Wiener Filter as an Optimal MMSE Interpolator

Tsai-Sheng Kao

Abstract—The ideal sinc filter, ignoring the noise statistics, is often applied for generating an arbitrary sample of a bandlimited signal by using the uniformly sampled data. In this article, an optimal interpolator is proposed; it reaches a minimum mean square error (MMSE) at its output in the presence of noise. The resulting interpolator is thus a Wiener filter, and both the optimal infinite impulse response (IIR) and finite impulse response (FIR) filters are presented. The mean square errors (MSE's) for the interpolator of different length impulse responses are obtained by computer simulations; it shows that the MSE's of the proposed interpolators with a reasonable length are improved about 0.4 dB under flat power spectra in noisy environment with signal-to-noise power ratio (SNR) equal 10 dB. As expected, the results also demonstrate the improvements for the MSE's with various fractional delays of the optimal interpolator against the ideal sinc filter under a fixed length impulse response.

Keywords—Interpolator, minimum mean square error, Wiener filter.

I. INTRODUCTION

INTERPOLATION of a bandlimited discrete-time signal [1] has found numerous applications in the fields of signal processing, including communications, speech processing, and music technology [2]-[5]. The impulse response of an ideal interpolator is basically the sinc function, which is noncausal and of infinite length; it is not physically realizable [6]. Thus, one practical design class of an interpolation filter is either by directly truncating or by windowing the ideal sinc function [2], [7], [8]. These interpolators only approximate the frequency response of the ideal one, and they are in general not optimal. However, few researchers have studied on the design of an interpolator in the presence of noise. In this study, we propose an optimal interpolator that minimizes the mean square error (MSE) with the knowledge of the signal and noise characteristics. The resulting interpolator is thus a Wiener filter, and both the optimal infinite impulse response (IIR) and finite impulse response (FIR) filters are presented. It is also shown that the optimal interpolator is a scaled version of the ideal one when the signal and noise have flat power spectral densities (PSD's). Finally, computer simulations show the benefits of the proposed interpolator under noisy environments and with different fractional delays.

II. OPTIMAL IIR INTERPOLATOR

For bandlimited signal processing, a fractional delay $0 \leq \mu < 1$ of a discrete-time signal $x(n)$ can be interpolated by the ideal filter with impulse response given by

$$
\begin{aligned}
h(n; \mu) &= \frac{\sin \pi(n - \mu)}{\pi(n - \mu)} \\
&= \text{sinc}(n - \mu), \quad \text{for all } n
\end{aligned}
\tag{1}
$$

The corresponding frequency response is thus denoted by

$$
H(\omega; \mu) = e^{-j\omega\mu}
\tag{2}
$$

where the magnitude response $|H(\omega; \mu)| = 1$ for all ω and the group delay defined by $\frac{-\partial arg\{H(\omega;\mu)\}}{\partial \omega}$ equals μ. However, as shown in Fig. 1, the received data $r(n)$ is always noise-corrupted, and it is impossible to access the noise-free data $x(n)$ to produce its delayed version $x_{id}(n)$ through the ideal filter $h(n; \mu)$. Hence, an optimal interpolator $c(n)$ is proposed to minimize the output mean square error $E[e^2(n)] = E[(y(n) - x_{id}(n))^2]$, where $E[\cdot]$ denotes the expectation operation and $y(n)$ is the convoluted output of $r(n)$ and $c(n)$.
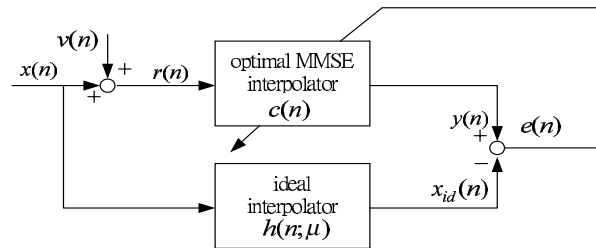

Fig. 1 Block diagram of the MMSE interpolator

Assume the data $x(n)$ and noise $v(n)$ are statistically independent and stationary, and the PSD $\varepsilon(\omega)$ of the output error $e(n)$ is given by

$$
\varepsilon(\omega) = S_x(\omega)||C(\omega) - H(\omega; u)||^2 + V(\omega)||C(\omega)||^2
\tag{3}
$$

where $S_x(\omega)$ and $V(\omega)$ are respectively the PSDs of $x(n)$ and $v(n)$, and $C(\omega)$ is the Fourier transformation of $c(n)$. Via Parseval relation, the MSE $E[e^2(n)]$ can be further expressed by

$$
E[e^2(n)] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varepsilon(\omega) d\omega
\tag{4}
$$

The optimal frequency response $C(\omega)$ that minimizes (4) is obtained by using a technique known as calculus of variations [9], and it is obtained as

$$
C(\omega) = \frac{S_x(\omega)H(\omega; \mu)}{S_x(\omega) + V(\omega)}
\tag{5}
$$

The minimum mean square error (MMSE) ϵ_{mmse} is computed by substituting (5) into (3-4) and is expressed as

$$
\epsilon_{mmse} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{\frac{1}{S_x(\omega)} + \frac{1}{V(\omega)}} d\omega
\tag{6}
$$

Notably, when the noise is negligible, i.e., $S_x(\omega) \gg V(\omega)$, the optimal frequency response of the interpolator is given by $C(\omega) = H(\omega; \mu)$, which is identical to the ideal one in (2).

Ideally, the MMSE is contributed by the noise and is given by $\epsilon_{mmse} = \frac{1}{2\pi}\int_{-\pi}^{\pi}V(\omega)d\omega$. Furthermore, when $x(n)$ and $v(n)$ have flat PSD's with $\sigma_x^2 = E[x^2(n)]$ and $\sigma_v^2 = E[v^2(n)]$, the frequency response given in (5) is reduced to $\frac{\sigma_x^2}{\sigma_x^2+\sigma_v^2}H(\omega;\mu)$ and the optimal impulse response is thus written as

$$c(n) = \frac{\sigma_x^2}{\sigma_x^2+\sigma_v^2}h(n;\mu), \text{ for all } n \qquad (7)$$

It is a scaled version of the ideal sinc filter described in (1). Physically, either the proposed IIR interpolator or the ideal one can not be directly implemented. Hence, a realizable interpolator of finite length is mandatory and is derived in the following section.

III. OPTIMAL FIR INTERPOLATOR

To be physically realizable, the optimal FIR interpolator $\hat{c}(n)$ is assumed to be finite length of $2N$ and its frequency response is given by $\hat{C}(\omega) = \sum_{n=-I_1}^{I_2}\hat{c}(n)e^{-j\omega n}$, where $I_1 = N$ and $I_2 = N-1$ [2]. The MSE is consequently defined as $E[\hat{e}^2(n)] = E[(\hat{y}(n) - x_{id}(n))^2]$, where $\hat{y}(n)$ is the convolution of $r(n)$ and $\hat{c}(n)$. By orthogonal principle, the minimization problem turns out to solving the following normal equations [10].

$$\sum_{k=-I_1}^{I_2}\hat{c}(k)R_r(k-m) = h_\mu(m), \text{ for } -I_1 \le m \le I_2 \qquad (8)$$

where $R_r(k) = R_x(k) + R_v(k)$ and $h_\mu(m) = \sum_{k=-\infty}^{\infty}h(k;\mu)R_x(k-m) = \frac{1}{2\pi}\int_{-\pi}^{\pi}e^{j\omega(m-\mu)}S_x(\omega)d\omega$. The terms $R_x(k) = E[x(n)x(n+k)]$ and $R_v(k) = E[v(n)v(n+k)]$ are the autocorrelation functions of $x(n)$ and $v(n)$, respectively. In a matrix form, (8) is rewritten as

$$\boldsymbol{R_r}\boldsymbol{\hat{c}} = \boldsymbol{h_\mu} \qquad (9)$$

where the (i,j) element of $\boldsymbol{R_r}$ is given by $R_r(i-j)$ for $1 \le i,j \le 2N$, $\boldsymbol{\hat{c}} = [\hat{c}(-I_1),\ldots,\hat{c}(I_2)]^T$, and $\boldsymbol{h_\mu} = [h_\mu(-I_1),\ldots,h_\mu(I_2)]^T$ (the superscript T denotes the transpose of a vector). Hence, the coefficients of the optimal FIR interpolator is solved by

$$\boldsymbol{\hat{c}} = \boldsymbol{R_r}^{-1}\boldsymbol{h_\mu} \qquad (10)$$

The MMSE $\hat{\epsilon}_{mmse}$, by substituting (10) into (3-4), is obtained by

$$\hat{\epsilon}_{mmse} = \frac{1}{2\pi}\int_{-\pi}^{\pi}S_x(\omega)||\hat{C}(\omega)-H(\omega;u)||^2 + V(\omega)||\hat{C}(\omega)||^2 d\omega \qquad (11)$$

We observe that the FIR interpolator with a smaller length of impulse response can not well reconstruct the delayed sampled, and the MSE could be mainly contributed by the interpolation error in (11), i.e., the term $S_x(\omega)||\hat{C}(\omega)-H(\omega;u)||^2$. Compared with the ideal sinc filter, an optimal interpolator of a longer impulse response can, however, obtain a lower mean square error. Similarly, when the signal and noise have
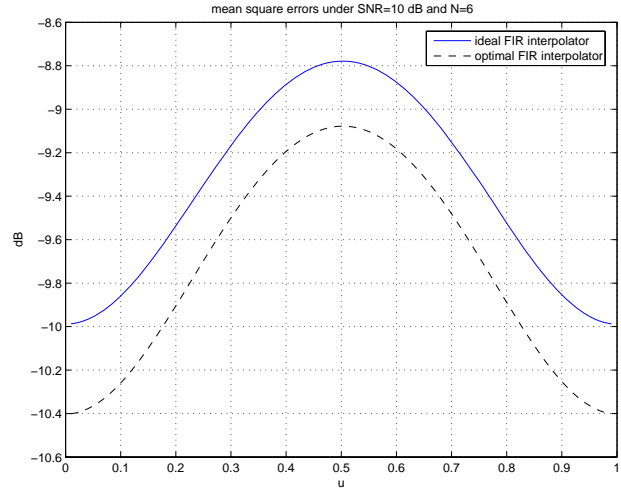

Fig. 2 Mean square errors for the different parameter settings of N

flat PSD's, it can be easily verified that equation (10) can be expressed as

$$\hat{c}(n) = \frac{\sigma_x^2}{\sigma_x^2+\sigma_v^2}h(n;\mu), \text{ for } -I_1 \le n \le I_2 \qquad (12)$$

It is also a scalar version of the ideal one when the noise power is taken into account.

IV. SIMULATION RESULTS

For simplicity, we assume $x(n)$ and $v(n)$ have flat power spectra and the signal-to-noise power ratio (SNR) is defined by $10\log\frac{\sigma_x^2}{\sigma_v^2}$. The parameter settings are $\sigma_x^2 = 1$, SNR=10 dB, and $\mu = 0.5$. Fig. 2 shows the output mean square errors for increasing the parameter N, where the ideal FIR interploator is depicted in solid line and the optimal one in dased line. When $N \le 3$, the MSE's of the both interpolators are almost the same, and the errors could be mainly contributed by the interpolation errors. However, these filters are more capable of reconstructing the delayed signal for a large setting of the parameter N; the MSE's of these proposed interpolators are 0.4 dB less than those of the ideal one's. Since the fractional delay μ is time-variant, the MSE's for different μ's are simulated for SNR=10 dB and $N = 6$. The results are depicted in Fig. 3, which illustrates the MSE's are improved, on average, about 0.35 dB.

V. CONCLUSIONS

We have proposed an optimal interpolator that minimizes the output mean square error in the presence of noise. The MSE of the proposed interpolator is improved about 0.4 dB under flat power spectra in noisy environments. Although it is verified for flat PSD's of the signal and noise, it can still be applied to a system with nonflat spectra when the characteristics can be empirically obtained. Analytic power
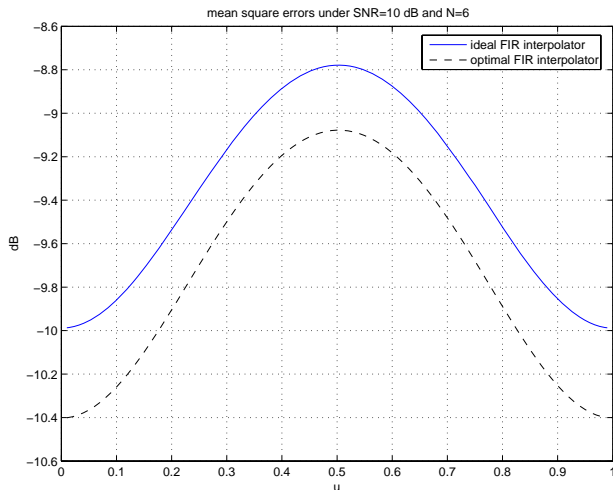
Fig. 3 Mean square errors for various fractional delays μ's

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spectral densities, for example, noise shaping or quantization error, can often be obtained or be modeled in the application. Unfortunately, most applications assume flat power spectral densities due to the difficulties of identifying the spectra. Nevertheless, the proposed interpolator can be applied in both cases.

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