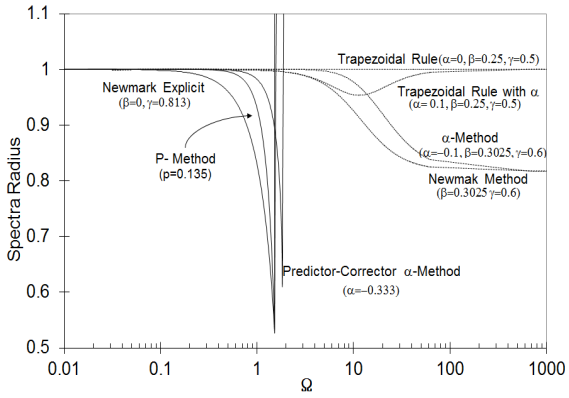


Fig. 9 Comparison of the Spectra Radius

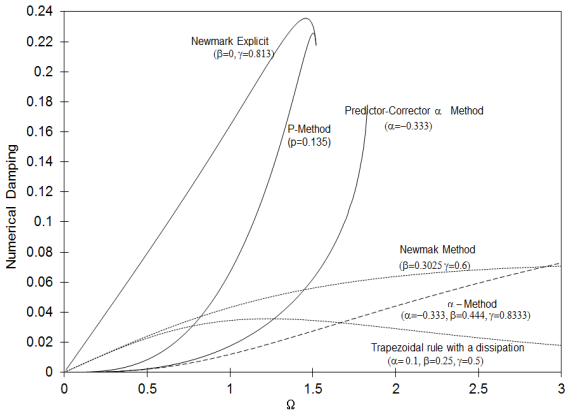


Fig. 10 Comparison of the Numerical Damping

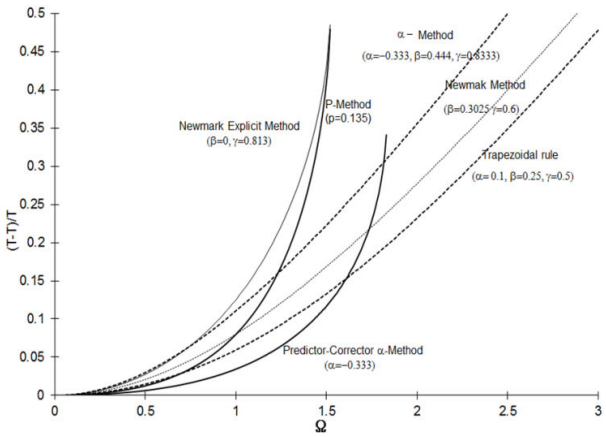
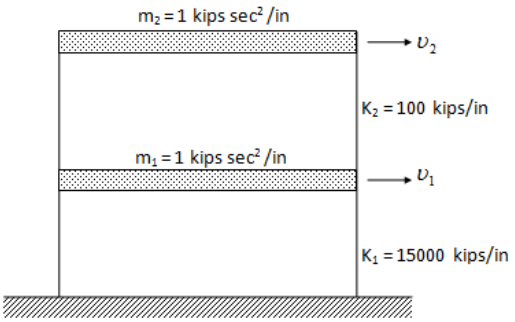


Fig. 11 Comparison of the Period Error

VI. Numerical Example

A two-story linear elastic shear building is considered. The unusual structure is intentionally chosen to have a high natural frequency in order to demonstrate the characteristic of the numerical dissipation. Fig. 12 shows the mode shapes and frequencies of the structure. It is subjected to initial displacement and the response becomes free vibration.



$$K = \begin{bmatrix} 15100 & -100 \\ -100 & 100 \end{bmatrix}, \quad m = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Phi = \begin{bmatrix} 0.00667 & 0.99998 \\ 0.99998 & -0.00667 \end{bmatrix}$$

$$\omega_{1,2} = 9.967,\ 122.885\ rad/\sec$$

Fig. 12 Examples Structure and Properties

The initial condition for given structure is selected as the first mode shape add to 100 times of the second mode such as $\upsilon(0) = \left\{ \begin{array}{cc} 100.00467 & 0.3398 \end{array} \right\}^T$.

The exact solution for second floor displacement is in Fig. 13 and the numerical solutions are in Fig. 14, 15 and 16. The time step used in all the logarithms is 0.01 sec. Fig. 15 demonstrates that P-method and PC α-method can effectively damp out the second mode. After about 0.2 sec, P-method eliminates the second mode completely. PC α-method damp out the second mode after about 0.5 sec.

The Newmark ($\gamma = 0.6$, $\gamma = 0.8$) method takes 0.2 and 0.4 sec to damp out the second mode. When $\gamma = 0.5$ is used in Newmark method, the numerical dissipation is not appeared as discussed before. On the other hand, the α-method takes 1.4 sec to eliminate the second mode. Even though Newmark method damp out the second mode very quickly, the first mode also damped out strongly as Fig. 16 Obviously, the P-method can filter out the second mode very quickly and hardly affects the first mode at all. Actually, this result can be completely explained by the Fig. 8 since the Ωvalues for each mode are about 0.01 and 1.22 that correspond to about zero and 12% numerical damping. Thus, the first mode is almost not affected and the second mode can be damped out very quickly.

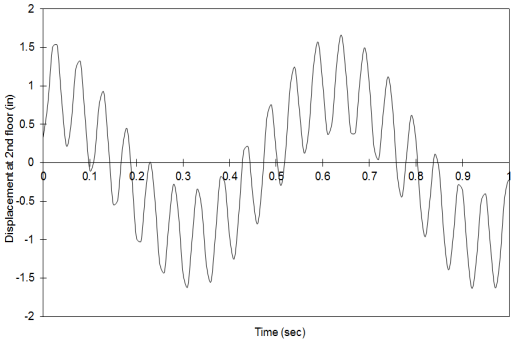


Fig. 13 Exact Response of 2nd Floor

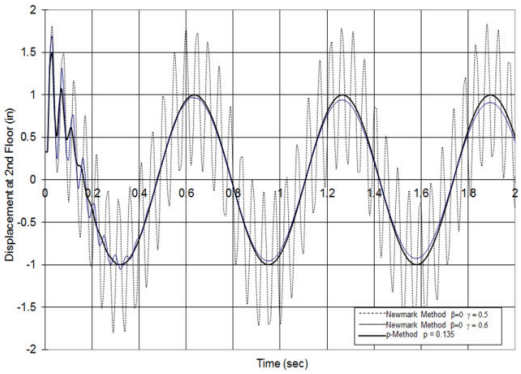


Fig. 14 Numerical Solution by Newmark(Y=0.5) and P-method

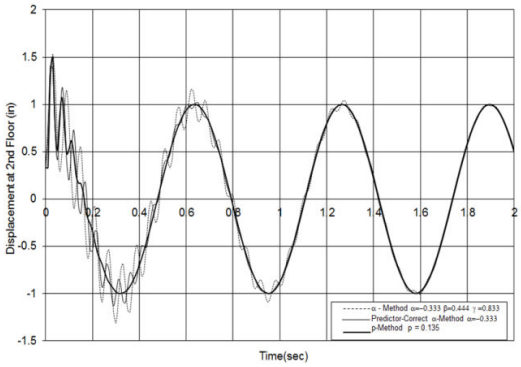


Fig. 15 Numerical Solution by PC α-method and α-method

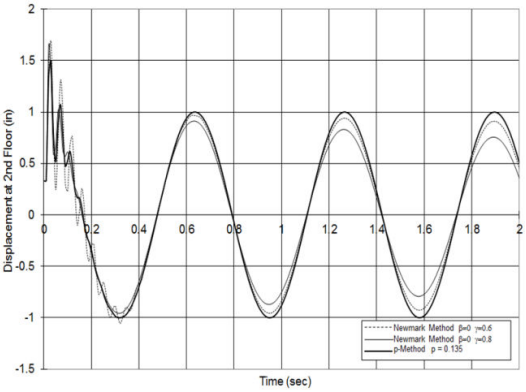


Fig. 16 Numerical Solution by Newmark Explicit and P-method

VII. Conclusion

P-method and PC α-method, conditionally stable, have been introduced and are shown to possess significantly improved numerical damping. In particular, those methods are of second-order-accuracy and they are possible to achieve zero damping. It was shown that P-method and PC α-method possess better accuracy than the Newmark explicit method since all the introduced algorithms are second-order methods while the Newmark explicit method is first-order method.

PC α-method gives more accuracy than other methods because it based on the α-method inherits the superior properties of the implicit α-method.

Even though P-method has the second order accuracy and numerical damping, it is not efficient to be implemented in nonlinear MDOF system. Because the parameter α is expressed in terms of p1, time step, and natural frequencies that are always changing during the time history analysis, i.e., the parameter α is not a constant and have to compute at the each time step. However, P-method can be applied to solve linear elastic MDOF system using modal superposition method. In spite of this limitation, it is still useful for pseudo-dynamic test methods since the spurious growth of higher mode response can be eliminated quickly by the numerical damping while lower modes are obtained accurately.

In finite element analysis, the PC α-method is more useful than other methods because it is the explicit scheme and it achieves the second order accuracy and numerical damping simultaneously.

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