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# The Spanning Laceability of $k$-ary $n$-cubes when $k$ is Even

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*Abstract*—$Q_n^k$ has been shown as an alternative to the hypercube family. For any even integer $k \geq 4$ and any integer $n \geq 2$, $Q_n^k$ is a bipartite graph. In this paper, we will prove that given any pair of vertices, $w$ and $b$, from different partite sets of $Q_n^k$, there exist $2n$ internally disjoint paths between $w$ and $b$, denoted by $\{P_i \mid 0 \leq i \leq 2n-1\}$, such that $\bigcup_{i=0}^{2n-1} P_i$ covers all vertices of $Q_n^k$. The result is optimal since each vertex of $Q_n^k$ has exactly $2n$ neighbors.

*Keywords*—container, Hamiltonian, $k$-ary $n$-cube, $m^*$-connected.

## I. Introduction

The $k$-ary $n$-cube, denoted by $Q_n^k$, has been proposed as an alternative to the hypercube since it shares many nice properties of $Q_n$ such as regular degrees, vertex symmetry, edge symmetry, recursive structure, etc.. The underlying topology of many machines is based on $k$-ary $n$-cubes, such as the Cray T3E, the iWARP, the Cray T3D and so on. Please see [1], [4], [11], [17]. Many reseachers have been working on $k$-ary $n$-cubes. For example, Stewart and Xiang [20] proved that the $k$-ary $n$-cube is edge-bipancyclic and bipanconnected for $k \geq 3$ and $n \geq 2$ and $k$ being even. Namely, any edge of a $k$-ary $n$-cube $Q_n^k$ lies on a cycle of any even length r for $4 \leq r \leq |Q_n^k|$, where $|Q_n^k|$ is the total number of vertices of $Q_n^k$. Besides, given two vertices $u$ and $v$ of $Q_n^k$, there exists a path of any even length $r$ between $u$ and $v$ for $d(u,v) \leq r \leq |Q_n^k|$, where $d(u,v)$ is the distance between $u$ and $v$. Other studies about fault tolerance on $k$-ary $n$-cubes can be found in [8], [23]. Recently, there are many studies about the spanning connectivity for interconnection networks and graphs [9]. A graph $H = (B \bigcup W, E)$ is *bipartite* if $V(H)$ is the union of two disjoint sets $B$ and $W$ such that every edge joins $B$ with $W$. It is easy to see that any bipartite graph with at least three vertices is not hamiltonian connected except $K_2$. Note that any (nontrivial) bipartite graph except $K_2$ cannot be hamiltonian connected, whereas a bipartite graph is *hamiltonian laceable* if there exists a hamiltonian path between any two vertices $u$, $v$ with $u \in B$ and $v \in W$ [22]. A graph $H = (B \bigcup W, E)$ is a *balanced* bipartite graph if $|V(B)| = |V(W)|$. Throughout this thesis, we only work on $Q_n^k$ with $k \geq 4$ an even integer and $n \geq 2$, which are balanced bipartite graphs. A bipartite graph $H = (B \bigcup W, E)$ is $m^*$-*laceable* if given a white vertex $w \in W$ and a black vertex $b \in B$, there exist(s) $m$ internal disjoint paths between $w$ and $b$, denoted by $P_i$ for $0 \leq i \leq m-1$, such that $\bigcup_0^{m-1} P_i$ covers $V$. The *spanning laceability* of a graph $H$, $\kappa^*(H)$, is the largest integer $k$ such that $H$ is $m^*$-laceable for every $m$ with $1 \leq m \leq k$. A higher spanning connectivity/laceability of the interconnection network implies a more efficient communication between processors. About the spanning connectivity and the spanning laceability, readers can refer to [6], [7], [12]–[15].

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In this paper, we want to show the spanning laceability of $k$-ary $n$-cubes for any even integer $k \geq 4$. More precisely, we show that given a white vertex $w$ and a black vertex $b$ of a $k$-ary $n$-cube $Q_n^k$, there exist(s) $m$ internally disjoint path(s) between $w$ and $b$ whose union covers all vertices of $Q_n^k$ for $1 \leq m \leq 2n$. The result is optimal since any vertex in $Q_n^k$ has exactly $2n$ neighbors. This paper is organized as follows. In Section 2, we introduce the graph terminologies and symbols that will be used in the paper and the definition of $Q_n^k$. In Section 3, we show our main results.

## II. Preliminaries

Throughout this paper, we follow [3] for the graph definitions and notations. The sets of vertices and edges of a graph $G$ are denoted by $V(G)$ and $E(G)$, respectively. If $u$, $v$ are vertices of a graph $G$ such that there is an edge $e = (u,v) \in E(G)$ between $u$ and $v$, then we say that the vertices $u$ and $v$ are *adjacent* in $G$. The *degree* of any vertex $x$ is the number of distinct vertices adjacent to $x$. A *path* $P$ between two vertices $v_0$ and $v_k$ is represented by $P = \langle v_0, v_1, \ldots, v_k \rangle$, where each pair of consecutive vertices are connected by an edge. We use $P^{-1}$ to denote the path $\langle v_k, v_{k-1}, v_{k-2}, \ldots, v_0 \rangle$. We also write the path $P = \langle v_0, v_1, \ldots, v_k \rangle$ as $\langle v_0, v_1, \ldots, v_i, Q, v_j, v_{j+1}, \ldots, v_k \rangle$, where $Q$ denotes the path $\langle v_i, v_{i+1}, \ldots, v_j \rangle$. A *hamiltonian path* between $u$ and $v$, where $u$ and $v$ are two distinct vertices of $G$, is a path joining $u$ to $v$ that visits every vertex of $G$ exactly once. A *cycle* is a path of at least three vertices such that the first vertex is the same as the last vertex. A *hamiltonian cycle* of $G$ is a cycle that traverses every vertex of $G$ exactly once. A *hamiltonian graph* is a graph with a hamiltonian cycle. A graph $G$ is *connected* if there is a path between any two distinct vertices in $G$ and is *hamiltonian connected* if there is a hamiltonian path between any two distinct vertices in $G$ [18]. A graph $H = (W \cup B, E)$ is *bipartite* if $V(H) = W \cup B$ and $E(H)$ is a subset of $\{(w,b) | w \in W, b \in B\}$. A bipartite graph $H$ is *hamiltonian laceable* if there is a hamiltonian path between any two distinct vertices from different partite sets in $H$.

A graph $G$ is $k$-*connected* if there exists $V' \subseteq V(G)$ with $|V'| = k$ such that $G - V'$ is disconnected and $G - V''$ is connected for any $V'' \subseteq V(G)$ with $|V''| < k$. It follows from Menger's Theorem [16] that for every $k$-connected graph $G$, there exist $k$ internally vertex-disjoint paths between any pair of distinct vertices of $G$. A *$k$-container* $C(u, v)$ in a graph $G$ is a set of $k$ internally vertex-disjoint paths between two distinct vertices $u$ and $v$. We say that a graph $G$ has a *spanning $k$-container* between $u$ and $v$, denoted by $C(u, v)$, if $C(u, v)$ is a $k$-container that covers all vertices of $G$. A spanning $k$-container is also abbreviated as a *$k^*$-container* for simplicity. A graph $G$ is *$k^*$-connected* if there is a $k^*$-container between any pair of vertices of $G$. Obviously, a graph $G$ is hamiltonian connected if and only if $G$ is $1^*$-connected, and $G$ is hamiltonian if and only if $G$ is $2^*$-connected. Lin et al. [13] defined the concept of *spanning connectivity*. The *spanning connectivity* of a graph $G$, $\kappa^*(G)$, is the largest integer $k$ such that $G$ is $w^*$-connected for all $1 \leq w \leq k$. Similarly, a bipartite graph $H$ is *$k^*$-laceable* if there is a $k^*$-container between any pair of two vertices from different partite sets of $H$. Also, a bipartite graph $H$ is hamiltonian laceable if and only if $H$ is $1^*$-laceable, and $H$ is hamiltonian if and only if $H$ is $2^*$-laceable. So, the *spanning laceability* of a bipartite graph $H$, $\kappa^*(H)$, is the largest integer $k$ such that $H$ is $m^*$-laceable for all $1 \leq m \leq k$.

The $k$-ary $n$-cube, $Q_n^k$, is defined for all integers $k \geq 2$ and $n \geq 1$. The subclass $Q_n^2$ is the well-studied hypercube family. The subclass $Q_1^k$ with $k \geq 3$ is defined as the cycle of length $k$. The $k$-ary $n$-cube, $Q_n^k$, for $k \geq 3$ and $n \geq 2$ is defined as follows. Let $u \in V(Q_n^k)$ be represented by $(u(0), u(1), ..., u(n-1))$, where $0 \leq u(i) \leq k-1$. Two vertices $u$ and $v$ are adjacent if and only if $|u(i) - v(i)| = 1$ or $k-1$ for some $i$ and $u(j) = v(j)$ for any $0 \leq j \leq n-1$ with $j \neq i$. It is shown that $Q_n^k$ is bipartite if $k$ is even [10]. Here we mention some properties of $Q_n^k$ that will be used in this paper.

$Q_n^k$ is *vertex symmetric* (and *edge symmetric*) [10]. It means that given any two distinct vertices $v$ and $v'$ of $Q_n^k$, there is an automorphism of $Q_n^k$ mapping $v$ to $v'$. Note that each vertex of $Q_n^k$ is represented by a $n$-bit tuple. We will call the $d$th-bit *the dth dimension*. We can partition $Q_n^k$ over dimension $d$ by fixing the $d$th element of any vertex tuple at some value $a$ for every $a \in \{0, 1, ..., k-1\}$. This results in $k$ copies of $Q_{n-1}^k$, denoted by $Q_{n-1}^{k,0}$, $Q_{n-1}^{k,1}$, ..., $Q_{n-1}^{k,k-1}$, with corresponding vertices in $Q_{n-1}^{k,0}, Q_{n-1}^{k,1}, ..., Q_{n-1}^{k,k-1}$ joined in a cycle of length $k$ (in dimension $d$) [19].

In this article, we always partition $Q_n^k$ over the 0-th dimension by letting $V(Q_{n-1}^{k,i}) = \{((i), v(1), v(2), \ldots, v(n-1)) \mid 0 \leq v(j) \leq k-1, \forall 1 \leq j \leq n-1\}$ for $0 \leq i \leq k-1$. Given a vertex $x = (x(0), x(1), \ldots, x(n-1)) \in V(Q_n^k)$, the symbol $x^j = ((j), x(1), x(2), \ldots, x(n-1))$, where $0 \leq j \leq k-1$, is defined to be the vertex corresponding to $x$ in $Q_{n-1}^{k,j}$ for simplicity. So, if $P = \langle x_0, x_1, \ldots, x_{n-1} \rangle$, $P^j$ is represented by $\langle x_0^j, x_1^j, \ldots, x_{n-1}^j \rangle$. Throughout this paper, let $n \geq 2$ be an integer and $k \geq 4$ an even integer.

**Theorem 1.** *[10] For any even integer $k \geq 4$, $Q_n^k$ is hamiltonian laceable for $n \geq 2$. In other words, $Q_n^k$ is $1^*$-laceable.*

**Theorem 2.** *[5] The graph $Q_n^k$ is hamiltonian. In other words, $Q_n^k$ is $2^*$-laceable.*

## III. Main Results

**Lemma 1.** *Given $Q_n^k$ and its $k$ subcubes, $Q_{n-1}^{k,i}$, where $0 \leq i \leq k-1$. Let $j$ and $j'$ be two integers satisfying $0 \leq j \leq j' \leq k-1$, $w \in V(Q_{n-1}^{k,j})$ an arbitrary white vertex, and $b \in V(Q_{n-1}^{k,j'})$ an arbitrary black vertex. Then there exists a path between $w$ and $b$ that visits each vertex in $Q_{n-1}^{k,j}$, $Q_{n-1}^{k,j+1}$, $Q_{n-1}^{k,j+2}$, ..., $Q_{n-1}^{k,j'}$ exactly once.*

*Proof:* There are three cases.

**Case 1.** $j = j'$. W.L.O.G., let $j = j' = 0$. By Theorem 1, $Q_{n-1}^{k,0}$ is hamiltonian laceable. Thus, there is a hamiltonian path between $w$ and $b$ that visits each vertex of $Q_{n-1}^{k,0}$ exactly once.

**Case 2.** $j - j' = 1$. W.L.O.G., we can let $j = 0$ and $j' = 1$. Let $w$ be a white vertex in $Q_{n-1}^{k,0}$ and $b$ a black vertex in $Q_{n-1}^{k,1}$. We can find a pair of adjacent vertices $x^0$ and $x^1$ where $x^0$ is a black vertex of $Q_{n-1}^{k,0}$ and $x^1$ is a white vertex of $Q_{n-1}^{k,1}$. By Theorem 1, there exists a hamiltonian path $P_0$ of $Q_{n-1}^{k,0}$ between $w$ and $x^0$, and a hamiltonian path $P_1$ of $Q_{n-1}^{k,1}$ between $x^1$ and $b$. Let $P = \langle w, P_0, x^0, x^1, P_1, b \rangle$. Hence $P$ is the path between $w$ and $b$ that visits every vertex of $Q_{n-1}^{k,0}$ and $Q_{n-1}^{k,1}$ exactly once.

**Case 3.** $j - j' \geq 2$. Let $w$ be a white vertex in $Q_{n-1}^{k,j}$ and $b$ be a black vertex in $Q_{n-1}^{k,j'}$. There are $j - j' + 1$ $k$-ary $n-1$-cubes, $Q_{n-1}^{k,j}, Q_{n-1}^{j,j+1}, Q_{n-1}^{k,j+2}, \ldots, Q_{n-1}^{k,j'-1}$ and $Q_{n-1}^{k,j'}$. There are $j' - j$ pairs of adjacent vertices $x^r \in Q_{n-1}^{k,r}$ and $y^{r+1} \in Q_{n-1}^{k,r+1}$ where $x^r$ is a black vertex and $y^{r+1}$ is a white vertex for $j \leq r \leq j'-1$. By Theorem 1, there is a hamiltonian path $R_r$ of $Q_{n-1}^{k,r}$ joining $y^r$ to $x^r$, where $j+1 \leq r \leq j'-1$. Again, with Theorem 1, there exists a hamiltonian path $T$ of $Q_{n-1}^{k,j}$ joining $w$ to $x^j$, and a hamiltonian path $U$ of $Q_{n-1}^{k,j'}$ joining $y^{j'}$ to $b$. Let $P = \langle w, T, x^j, y^{j+1}, R_{j+1}, x^{j+1}, y^{j+2}, R_{j+2}, x^{j+2}, \ldots, y^{j'-1}, R_{j'-1}, x^{j-1}, y^{j'}, U, b \rangle$. Therefore, $P$ is a path covering all the vertices of $Q_{n-1}^{k,j}, Q_{n-1}^{j,j+1}, Q_{n-1}^{k,j+2}, \ldots, Q_{n-1}^{k,j'}$ for $0 \leq j \leq j' \leq k-1$ between $w$ and $b$. Please see Figure 1 for an illustration.

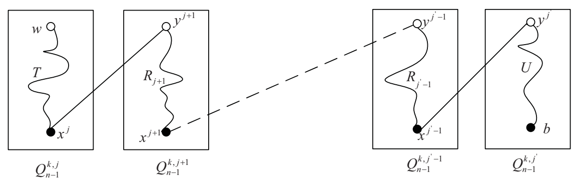

Fig. 1. The illustration for Case 3 of Lemma 1.

■

**Lemma 2.** *Given $Q_n^k$ and its $k$ subcubes $Q_{n-1}^{k,i}$ for $0 \leq i \leq k-1$. Let $w$ be a white vertex, $b$ a black vertex in $Q_{n-1}^{k,i}$, and $j$ an integer with $0 \leq i \leq j \leq k-1$. There exists a path between $w$ and $b$ that covers all the vertices of $Q_{n-1}^{k,i}$, $Q_{n-1}^{k,i+1}$, ..., and $Q_{n-1}^{k,j}$.*

*Proof:* We consider the following two cases.

**Case 1.** $j = i$. There is only one $k$-ary $(n-1)$-cube $Q_{n-1}^{k,i}$. By Theorem 1, the lemma holds in this case.

**Case 2.** $j \neq i$. There are $j - i + 1$ $k$-ary $(n-1)$-cubes. According to Theorem 1, there is hamiltonian path $P_i$ that covers all the vertices of $Q_{n-1}^{k,i}$ between $w$ and $b$ of the form $\langle w, S_i, x^i, y^i, T_i, b\rangle$, where $\{x^i, y^i\}$ is an edge of $Q_{n-1}^{k,i}$ with $\{x^i, y^i\} \bigcap \{w, b\} = \emptyset$. Notice that by Theorem 1, $Q_{n-1}^{k,r}$ is hamiltonian laceable and hence there exists a hamiltonian path $P_r$ between $x^r$ and $y^r$ of the form $\langle x^r, S_r, z^r, w^r, T_r, y^r\rangle$ for $i+1 \leq r \leq j$. Let the required path between $w$ and $b$ be $R$, we have the following two subcubes.

**Case 2.1.** If $j - i + 1$ is even, then
$R = \langle w, S_i, x^i, x^{i+1}, S_{i+1}, z^{i+1}, z^{i+2}, (S_{i+2})^{-1}, x^{i+2}, x^{i+3}, S_{i+3}, z^{i+3}, z^{i+4}, (S_{i+4})^{-1}, x^{i+4}, \ldots, x^j, S_j, z^j, w^j, T_j, y^j, y^{j-1}, (T_{j-1})^{-1}, w^{j-1}, w^{j-2}, T_{j-2}, y^{j-2}, y^{j-3}, (T_{j-3})^{-1}, w^{j-3}, \ldots, y^{i+1}, y^i, T_i, b\rangle$. Please see Figure 2 for an illustration.

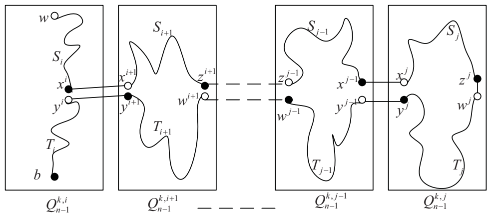

Fig. 2. The illustration for Lemma 2 when $j - i + 1$ is even.

**Case 2.2.** If $j - i + 1$ is odd, then
$R = \langle w, S_i, x^i, x^{i+1}, S_{i+1}, z^{i+1}, z^{i+2}, (S_{i+2})^{-1}, x^{i+2}, x^{i+3}, S_{i+3}, z^{i+3}, z^{i+4}, (S_{i+4})^{-1}, x^{i+4}, \ldots, z^j, (S_j)^{-1}, x^j, y^j, (T_j)^{-1}, w^j, w^{j-1}, T_{j-1}, y^{j-1}, y^{j-2}, (T_{j-2})^{-1}, w^{j-2}, w^{j-3}, T_{j-3}, y^{j-3}, \ldots, y^{i+1}, y^i, T_i, b\rangle$. Please see Figure 2 for an illustration. ■

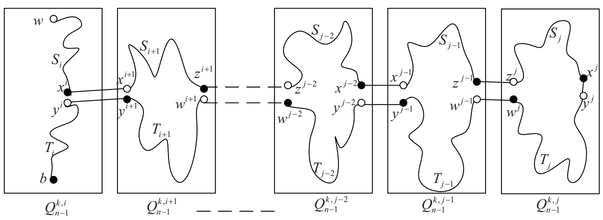

Fig. 3. The illustration for Lemma 2 when $j - i + 1$ is odd.

**Lemma 3.** *The graph $Q_2^4$ is $3^*$-laceable and $4^*$-laceable.*

*Proof:* The proof is by brute force. Reader can refer to Appendix A. ■

**Lemma 4.** *The graph $Q_2^6$ is $3^*$-laceable and $4^*$-laceable.*

*Proof:* By brute force, we constructed all spanning containers. Please see Appendix B. ■

**Lemma 5.** *The graph $Q_2^k$ is $3^*$-laceable and $4^*$-laceable for any even integer $k \geq 6$.*

*Proof:* With Lemma 4, we have shown that $Q_2^6$ is $3^*$-laceable and $4^*$-laceable. Now we will present a recursive algorithm that uses a $3^*$-container (resp. $4^*$-container) of $Q_2^k$ to construct a $3^*$-container (resp. $4^*$-container) of $Q_2^{k+2}$. Let $R$ be a subset of $V(Q_2^k) \cup E(Q_2^k)$. We define a function, $f$, which maps $R$ from $Q_2^k$ into $Q_2^{k+2}$ in the following way:
(1) If $(i,j) \in R \cap V(Q_2^k)$, where $0 \leq i,j \leq k-1$, then

$$f((i,j)) = \begin{cases} (i,j) & \text{if } 0 \leq i,j \leq k-2; \\ (i+2,j) & \text{if } i = k-1, 0 \leq j \leq k-2; \\ (i,j+2) & \text{if } j = k-1, 0 \leq i \leq k-2; \\ (i+2,j+2) & \text{if } i = k-1 = j. \end{cases}$$

(2) If $((i,j),(i',j')) \in R \cap E(Q_2^k)$, where $i \leq i', j \leq j'$, then $f(((i,j),(i',j')))$

$$= \begin{cases} ((i,j),(i',j')) & \text{if } 0 \leq i,j \leq k-3, \\ & \quad 1 \leq i',j' \leq k-2; \\ ((i+2,j),(i'+2,j)) & \text{if } i = i' = k-1, \\ & \quad 0 \leq j \leq k-3, \\ & \quad 1 \leq j' \leq k-2; \\ ((i,j+2),(i',j'+2)) & \text{if } j = j' = k-1, \\ & \quad 0 \leq i \leq k-3, \\ & \quad 1 \leq i' \leq k-2; \\ ((i,j),(i',j'+2)) & \text{if } 0 \leq i = i' \leq k-2, \\ & \quad j = 0, j' = k-1; \\ ((i,j),(i'+2,j')) & \text{if } 0 \leq j = j' \leq k-2, \\ & \quad i = 0, i' = k-1; \\ ((i,j+2),(i'+2,j'+2)) & \text{if } i = 0, i' = k-1, \\ & \quad j = j' = k-1; \\ ((i+2,j),(i'+2,j'+2)) & \text{if } j = 0, j' = k-1, \\ & \quad i = i' = k-1. \end{cases}$$

Let $w$ be a white vertex and $b$ be a black vertex of $Q_2^k$. We say that a $3^*$-container (resp. $4^*$-container) $C(u,v)$ of $Q_2^k$ is *regular* if $C(w,b)$ contains some edges in $\{((\alpha, k-2), (\alpha, k-1)) \mid 0 \leq \alpha \leq k-1\}$ and $\{((k-2,\beta),(k-1,\beta)) \mid 0 \leq \beta \leq k-1\}$. For example, all $3^*$-containers and $4^*$-containers of $Q_2^6$ constructed in Lemma 4 are regular. Let $C(w,b)$ be a regular $3^*$-container (resp. $4^*$-container) of $Q_2^k$ with the endvertex set $P = \{w = (0,0), b = (x,y)\}$. We construct a regular $3^*$-container (resp. $4^*$-container) of $Q_2^{k+2}$ with the endvertex set $f(P)$ using the following algorithm. Please see Figure 4 for an illustration.

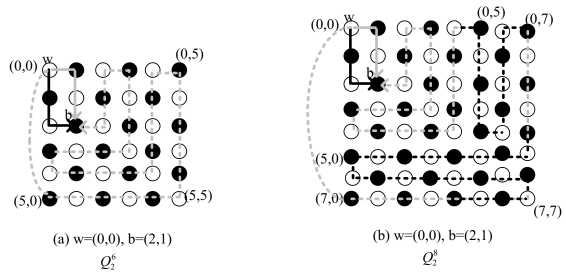


Fig. 4. Using the $4^*$-container of $Q_2^6$ to construct the $4^*$-container of $Q_2^8$.

**Step 1.** In $Q_2^k$, let $\{v_0, v_1, \ldots, v_{t-1}\}$ and $\{h_0, h_1, \ldots, h_{s-1}\}$ be finite sequences of indices satisfying the following requirements:
(1) $0 \leq v_0 < v_1 < \ldots < v_{t-1} \leq k-1$ and $k-1 \geq h_0 > h_1 > \ldots > h_{s-1} \geq 0$;

(2) for $0 \leq i \leq k-1$, $((v_i, k-2), (v_i, k-1))$ is an edge of $C(w,b)$; for $0 \leq j \leq k-1$, $((k-2, h_j), (k-1, h_j))$ is an edge of $C(w,b)$.

**Step 2.** Let $\overline{C}(w,b)$ be the image in $Q_2^{k+2}$ of $C(w,b) - (\{((v_i, k-2), (v_i, k-1)) \mid 0 \leq i \leq k-1\} \cup \{((k-2, h_j), (k-1, h_j)) \mid 0 \leq j \leq k-1\})$ under the function $f$. Please see Figure 5 for an illustration.

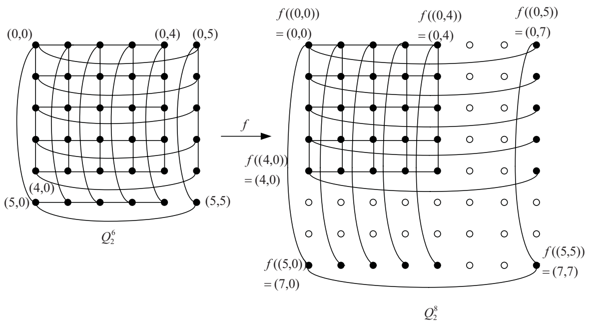


Fig. 5. Using function $f$ to map a subset of edges and vertices of $Q_2^6$ into $Q_2^8$.

**Step 3.** For any two positive integers $r$ and $d$, we use $[r]_d$ to denote $r(\textbf{mod}\ d)$. In $Q_2^{k+2}$, define the following path patterns, where $r_1$, $r_2$ are integers:

$$\begin{aligned} I_\alpha(r_1, r_2) &= \langle (r_1, \alpha), ([r_1+1]_{k+2}, \alpha), \ldots, (r_2, \alpha)\rangle; \\ I_\alpha^{-1}(r_2, r_1) &= \langle (r_2, \alpha), ([r_2-1]_{k+2}, \alpha), \ldots, (r_1, \alpha)\rangle; \\ H_\beta(r_1, r_2) &= \langle (\beta, r_1), (\beta, [r_1+1]_{k+2}), \ldots, (\beta, r_2)\rangle; \\ H_\beta^{-1}(r_2, r_1) &= \langle (\beta, r_2), (\beta, [r_2-1]_{k+2}), \ldots, (\beta, r_1)\rangle. \end{aligned}$$

Let $\overline{v}_i = v_i + 2$ if $v_i = k-1$ and $\overline{v}_i = v_i$ if $0 \leq v_i \leq k-2$, and $\overline{h}_j = h_j + 2$ if $h_j = k-1$ and $\overline{h}_j = h_j$ if $0 \leq h_j \leq k-2$.

**Case 1.** $v_0 = k-1$.
Let $P_0 = \langle (k+1, k-2), (k+1, k-1), (0, k-1), I_{k-1}(0, k-2), (k-2, k-1), (k-2, k), I_k^{-1}(k-2, 0), (0, k), (k+1, k), (k+1, k+1)\rangle$.

<u>Case 1.1.</u> $s = 1$.
Let $\overline{P}_0 = \langle (k-2, \overline{h}_0), (k-1, \overline{h}_0), H_{k-1}^{-1}(\overline{h}_0, [\overline{h}_0+1]_{k+2}), (k-1, [\overline{h}_0+1]_{k+2}), (k, [\overline{h}_0+1]_{k+2}), H_k([\overline{h}_0+1]_{k+2}, \overline{h}_0), (k, \overline{h}_0), (k+1, \overline{h}_0)\rangle$. Then $\overline{C}(w,b) \cup P_0 \cup \overline{P}_0$ is the $3^*$-container (or $4^*$-container) of $Q_2^{k+2}$.

<u>Case 1.2.</u> $s \geq 2$.
Let $\overline{P}_i = \langle (k-2, \overline{h}_i), (k-1, \overline{h}_i), H_{k-1}^{-1}(\overline{h}_i, \overline{h}_{i+1}+1), (k-1, \overline{h}_{i+1}+1), (k, \overline{h}_{i+1}+1), H_k(\overline{h}_{i+1}+1, \overline{h}_i), (k, \overline{h}_i), (k+1, \overline{h}_i)\rangle$ for $0 \leq i \leq s-2$, and $\overline{P}_{s-1} = \langle (k-2, \overline{h}_{s-1}), (k-1, \overline{h}_{s-1}), H_{k-1}^{-1}(\overline{h}_{s-1}, [\overline{h}_0+1]_{k+2}), (k-1, [\overline{h}_0+1]_{k+2}), (k, [\overline{h}_0+1]_{k+2}), H_k([\overline{h}_0+1]_{k+2}, \overline{h}_{s-1}), (k, \overline{h}_{s-1}), (k+1, \overline{h}_{s-1})\rangle$. Then $\overline{C}(w,b) \cup P_0 \cup \{\overline{P}_i \mid 0 \leq i \leq s-1\}$ is the $3^*$-container (or $4^*$-container) of $Q_2^{k+2}$.

**Case 2.** $v_{t-1} \leq k-2$ and $((k-2, k-1), (k-1, k-1)) \in E(C(w,b))$ in $Q_2^k$.

<u>Case 2.1.</u> $t = 1$.
Let $P_0 = \langle (\overline{v}_0, k-2), (\overline{v}_0, k-1), I_{k-1}(\overline{v}_0, k-2), (k-2, k-1), (k-2, k), I_k^{-1}(k-2, \overline{v}_0), (\overline{v}_0, k), (\overline{v}_0, k+1)\rangle$.

<u>Case 2.1.1</u> $s = 1$.
Let $\overline{P}_0 = \langle (k-2, \overline{h}_0), (k-1, \overline{h}_0), H_{k-1}^{-1}(\overline{h}_0, 0), (k-1, 0), (k, 0), H_k(0, k-1), (k, k-1), (k+1, k-1), I_{k-1}(k+1, [\overline{v}_0-1]_{k+2}), ([\overline{v}_0-1]_{k+2}, k-1), ([\overline{v}_0-1]_{k+2}, k), I_k^{-1}([\overline{v}_0-1]_{k+2}, k+1), (k+1, k), (k, k), (k, \overline{h}_0), (k+1, \overline{h}_0)\rangle$. Then $\overline{C}(w,b) \cup P_0 \cup \overline{P}_0$ is the $3^*$-container (or $4^*$-container) of $Q_2^{k+2}$.

<u>Case 2.1.2</u> $s = 2$.
Let $\overline{P}_0 = \langle (k-2, \overline{h}_0), (k-1, \overline{h}_0), H_{k-1}^{-1}(\overline{h}_0, \overline{h}_1+1), (k-1, \overline{h}_1+1), (k, \overline{h}_1+1), H_k(\overline{h}_1+1, k-1), (k, k-1), (k+1, k-1), I_{k-1}(k+1, [\overline{v}_0-1]_{k+2}), ([\overline{v}_0-1]_{k+2}, k-1), ([\overline{v}_0-1]_{k+2}, k), I_k^{-1}([\overline{v}_0-1]_{k+2}, k+1), (k+1, k), (k, k), (k, \overline{h}_0), (k+1, \overline{h}_0)\rangle$, and $\overline{P}_1 = \langle (k-2, \overline{h}_1), (k-1, \overline{h}_1), H_{k-1}^{-1}(\overline{h}_1, 0), (k-1, 0), (k, 0), H_k(0, \overline{h}_1), (k, \overline{h}_1), (k+1, \overline{h}_1)\rangle$. Then $\overline{C}(w,b) \cup P_0 \cup \overline{P}_0 \cup \overline{P}_1$ is the $3^*$-container (or $4^*$-container) of $Q_2^{k+2}$.

<u>Case 2.1.3</u> $s \geq 3$.
Let $\overline{P}_0 = \langle (k-2, \overline{h}_0), (k-1, \overline{h}_0), H_{k-1}^{-1}(\overline{h}_0, \overline{h}_1+1), (k-1, \overline{h}_1+1), (k, \overline{h}_1+1), H_k(\overline{h}_1+1, k-1), (k, k-1), (k+1, k-1), I_{k-1}(k+1, [\overline{v}_0-1]_{k+2}), ([\overline{v}_0-1]_{k+2}, k-1), ([\overline{v}_0-1]_{k+2}, k), I_k^{-1}([\overline{v}_0-1]_{k+2}, k+1), (k+1, k), (k, k), (k, \overline{h}_0), (k+1, \overline{h}_0)\rangle$, $\overline{P}_i = \langle (k-2, \overline{h}_i), (k-1, \overline{h}_i), H_{k-1}^{-1}(\overline{h}_i, \overline{h}_{i+1}+1), (k-1, \overline{h}_{i+1}+1), (k, \overline{h}_{i+1}+1), H_k(\overline{h}_{i+1}+1, \overline{h}_i), (k, \overline{h}_i), (k+1, \overline{h}_i)\rangle$ for $1 \leq i \leq s-2$, and $\overline{P}_{s-1} = \langle (k-2, \overline{h}_{s-1}), (k-1, \overline{h}_{s-1}), H_{k-1}^{-1}(\overline{h}_{s-1}, 0), (k-1, 0), (k, 0), H_k(0, \overline{h}_{s-1}), (k, \overline{h}_{s-1}), (k+1, \overline{h}_{s-1})\rangle$. Then $\overline{C}(w,b) \cup P_0 \cup \{\overline{P}_i \mid 0 \leq i \leq s-1\}$ is the $3^*$-container (or $4^*$-container) of $Q_2^{k+2}$.

<u>Case 2.2.</u> $t \geq 2$.
Let $P_i = \langle (\overline{v}_i, k-2), (\overline{v}_i, k-1), I_{k-1}(\overline{v}_i, \overline{v}_{i+1}-1), (\overline{v}_{i+1}-1, k-1), (\overline{v}_{i+1}-1, k), I_k^{-1}(\overline{v}_{i+1}-1, \overline{v}_i), (\overline{v}_i, k), (\overline{v}_i, k+1)\rangle$ for $0 \leq i \leq t-2$, and $P_{t-1} = \langle (\overline{v}_{t-1}, k-2), (\overline{v}_{t-1}, k-1), I_{k-1}(\overline{v}_{t-1}, k-2), (k-2, k-1), (k-2, k), I_k^{-1}(k-2, \overline{v}_{t-1}), (\overline{v}_{t-1}, k), (\overline{v}_{t-1}, k+1)\rangle$.

<u>Case 2.2.1</u> $s = 1$.
Using the same $\overline{P}_0$ as in Case 2.1.1, then $\overline{C}(w,b) \cup \{P_i \mid 0 \leq i \leq t-1\} \cup \overline{P}_0$ is the $3^*$-container (or $4^*$-container) of $Q_2^{k+2}$.

<u>Case 2.2.2</u> $s = 2$.
Using the same $\overline{P}_0$ and $\overline{P}_1$ as in Case 2.1.2., then $\overline{C}(w,b) \cup \{P_i \mid 0 \leq i \leq t-1\} \cup \overline{P}_0 \cup \overline{P}_1$ is the $3^*$-container (or $4^*$-container) of $Q_2^{k+2}$.

<u>Case 2.2.3</u> $s \geq 3$.
Using the same $\{\overline{P}_i \mid 0 \leq i \leq s-1\}$ as in Case 2.1.3., then $\overline{C}(w,b) \cup \{P_i \mid 0 \leq i \leq t-1\} \cup \{\overline{P}_i \mid 0 \leq i \leq s-1\}$ is the $3^*$-container (or $4^*$-container) of $Q_2^{k+2}$.

**Case 3.** $v_{t-1} \leq k-2$ and $((k-2, k-1), (k-1, k-1)) \notin E(C(w,b))$ in $Q_2^k$.

<u>Case 3.1.</u> $t = 1$.
Let $P_0 = \langle (\overline{v}_0, k-2), (\overline{v}_0, k-1), I_{k-1}(\overline{v}_0, k-1), (k-1, k-1), H_{k-1}^{-1}(k-1, \overline{h}_0+1), (k-1, \overline{h}_0+1), (k, \overline{h}_0+1), H_k(\overline{h}_0+1, k-1), (k, k-1), (k+1, k-1), (0, k-1), I_{k-1}(0, \overline{v}_0-1), (\overline{v}_0-1, k-1), (\overline{v}_0-1, k), I_k^{-1}(\overline{v}_0-1, 0), (0, k), (k+1, k), (k, k), (k, k+1), (k-1, k+1), (k-1, k), I_k^{-1}(k-1, \overline{v}_0), (\overline{v}_0, k), (\overline{v}_0, k+1)\rangle$.

<u>Case 3.1.1</u> $s = 1$.
Let $\overline{P}_0 = \langle (k-2, \overline{h}_0), (k-1, \overline{h}_0), H_{k-1}^{-1}(\overline{h}_0, 0), (k-1, 0), (k, 0), H_k(0, \overline{h}_0), (k, \overline{h}_0), (k+1, \overline{h}_0)\rangle$. Then $\overline{C}(w,b) \cup P_0 \cup \overline{P}_0$ is the $3^*$-container (or $4^*$-container) of $Q_2^{k+2}$.

<u>Case 3.1.2</u> $s \geq 2$.
Let $\overline{P}_i = \langle (k-2, \overline{h}_i), (k-1, \overline{h}_i), H_{k-1}^{-1}(\overline{h}_i, \overline{h}_{i+1}+1), (k-1, \overline{h}_{i+1}+1), (k, \overline{h}_{i+1}+1), H_k(\overline{h}_{i+1}+1, \overline{h}_i), (k, \overline{h}_i), (k+1, \overline{h}_i)\rangle$

for $0 \le i \le s-2$, and $\overline{P}_{s-1} = \langle (k-2, \overline{h}_{s-1}), (k-1, \overline{h}_{s-1}), H_{k-1}^{-1}(\overline{h}_{s-1}, 0), (k-1, 0), (k, 0), H_k(0, \overline{h}_{s-1}), (k, \overline{h}_{s-1}), (k+1, \overline{h}_{s-1}) \rangle$. Then $\overline{C}(w,b) \cup P_0 \cup \{\overline{P}_i \mid 0 \le i \le s-1\}$ is the $3^*$-container (or $4^*$-container) of $Q_2^{k+2}$.

Case 3.2. $t \ge 2$.

Let $P_i = \langle (\overline{v}_i, k-2), (\overline{v}_i, k-1), I_{k-1}(\overline{v}_i, \overline{v}_{i+1}-1), (\overline{v}_{i+1}-1, k-1), (\overline{v}_{i+1}-1, k), I_k^{-1}(\overline{v}_{i+1}-1, \overline{v}_i), (\overline{v}_i, k), (\overline{v}_i, k+1) \rangle$ for $0 \le i \le t-2$, and $P_{t-1} = \langle (\overline{v}_{t-1}, k-2), (\overline{v}_{t-1}, k-1), I_{k-1}(\overline{v}_{t-1}, k-1), (k-1, k-1), H_{k-1}^{-1}(k-1, \overline{h}_0+1), (k-1, \overline{h}_0+1), (k, \overline{h}_0+1), H_k(\overline{h}_0+1, k-1), (k, k-1), (k+1, k-1), (0, k-1), I_{k-1}(0, \overline{v}_0-1), (\overline{v}_0-1, k-1), (\overline{v}_0-1, k), I_k^{-1}(\overline{v}_0-1, 0), (0, k), (k+1, k), (k, k), (k, k+1), (k-1, k+1), (k-1, k), I_k^{-1}(k-1, \overline{v}_{t-1}), (\overline{v}_{t-1}, k), (\overline{v}_{t-1}, k+1) \rangle$.

Case 3.2.1 $s = 1$.

Using the same $\overline{P}_0$ as in Case 3.1.1, then $\overline{C}(w,b) \cup \{P_i \mid 0 \le i \le t-1\} \cup \overline{P}_0$ is the $3^*$-container (or $4^*$-container) of $Q_2^{k+2}$.

Case 3.2.2 $s \ge 2$.

Using the same $\{\overline{P}_i \mid 0 \le i \le s-1\}$ as in Case 3.1.2., then $\overline{C}(w,b) \cup \{P_i \mid 0 \le i \le t-1\} \cup \{\overline{P}_i \mid 0 \le i \le s-1\}$ is the $3^*$-container (or $4^*$-container) of $Q_2^{k+2}$.

**Case 4.** $v_{t-1} = k-1$ for some $t \ge 2$ and $v_0 = 0$.

Case 4.1. $t = 2$.

Let $P_0 = \langle (\overline{v}_0, k-2), (\overline{v}_0, k-1), I_{k-1}(\overline{v}_0, k-2), (k-2, k-1), (k-2, k), I_k^{-1}(k-2, \overline{v}_0), (\overline{v}_0, k), (\overline{v}_0, k+1) \rangle$, and $P_1 = \langle (k+1, k-2), (k+1, k-1), (k+1, k), (k+1, k+1) \rangle$.

Case 4.1.1. $s = 1$.

Using the same $\overline{P}_0$ as in Case 1.1., then $\overline{C}(w,b) \cup P_0 \cup P_1 \cup \overline{P}_0$ is the $3^*$-container (or $4^*$-container) of $Q_2^{k+2}$.

Case 4.1.2. $s \ge 2$.

Using the same $\{\overline{P}_i \mid 0 \le i \le s-1\}$ as in Case 1.2., then $\overline{C}(w,b) \cup P_0 \cup P_1 \cup \{\overline{P}_i \mid 0 \le i \le s-1\}$ is the $3^*$-container (or $4^*$-container) of $Q_2^{k+2}$.

Case 4.2. $t \ge 3$.

Let $P_i = \langle (\overline{v}_i, k-2), (\overline{v}_i, k-1), I_{k-1}(\overline{v}_i, \overline{v}_{i+1}-1), (\overline{v}_{i+1}-1, k-1), (\overline{v}_{i+1}-1, k), I_k^{-1}(\overline{v}_{i+1}-1, \overline{v}_i), (\overline{v}_i, k), (\overline{v}_i, k+1) \rangle$ for $0 \le i \le t-3$, $P_{t-2} = \langle (\overline{v}_{t-2}, k-2), (\overline{v}_{t-2}, k-1), I_{k-1}(\overline{v}_{t-2}, k-2), (k-2, k-1), (k-2, k), I_k^{-1}(k-2, \overline{v}_{t-2}), (\overline{v}_{t-2}, k), (\overline{v}_{t-2}, k+1) \rangle$, and $P_{t-1} = \langle (k+1, k-2), (k+1, k-1), (k+1, k), (k+1, k+1) \rangle$.

Case 4.2.1. $s = 1$.

Using the same $\overline{P}_0$ as in Case 1.1., then $\overline{C}(w,b) \cup \{P_i \mid 0 \le i \le t-1\} \cup \overline{P}_0$ is the $3^*$-container (or $4^*$-container) of $Q_2^{k+2}$.

Case 4.2.2. $s \ge 2$.

Using the same $\{\overline{P}_i \mid 0 \le i \le s-1\}$ as in Case 1.2., then $\overline{C}(w,b) \cup \{P_i \mid 0 \le i \le t-1\} \cup \{\overline{P}_i \mid 0 \le i \le s-1\}$ is the $3^*$-container (or $4^*$-container) of $Q_2^{k+2}$. ■

**Theorem 3.** *For any integer $n \ge 2$ and any even integer $k \ge 4$, the graph $Q_n^k$ is $m^*$-laceable where $1 \le m \le 2n$.*

*Proof:* According to Theorem 2-3 and Lemma 3-5, the theorem holds for any even integer $k \ge 4$ when $n = 2$. We will give the proof of the theorem by mathematical induction on $n$. By induction hypothesis, assume that $Q_{n-1}^{k,i}$ is $m^*$-laceable for $1 \le m \le 2n-2$, where $0 \le i \le k-1$. Given a white vertex $w \in V(Q_{n-1}^{k,j})$ and a black vertex $b \in V(Q_{n-1}^{k,j'})$. We will show that we can use the $m^*$-containers of $Q_{n-1}^{k,j}$ to construct a $(m+2)^*$-container of $Q_n^k$ between $w$ and $b$.

**Case 1.** For $j = j'$. Without loss of generality, we let $j = j' = 0$.

In this case, we have $\{w, b\} \in Q_{n-1}^{k,0}$. By induction hypothesis, there are $m$ internal disjoint paths $\{P_i\}_{i=0}^{m-1}$ whose union covers all vertices of $Q_{n-1}^{k,0}$ between $w$ and $b$ for $1 \le m \le 2n-2$. By Lemma 2, the exists a path $S$ covering all vertices of $Q_{n-1}^{k,i}$ for $1 \le i \le k-2$ between $w^1$ and $b^1$. We can let $P_m = \langle w, w^1, S, b^1, b \rangle$. In $Q_{n-1}^{k,k-1}$, there exist a hamiltonian path $R$ joining from $w^{k-1}$ to $b^{k-1}$ by Theorem 1. Also, we can let $P_{m+1} = \langle w, w^{k-1}, R, b^{k-1}, b \rangle$. Therefore, there are $m+2$ internal disjoint paths $\{P_i\}_{i=0}^{m+1}$ whose union covers all vertices of $Q_n^k$ between $w$ and $b$. Please see Figure 6 for an illustration.

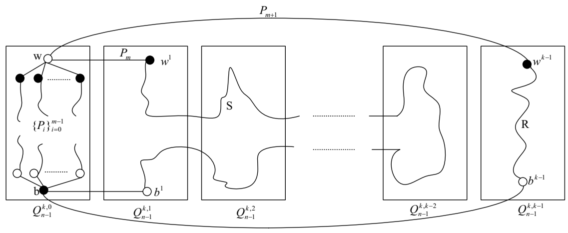


Fig. 6. The illustration for Case 1 of Theorem 3.

**Case 2.** For $|j' - j| = 1$. Without loss of generality, we let $j = 0$ and $j' = 1$.

We have the following two cases.

Case 2.1. Suppose that $d(w,b) = 1$. It is easy to see that we can let $P_{m+1} = \langle w, b \rangle$.

Case 2.1.1. If $m = 1$.

Let $z$ be any black vertex of $Q_{n-1}^{k,0}$. By Theorem 1, there exist a hamiltonian path $S$ of $Q_{n-1}^{k,0}$ from $w$ to $z$, and a hamiltonian path $T$ of $Q_{n-1}^{k,1}$ from $z^1$ to $b$. So we set $P_0 = \langle w, S, z, z^1, T, b \rangle$. According to Lemma 1, a hamiltonian path $R$ between $w^{k-1} \in Q_{n-1}^{k,k-1}$ and $b^2 \in Q_{n-1}^{k,2}$ covers all vertices of $Q_{n-1}^{k,i}$ for $2 \le i \le k-1$. We can write $P_1$ as $\langle w, w^{k-1}, R, b^2, b \rangle$. Hence, there are 3 internal disjoint paths $\{P_0, P_1, P_2\}$ whose union covers all vertices of $Q_n^k$ between $w$ and $b$. Please see Figure 7 for an illustration.

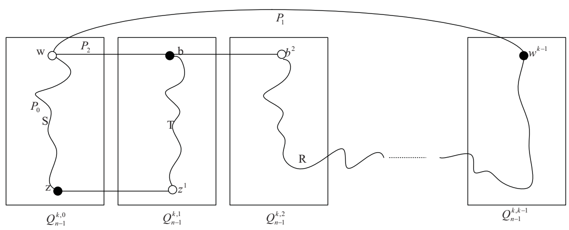


Fig. 7. The illustration for Case 2.1.1 of Theorem 3.

Case 2.1.2. If $m \ge 2$.

According to the induction hypothesis, given any black vertex $z \in V(Q_{n-1}^{k,0} - N(w))$, there exist $m$ internal disjoint paths $\{R_i\}_{i=0}^{m-1}$ whose union covers all vertices of $Q_{n-1}^{k,0}$ between $w$ and $z$ for $2 \le m \le 2n-2$. Let $R_i = \langle w, S_i, y_i, z \rangle$ for $0 \le i \le m-1$. We set $P_0 = \langle w, S_0, y_0, z, z^1, y_0^1, (S_0^1)^{-1}, b \rangle$ and $P_i = \langle w, S_i, y_i, y_i^1, (S_i^1)^{-1}, b \rangle$ for $1 \le i \le m-1$. By

Lemma 1, there is a hamiltonian path $T$ between $w^{k-1} \in Q^{k,k-1}_{n-1}$ and $b^2 \in Q^{k,2}_{n-1}$ covering all vertices of $Q^{k,i}_{n-1}$ for $2 \le i \le k-1$. Set $P_m = \langle w, w^{k-1}, T, b^2, b\rangle$. Consequently, there are $m+2$ internal disjoint paths $\{P_i\}_{i=0}^{m+1}$ whose union covers all vertices of $Q^k_n$ between $w$ and $b$. Please see Figure 8 for an illustration.

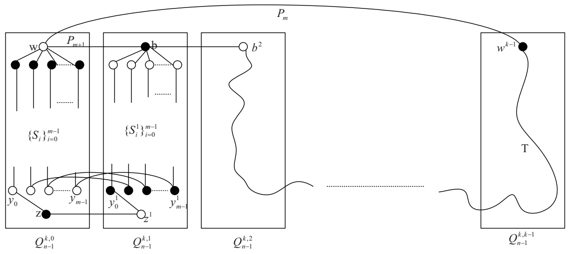

Fig. 8. The illustration for Case 2.1.2 of Theorem 3.

Case 2.2. Suppose that $d(w,b) \ge 3$.
Case 2.2.1. If $m = 1$.
Given any black vertex $z$ in $Q^{k,0}_{n-1}$, by Theorem 1, there is a hamiltonian path $R$ of $Q^{k,0}_{n-1}$ joining from $w$ to $z$. So there is also a hamiltonian path $S$ of $Q^{k,1}_{n-1}$ between $w^1$ to $z^1$. We can set $S = \langle w^1, S'_1, b, S'_2, z^1\rangle$. By Lemma 1, there exists a hamiltonian path $T$ between $w^{k-1} \in Q^{k,k-1}_{n-1}$ and $b^2 \in Q^{k,2}_{n-1}$ covering all vertices of $Q^{k,i}_{n-1}$ for $2 \le i \le k-1$. We let $P_0 = \langle w, R, z, z^1, (S'_2)^{-1}, b\rangle$, $P_1 = \langle w, w^1, S'_1, b\rangle$, and $P_2 = \langle w, w^{k-1}, T, b^2, b\rangle$. Therefore, there are 3 internal disjoint paths $\{P_0, P_1, P_2\}$ whose union covers all vertices of $Q^k_n$ between $w$ and $b$. Please see Figure 9 for an illustration.

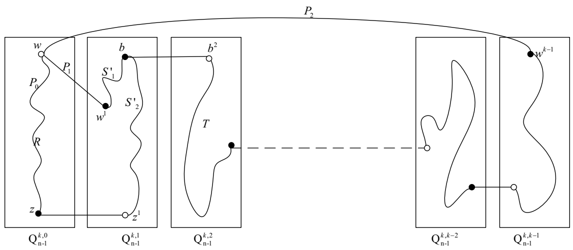

Fig. 9. The illustration for Case 2.2.1 of Theorem 3.

Case 2.2.2. If $m \ge 2$.
Let $z$ be a black vertex of $V(Q^{k,0}_{n-1} - N(w))$. In $Q^{k,0}_{n-1}$, according to the induction hypothesis, there exist $m$ internal disjoint paths $\{S_i\}_{i=0}^{m-1}$ whose union covers all vertices of $Q^{k,0}_{n-1}$ between $w$ and $z$ for $2 \le m \le 2n-2$. So as in $Q^{k,1}_{n-1}$, there exist $m$ internal disjoint paths $\{T_i\}_{i=0}^{m-1}$ whose union covers all vertices of $Q^{k,1}_{n-1}$ between $z^1$ and $b$ for $2 \le m \le 2n-2$. Let $T_0 = \langle z^1, y_0, T'_0, x_0, w^1, T''_0, b\rangle$ and $T_i = \langle z^1, y_i, T'_i, b\rangle$ for $1 \le i \le m-1$ in $Q^{k,1}_{n-1}$.
Case 2.2.2.1. If $b^0 \notin V(S_0)$.
Without loss of generality, let $b^0 \in V(S_{m-1})$. In $Q^{k,0}_{n-1}$, we also let $S_0 = \langle w, x^0_0, e, S'_0, y^0_0, z\rangle$, $S_i = \langle w, S'_i, y^0_i, z\rangle$ for $1 \le i \le m-2$, and $S_{m-1} = \langle w, S'_{m-1}, b^0, f, S''_{m-1}, y^0_{m-1}, z\rangle$. A hamiltonian path $R$ is embedded in $Q^{k,k-1}_{n-1}$ between $w^{k-1}$ and $f^{k-1}$ by Theorem 1. Write $R$ as $\langle w^{k-1}, R', e^{k-1}, g, R'', f^{k-1}\rangle$. Notice that $g^{k-2}$ is a black vertex and $b^2$ is a white vertex. According to Lemma 1, there is a hamiltonian path $U$ between $g^{k-2}$ and $b^2$ covering all vertices of $Q^{k,i}_{n-1}$ for $2 \le i \le k-2$. We can set $P_0 = \langle w, x^0_0, x_0, (T'_0)^{-1}, y_0, z^1, y_{m-1}, T_{m-1}, b\rangle$, $P_1 = \langle w, w^1, T''_0, b\rangle$, $P_2 = \langle w, w^{k-1}, R', e^{k-1}, e, S'_0, y^0_0, z, y^0_{m-1}, (S''_{m-1})^{-1}, f, f^{k-1}, (R'')^{-1}, g, g^{k-2}, U, b^2, b\rangle$, $P_3 = \langle w, S'_{m-1}, b^0, b\rangle$, and $P_i = \langle w, S'_{i-3}, y^0_{i-3}, y_{i-3}, T'_{i-3}, b\rangle$ for $4 \le i \le m+1$. So, there are $m+2$ internal disjoint paths $\{P_i\}_{i=0}^{m+1}$ whose union covers all vertices of $Q^k_n$ between $w$ and $b$. Please see Figure 10 for an illustration.

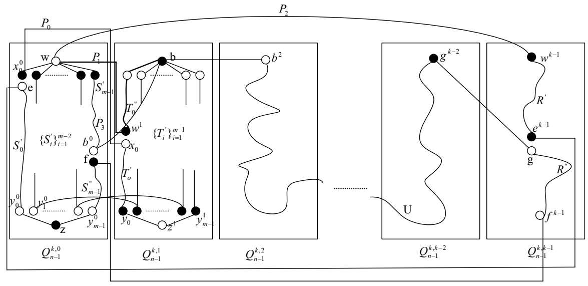

Fig. 10. The illustration for Case 2.2.2.1 of Theorem 3.

Case 2.2.2.2. If $b^0 \in V(S_0)$.
Let $S_0 = \langle w, x^0_0, e, S'_0, b^0, f, S''_0, y^0_0, z\rangle$, and $S_i = \langle w, S'_i, y^0_i, z\rangle$ for $1 \le i \le m-1$. A hamiltonian path $R$ is embedded in $Q^{k,k-1}_{n-1}$ between $w^{k-1}$ and $f^{k-1}$ by Theorem 1. $R$ is written as $\langle w^{k-1}, R', e^{k-1}, g, R'', f^{k-1}\rangle$. Notice that $g^{k-2}$ is a black vertex and $b^2$ is a white vertex. According to Lemma 1, there is a hamiltonian path $U$ between $g^{k-2}$ and $b^2$ covering all vertices of $Q^{k,i}_{n-1}$ for $2 \le i \le k-2$. We let $P_0 = \langle w, x^0_0, x_0, (T'_0)^{-1}, y_0, z^1, y_{m-1}, T'_{m-1}, b\rangle$, $P_1 = \langle w, w^1, T''_0, b\rangle$, $P_2 = \langle w, w^{k-1}, R', e^{k-1}, e, S'_0, b^0, b\rangle$, $P_3 = \langle w, S'_{m-1}, y^0_{m-1}, z, y^0_0, (S''_0)^{-1}, f, f^{k-1}, (R'')^{-1}, g, g^{k-2}, U, b^2, b\rangle$, and $P_i = \langle w, S'_{i-3}, y^0_{i-3}, y_{i-3}, T'_{i-3}, b\rangle$ for $4 \le i \le m+1$. Hence, there are $m+2$ internal disjoint paths $\{P_i\}_{i=0}^{m+1}$ whose union covers all vertices of $Q^k_n$ between $w$ and $b$. Please see Figure 11 for an illustration.

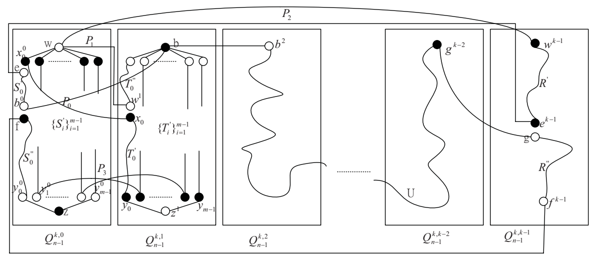

Fig. 11. The illustration for Case 2.2.2.2 of Theorem 3.

**Case 3.** For $|j'-j| \ge 2$. Without loss of generality, we let $j = 0$ and $2 \le j' \le \frac{k}{2}$ be even.
Because $b \in Q^{k,j'}_{n-1}$ where $j'$ is even, $b^i$ is a white (resp. black) vertex in $Q^{k,i}_{n-1}$ for $0 \le i \le k-1$ when $i$ is odd (resp. even). It is easy to see that $w^i$ is a black (resp. white) vertex in $Q^{k,i}_{n-1}$ for $0 \le i \le k-1$ when $i$ is odd (resp. even). By the induction hypothesis, there exist $m$ internal disjoint paths $\{R^i_p\}_{p=0}^{m-1}$ of $Q^{k,i}_{n-1}$ between $w^i$ and $b^i$ for $0 \le i \le j'$.

Let $R_p^i = \langle w^i, x_p^i, U_p^i, y_p^i, b^i \rangle$ for $0 \leq p \leq m-1$ and $0 \leq i \leq j'$. According to Lemma 2, a hamiltonian path $S$ covers all vertices of $Q_{n-1}^{k,i}$ for $j'+1 \leq i \leq k-2$ joining from $w^{j'+1}$ to $b^{j'+1}$. There is a hamiltonian path $T$ of $Q_{n-1}^{k,k-1}$ from $w^{k-1}$ to $b^{k-1}$ by Theorem 1. Hence, we can write $P_p = \langle w = w^0, x_p^0, U_p^0, y_p^0, y_p^1, (U_p^1)^{-1}, x_p^1, x_p^2, U_p^2, \ldots, (U_p^{j'-1})^{-1}, x_p^{j'-1}, x_p^{j'}, U_p^{j'}, y_p^{j'}, b^{j'} = b \rangle$ for $0 \leq p \leq m-1$, $P_m = \langle w = w^0, w^1, w^2, \ldots, w^{j'}, w^{j'+1}, S, b^{j'+1}, b^{j'} = b \rangle$, and $P_{m+1} = \langle w = w^0, w^{k-1}, T, b^{k-1}, b^0, b^1, \ldots, b^{j'-1}, b^{j'} = b \rangle$. Therefore, there are $m+2$ internal disjoint paths $\{P_i\}_{i=0}^{m+1}$ whose union covers all vertices of $Q_n^k$ between $w$ and $b$. Please see Figure 12 for an illustration.

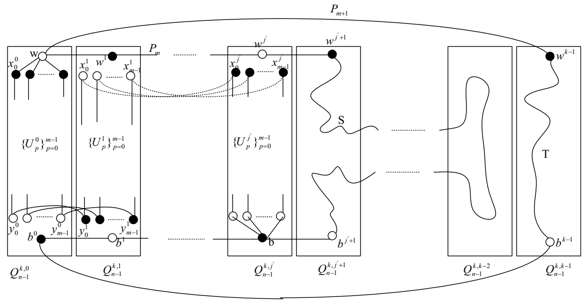

Fig. 12. The illustration for Case 3 of Theorem 3.

**Case 4.** For $|j' - j| \geq 2$. Without loss of generality, we let $j = 0$ and $3 \leq j' \leq \frac{k}{2} + 1$ be odd.

Case 4.1. If $m = 1$.

Choosing a black vertex $z$ of $Q_{n-1}^{k,0}$, by Theorem 1, there is a hamiltonian path $R$ of $Q_{n-1}^{k,0}$ joining from $w$ to $z$. In $Q_{n-1}^{k,k-1}$, there exists a hamiltonian path $S$ of $Q_{n-1}^{k,k-1}$ between $w^{k-1}$ and $z^{k-1}$. We can let $S = \langle w^{k-1}, S', e, b^{k-1}, S'', z^{k-1} \rangle$, where $b^{k-1}$ is a black vertex of $Q_{n-1}^{k,k-1}$, so $e$ is a white vertex of $Q_{n-1}^{k,k-1}$. By Theorem 1, there is a hamiltonian path $T$ of $Q_{n-1}^{k,k-2}$ joining from $e^{k-2}$ to $b^{k-2}$. Let $T = \langle e^{k-2}, W, f^{k-2}, b^{k-2} \rangle$. In $Q_{n-1}^{k,i}$, we also have a hamiltonian path $T^i$ between $e^i$ and $b^i$ for $j' \leq i \leq k-3$, so we let $T^i = \langle e^i, W^i, f^i, b^i \rangle$. According to Lemma 1, there is a hamiltonian path $U$ between a black vertex $w^1 \in Q_{n-1}^{k,1}$ and a white vertex $b^{j'-1} \in Q_{n-1}^{k,j'-1}$ covering all vertices of $Q_{n-1}^{k,i}$ for $2 \leq i \leq j'-1$. We set $P_0 = \langle w, w^1, U, b^{j'-1}, b \rangle$, $P_1 = \langle w, R, z, z^{k-1}, (S'')^{-1}, b^{k-1}, b^{k-2}, \ldots, b^{j'+1}, b^{j'} = b \rangle$, and $P_2 = \langle w, w^{k-1}, S', e, e^{k-2}, W, f^{k-2}, f^{k-3}, (W^{k-3})^{-1}, e^{k-3}, e^{k-4}, W^{k-4}, f^{k-4}, \ldots, e^{j'+1}, W^{j'+1}, f^{j'+1}, f^{j'}, W^{j'}, b^{j'} = b \rangle$. Hence, there are 3 internal disjoint paths $\{P_0, P_1, P_2\}$ whose union covers all vertices of $Q_n^k$ between $w$ and $b$. Please see Figure 13 for an illustration.

Case 4.2. If $m \geq 2$.

Given a white vertex $z$ in $Q_{n-1}^{k,j'}$ such that $z$ is adjacent to $b$. So $z^i$ is a black (resp. white) vertex and $w^i$ is a white (reps. black) vertex of $Q_{n-1}^{k,i}$ if $0 \leq i \leq j'-1$ when $i$ is even (resp. odd). By the induction hypothesis, there exist $m$ internal disjoint paths $\{R_i\}_{i=0}^{m-1}$ of $Q_{n-1}^{k,0}$ between $w$ and $z^0$. We write $R_0 = \langle w, x_0(1), x_0(2), \ldots, x_0(\alpha), z^0 \rangle$, and $R_p = \langle w, x_p, S_p, y_p, z^0 \rangle$ for $1 \leq p \leq m-1$. Again, by the induction hypothesis, there exist $m$ internal disjoint paths

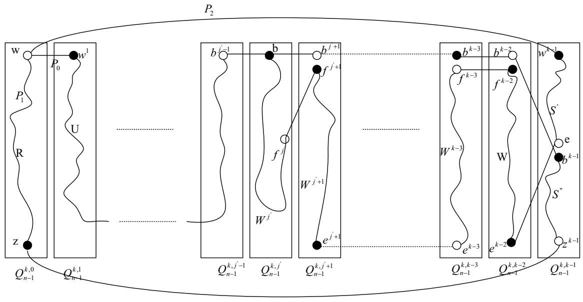

Fig. 13. The illustration for Case 4.1 of Theorem 3.

$\{T_p^i\}_{p=0}^{m-1}$ of $Q_{n-1}^{k,i}$ between $w^i$ and $z^i$ for $2 \leq i \leq j'-1$. We let $T_p^i = \langle w^i, x_p^i, U_p^i, t_p^i, z^i \rangle$ for $0 \leq p \leq m-1$ and $2 \leq i \leq j'-1$. Notice that $b^{j'-1}$ is adjacent to $z^{j'-1}$, without loss of generality, we let $t_{m-1}^{j'-1} = b^{j'-1}$. In $Q_{n-1}^{k,j'}$, there are $m$ internal disjoint paths $\{W_i\}_{i=0}^{m-1}$ from $b$ to $z$ by the induction hypothesis. We can write $W_p = \langle z, t_p^{j'}, Y_p, b \rangle$ for $0 \leq p \leq m-2$ and $W_{m-1} = \langle z, b \rangle$. According to Lemma 1, there is a hamiltonian path $V$ between $w^{k-1} \in Q_{n-1}^{k,k-1}$ and $b^{j'+1} \in Q_{n-1}^{k,j'+1}$ covering all vertices of $Q_{n-1}^{k,i}$ for $j'+1 \leq i \leq k-1$. Set $P_0 = \langle w, w^{k-1}, V, b^{j'+1}, b \rangle$, $P_1 = \langle w, w^1, w^2, x_0^2, U_0^2, t_0^2, t_0^3, (U_0^3)^{-1}, x_0^3, w^3, w^4, \ldots, w^{j'-1}, x_0^{j'-1}, U_0^{j'-1}, t_0^{j'-1}, t_0^{j'}, Y_0, b \rangle$, $P_2 = \langle w, x_0(1), x_0^1(1), x_0^1(2), x_0(2), \ldots, x_0(\alpha-1), x_0^1(\alpha-1), x_0^1(\alpha), x_0(\alpha), z^0, z^1, \ldots, z^{j'}, b \rangle$, $P_3 = \langle w, x_{m-1}, S_{m-1}, y_{m-1}, y_{m-1}^1, (S_{m-1}^1)^{-1}, x_{m-1}^1, x_{m-1}^2, U_{m-1}^2, t_{m-1}^2, t_{m-1}^3, (U_{m-1}^3)^{-1}, x_{m-1}^3, \ldots, x_{m-1}^{j'-1}, U_{m-1}^{j'-1}, t_{m-1}^{j'-1} = b^{j'-1}, b \rangle$, and $P_i = \langle w, x_{i-3}, S_{i-3}, y_{i-3}, y_{i-3}^1, (S_{i-3}^1)^{-1}, x_{i-3}^1, x_{i-3}^2, U_{i-3}^2, t_{i-3}^2, t_{i-3}^3, (U_{i-3}^3)^{-1}, x_{i-3}^3, \ldots, x_{i-3}^{j'-1}, U_{i-3}^{j'-1}, t_{i-3}^{j'-1}, t_{i-3}^{j'}, Y_{i-3}, b \rangle$ for $4 \leq i \leq m+1$. So, there are $m+2$ internal disjoint paths $\{P_i\}_{i=0}^{m+1}$ whose union covers all vertices of $Q_n^k$ between $w$ and $b$. Please see Figure 14 for an illustration.

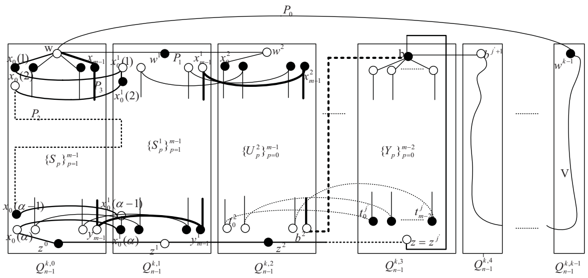

Fig. 14. The illustration for Case 4.2 of Theorem 3.

■

## Appendix A
## Proof of Lemma 3

Notice that $Q_2^4$ is vertex symmetric. W.L.O.G, let $w = (0,0)$. There are only two cases for $b$. That is, $b \in \{(1,0),(2,1)\}$.

**Case 1.** To prove that $Q_2^4$ is $3^*$-laceable.

Case 1.1. Let $b = (1,0)$.

The three disjoint paths $\{P_1, P_2, P_3\}$ between $w$ and $b$ whose

union covers all vertices of $Q_2^4$ are $P_1 = \langle(0,0),(1,0)\rangle$, $P_2 = \langle(0,0),(0,1),(1,1),(1,0)\rangle$, and $P_3 = \langle(0,0),(3,0),(3,1),(3,2),(3,3),(2,3),(1,3),(0,3),(0,2),(1,2),(2,2),(2,1),(2,0),(1,0)\rangle$.
Case 1.2. Let $b = (2,1)$.
The three disjoint paths $\{R_1,R_2,R_3\}$ between $w$ and $b$ whose union covers all vertices of $Q_2^4$ are $R_1 = \langle(0,0),(1,0),(2,0),(2,1)\rangle$, $R_2 = \langle(0,0),(0,1),(1,1),(2,1)\rangle$, and $R_3 = \langle(0,0),(3,0),(3,1),(3,2),(3,3),(2,3),(1,3),(0,3),(0,2),(1,2),(2,2),(2,1)\rangle$.
**Case 2.** To prove that $Q_2^4$ is $4^*$-laceable.
Case 2.1. Let $b = (1,0)$.
The four disjoint paths $\{P_1,P_2,P_3,P_4\}$ between $w$ and $b$ whose union covers all vertices of $Q_2^4$ are $P_1 = \langle(0,0),(1,0)\rangle$, $P_2 = \langle(0,0),(0,1),(1,1),(1,0)\rangle$, $P_3 = \langle(0,0),(0,3),(0,2),(1,2),(1,3),(1,0)\rangle$, and $P_4 = \langle(0,0),(3,0),(3,1),(3,2),(3,3),(2,3),(2,2),(2,1),(2,0),(1,0)\rangle$.
Case 2.2. Let $b = (2,1)$.
The four disjoint paths $\{R_1,R_2,R_3,R_4\}$ between $w$ and $b$ whose union covers all vertices of $Q_2^4$ are $R_1 = \langle(0,0),(3,0),(3,1),(2,1)\rangle$, $R_2 = \langle(0,0),(1,0),(2,0),(2,1)\rangle$, $R_3 = \langle(0,0),(0,1),(1,1),(2,1)\rangle$, and $R_4 = \langle(0,0),(0,3),(0,2),(1,2),(1,3),(2,3),(3,3),(3,2),(2,2),(2,1)\rangle$.

## APPENDIX B
## PROOF OF LEMMA 4

Notice that $Q_2^6$ is vertex symmetric. W.L.O.G, let $w = (0,0)$. There are four cases for $b$. That is, $b \in \{(1,0),(2,1),(3,0),(3,2)\}$.
**Case 1.** To prove that $Q_2^6$ is $3^*$-laceable.
Case 1.1. Let $b = (1,0)$.
The three disjoint paths $\{P_1,P_2,P_3\}$ between $w$ and $b$ whose union covers all vertices of $Q_2^6$ are $P_1 = \langle(0,0),(1,0)\rangle$, $P_2 = \langle(0,0),(0,1),(1,1),(1,0)\rangle$, and $P_3 = \langle(0,0),(5,0),(5,1),(5,2),(5,3),(5,4),(5,5),(4,5),(3,5),(2,5),(1,5),(0,5),(0,4),(1,4),(2,4),(3,4),(4,4),(4,3),(4,2),(4,1),(4,0),(3,0),(3,1),(3,2),(3,3),(2,3),(1,3),(0,3),(0,2),(1,2),(2,2),(2,1),(2,0),(1,0)\rangle$.
Case 1.2. Let $b = (2,1)$.
The three disjoint paths $\{R_1,R_2,R_3\}$ between $w$ and $b$ whose union covers all vertices of $Q_2^6$ are $R_1 = \langle(0,0),(1,0),(2,0),(2,1)\rangle$, $R_2 = \langle(0,0),(0,1),(1,1),(2,1)\rangle$, and $R_3 = \langle(0,0),(5,0),(5,1),(5,2),(5,3),(5,4),(5,5),(4,5),(3,5),(2,5),(1,5),(0,5),(0,4),(1,4),(2,4),(3,4),(4,4),(4,3),(4,2),(4,1),(4,0),(3,0),(3,1),(3,2),(3,3),(2,3),(1,3),(0,3),(0,2),(1,2),(2,2),(2,1)\rangle$.
Case 1.3. Let $b = (3,0)$.
The three disjoint paths $\{S_1,S_2,S_3\}$ between $w$ and $b$ whose union covers all vertices of $Q_2^6$ are $S_1 = \langle(0,0),(1,0),(2,0),(3,0)\rangle$, $S_2 = \langle(0,0),(5,0),(4,0),(3,0)\rangle$, and $S_3 = \langle(0,0),(0,5),(1,5),(2,5),(3,5),(4,5),(5,5),(5,4),(4,4),(3,4),(2,4),(1,4),(0,4),(0,3),(1,3),(2,3),(3,3),(4,3),(5,3),(5,2),(5,1),(4,1),(4,2),(3,2),(2,2),(1,2),(0,2),(0,1),(1,1),(2,1),(3,1),(3,0)\rangle$.
Case 1.4. Let $b = (3,2)$.
The three disjoint paths $\{T_1,T_2,T_3\}$ between $w$ and $b$ whose union covers all vertices of $Q_2^6$ are $T_1 = \langle(0,0),(1,0),(2,0),(3,0),(3,1),(3,2)\rangle$, $T_2 = \langle(0,0),(0,1),(0,2),(1,2),(1,1),(2,1),(2,2),(3,2)\rangle$, and $T_3 = \langle(0,0),(5,0),(4,0),(4,1),(5,1),(5,2),(5,3),(5,4),(5,5),(4,5),(3,5),(2,5),(1,5),(0,5),(0,4),(0,3),(1,3),(1,4),(2,4),(2,3),(3,3),(3,4),(4,4),(4,3),(4,2),(3,2)\rangle$.
**Case 2.** To prove that $Q_2^6$ is $4^*$-laceable.
Case 2.1. Let $b = (1,0)$.
The four disjoint paths $\{P_1,P_2,P_3,P_4\}$ between $w$ and $b$ whose union covers all vertices of $Q_2^6$ are $P_1 = \langle(0,0),(1,0)\rangle$, $P_2 = \langle(0,0),(0,1),(1,1),(1,0)\rangle$, $P_3 = \langle(0,0),(0,5),(0,4),(0,3),(0,2),(1,2),(1,3),(1,4),(1,5),(1,0)\rangle$, and $P_4 = \langle(0,0),(5,0),(5,1),(5,2),(5,3),(5,4),(5,5),(4,5),(4,4),(4,3),(4,2),(4,1),(4,0),(3,0),(3,1),(3,2),(3,3),(3,4),(3,5),(2,5),(2,4),(2,3),(2,2),(2,1),(2,0),(1,0)\rangle$.
Case 2.2. Let $b = (2,1)$.
The four disjoint paths $\{R_1,R_2,R_3,R_4\}$ between $w$ and $b$ whose union covers all vertices of $Q_2^6$ are $R_1 = \langle(0,0),(1,0),(2,0),(2,1)\rangle$, $R_2 = \langle(0,0),(0,1),(1,1),(2,1)\rangle$, $R_3 = \langle(0,0),(5,0),(5,1),(5,2),(5,3),(5,4),(5,5),(4,5),(4,4),(4,3),(4,2),(4,1),(4,0),(3,0),(3,1),(2,1)\rangle$, and $R_4 = \langle(0,0),(0,5),(1,5),(2,5),(3,5),(3,4),(2,4),(1,4),(0,4),(0,3),(0,2),(1,2),(1,3),(2,3),(3,3),(3,2),(2,2),(2,1)\rangle$.
Case 2.3. Let $b = (3,0)$.
The four disjoint paths $\{S_1,S_2,S_3,S_4\}$ between $w$ and $b$ whose union covers all vertices of $Q_2^6$ are $S_1 = \langle(0,0),(1,0),(2,0),(3,0)\rangle$, $S_2 = \langle(0,0),(0,1),(1,1),(2,1),(3,1),(3,0)\rangle$, $S_3 = \langle(0,0),(5,0),(5,1),(5,2),(5,3),(5,4),(5,5),(4,5),(4,4),(4,3),(4,2),(4,1),(4,0),(3,0)\rangle$, and $S_4 = \langle(0,0),(0,5),(0,4),(0,3),(0,2),(1,2),(1,3),(1,4),(1,5),(2,5),(2,4),(2,3),(2,2),(3,2),(3,3),(3,4),(3,5),(3,0)\rangle$.
Case 2.4. Let $b = (3,2)$.
The four disjoint paths $\{T_1,T_2,T_3,T_4\}$ between $w$ and $b$ whose union covers all vertices of $Q_2^6$ are $T_1 = \langle(0,0),(1,0),(2,0),(3,0),(3,1),(3,2)$, $T_2 = \langle(0,0),(0,1),(0,2),(1,2),(1,1),(2,1),(2,2),(3,2)\rangle$, $T_3 = \langle(0,0),(5,0),(4,0),(4,1),(5,1),(5,2),(4,2),(3,2)\rangle$, and $T_4 = \langle(0,0),(0,5),(1,5),(2,5),(3,5),(4,5),(5,5),(5,4),(5,3),(4,3),(4,4),(3,4),(2,4),(1,4),(0,4),(0,3),(1,3),(2,3),(3,3),(3,2)\rangle$.

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