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The Design of Axisymmetric Ducts for Incompressible Flow with a Parabolic Axial Velocity Inlet Profile

V.Pavlika

***Abstract*—**In this paper a numerical algorithm is described for solving the boundary value problem associated with axisymmetric, inviscid, incompressible, rotational (and irrotational) flow in order to obtain duct wall shapes from prescribed wall velocity distributions. The governing equations are formulated in terms of the stream function $\psi(x,y)$ and the function $\varphi(x,y)$ as independent variables where for irrotational flow $\varphi(x,y)$ can be recognized as the velocity potential function, for rotational flow $\varphi(x,y)$ ceases being the velocity potential function but does remain orthogonal to the stream lines. A numerical method based on the finite difference scheme on a uniform mesh is employed. The technique described is capable of tackling the so-called inverse problem where the velocity wall distributions are prescribed from which the duct wall shape is calculated, as well as the direct problem where the velocity distribution on the duct walls are calculated from prescribed duct geometries. The two different cases as outlined in this paper are in fact boundary value problems with Neumann and Dirichlet boundary conditions respectively. Even though both approaches are discussed, only numerical results for the case of the Dirichlet boundary conditions are given. A downstream condition is prescribed such that cylindrical flow, that is flow which is independent of the axial coordinate, exists.

***Keywords*—**Inverse problem, irrotational incompressible flow, Boundary value problem.

I. INTRODUCTION

DESIGNERS of ducts require numerical techniques for calculating duct geometries from a prescribed velocity distribution. The objective of the prescribed velocity is typically to avoid boundary layer separation. At inlet a velocity distribution is prescribed to allow for a vorticity vector to be present calculated from $\underline{\omega} = \underline{\nabla} \wedge \underline{v}$ where the $\wedge$ denotes the usual cross product of vectors, $\underline{\omega}$ is the vorticity vector and $\underline{v}$ the velocity vector respectively. This paper describes a numerical algorithm for solving the boundary value problem that arises when the independent variables are φ and ψ which have been previously defined. The dependent variable is y, is the radial coordinate.

Dr Vasos Pavlika is Senior Lecturer at the University of Westminster, +44-207-9115000, V.L.Pavlika@Westminster.ac.uk.

II. DESIGN PLANE

Defining

$w = u - iv = qe^{-i\vartheta}$ and $z = x + iy$.

Then using the Cauchy-Riemann equations the identity

$$\frac{\partial w}{\partial \bar{z}} = \frac{1}{2}(\eta - i\omega_\alpha)$$

is easily verified where

$$\eta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

and

$$\omega_\alpha = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

In application to steady plane flow, with rectilinear coordinates x, y and velocity components u, v in the x, y directions respectively q is the flow speed, ϑ is the flow direction measured from the x axis, ω_α is the component of vorticity normal to the plane and η is the rate of expansion or dilation. If η is zero everywhere apart from at isolated singularities e.g. point sources the velocity components can be derived from a stream function ψ level lines of which coincide with the streamlines. If ω_α is zero everywhere except at point vortices then the velocity components can also be derived from a velocity potential φ, level lines of which are orthogonal to the streamlines

III. THE FIRST AUXILIARY FLOW

Consider a flow of complex conjugate velocity $w^{(1)}$ where

$$w^{(1)} = \frac{\partial \psi}{\partial n} e^{-i\vartheta} \tag{1}$$

with ψ real and $\frac{\partial \psi}{\partial n}$ its derivative in a direction

$(\vartheta + \frac{\pi}{2})$ from the x-axis. This auxiliary flow and the actual flow (of complex conjugate velocity w) clearly share the direction ϑ and taking $\frac{\partial \psi}{\partial s}$, the derivative in the direction ϑ, to vanish over either flow field then

$$\frac{\partial \psi}{\partial s} = \cos\vartheta \frac{\partial \psi}{\partial x} + \sin\vartheta \frac{\partial \psi}{\partial y} \quad (2)$$
$$= 0$$

while

$$\frac{\partial \psi}{\partial n} = -\sin\vartheta \frac{\partial \psi}{\partial x} + \cos\vartheta \frac{\partial \psi}{\partial y}$$
$$= -\operatorname{cosec}\vartheta \frac{\partial \psi}{\partial x} = \sec\vartheta \frac{\partial \psi}{\partial y}$$

and substituting in definition (1)

$$w^{(i)} = \frac{\partial \psi}{\partial y} + i \frac{\partial \psi}{\partial x}$$

so that $\frac{\partial w^{(1)}}{\partial \bar{z}} = 2i \frac{\partial^2 \psi}{\partial z \partial \bar{z}}$ (3)

Certain observations can be made on this auxiliary flow characterized so far, by equation (2) and (3). From equation (3) it has zero rate of expansion and a vorticity given by

$$w^{(1)} = -\nabla^2 \psi$$

Level lines of $\psi(x, y)$ define its stream line pattern and also that of the actual flow, but the distribution of ψ across the stream has not yet been allocated.

IV. The Second Auxiliary Flow

Next consider a flow of complex conjugate velocity $w^{(2)}$, where

$$w^{(2)} = \frac{\partial \varphi}{\partial s} e^{-i\vartheta} \quad (4)$$

with $\varphi(x, y)$ real. This flow also shares direction and streamline pattern with the actual flow but in order to establish a family of curves orthogonal to the streamlines, this time $\frac{\partial \varphi}{\partial n}$ is taken to vanish over the flow field, i.e.,

$$\frac{\partial \varphi}{\partial n} = -\sin\vartheta \frac{\partial \varphi}{\partial x} + \cos\vartheta \frac{\partial \varphi}{\partial y} = 0$$

so that

$$\frac{\partial \varphi}{\partial s} = \cos\vartheta \frac{\partial \varphi}{\partial x} + \sin\vartheta \frac{\partial \varphi}{\partial y}$$

$$\therefore \frac{\partial \varphi}{\partial s} = \sec\vartheta \frac{\partial \varphi}{\partial x} = \operatorname{cosec}\vartheta \frac{\partial \varphi}{\partial y}$$

and substituting in definition (4)

$$w^{(2)} = \frac{\partial \varphi}{\partial x} - i \frac{\partial \varphi}{\partial y} \Rightarrow \frac{\partial w^{(2)}}{\partial \bar{z}} = 2 \frac{\partial^2 \varphi}{\partial z \partial \bar{z}}$$

This second auxiliary flow therefore has zero vorticity but a rate of expansion given by

$$\eta^{(2)} = \nabla^2 \varphi$$

level lines of $\varphi(x, y)$ define a family of curves orthogonal to the streamline pattern common to both auxiliary flows and the actual flow but the distribution of φ along the stream has yet to be allocated.

V. Intrinsic Flow Equations

The differential operator identity

$$\frac{\partial}{\partial s} + i \frac{\partial}{\partial n} = 2e^{-i\vartheta} \frac{\partial}{\partial \bar{z}}$$

is easily verified and when applied to the function $\log_e(w)$ there follows after some simple manipulation

$$\left(\frac{\partial}{\partial s} + i \frac{\partial}{\partial n}\right) \log_e(w) = \frac{1}{q}(\eta - i\omega_\alpha) \quad (5)$$

Applying equation (5) to the actual flow and each subsidiary flow gives

$$\left(\frac{\partial}{\partial s} + i \frac{\partial}{\partial n}\right)(\log_e(q) - i\vartheta) = \frac{1}{q}(\eta - i\omega_\alpha) \quad (6)$$

$$\left(\frac{\partial}{\partial s} + i \frac{\partial}{\partial n}\right)(\log_e(\Psi) + i\vartheta) = i\Psi\nabla^2\psi \quad (7)$$

$$\left(\frac{\partial}{\partial s} + i \frac{\partial}{\partial n}\right)(\log_e(\Phi) + i\vartheta) = \Phi\nabla^2\varphi \quad (8)$$

where Φ and Ψ represent the reciprocals of $\frac{\partial \phi}{\partial s}$ and $\frac{\partial \psi}{\partial n}$ respectively. The system of implicit flow equations comprises the real and imaginary parts of equation (6), the real part of equation (7) and the imaginary part of (8)

$$\frac{\partial}{\partial s}(\log_e(q)) + \frac{\partial \vartheta}{\partial n} = \frac{\eta}{q} \quad (9)$$

$$\frac{\partial}{\partial n}(\log_e(q)) - \frac{\partial \vartheta}{\partial s} = -\frac{\omega_\alpha}{q} \quad (10)$$

$$\frac{\partial}{\partial s}(\log_e(\Psi)) - \frac{\partial \vartheta}{\partial n} = 0 \quad (11)$$

$$\frac{\partial}{\partial s}(\log_e(\Phi)) + \frac{\partial \vartheta}{\partial s} = 0 \quad (12)$$

VI. The Fundamental Design Plane Equations

Eliminating ϑ between equations (9) and (10) and again between equations (11) and (12) gives

$$\frac{\partial}{\partial s}(\log_e(q\Psi)) = \frac{\eta}{q}$$

$$\frac{\partial}{\partial n}(\log_e(q\Phi)) = -\frac{\omega_\alpha}{q}$$

substituting $A = q\Psi$ and $B = q\Phi$. The last pair of equations can be written as

$$\frac{\partial}{\partial\varphi}(\log_e(A)) = \frac{\eta}{q^2}B \tag{13}$$

$$\frac{\partial}{\partial\psi}(\log_e(B)) = -\frac{\omega_\alpha}{q^2}A \tag{14}$$

whilst equations equation (11) and (12) similarly become

$$\frac{A}{B}\frac{\partial}{\partial\varphi}(\log(\frac{q}{A})) = -\frac{\partial\vartheta}{\partial\psi} \tag{15}$$

and

$$\frac{B}{A}\frac{\partial}{\partial\psi}(\log_e(\frac{q}{B})) = \frac{\partial\vartheta}{\partial\varphi} \tag{16}$$

eliminating ϑ between equations (15) and (16) gives

$$\frac{\partial}{\partial\phi}\left[\frac{A}{B}\frac{\partial}{\partial\varphi}\log_e\left(\frac{q}{A}\right)\right] + \frac{\partial}{\partial\psi}\left[\frac{B}{A}\frac{\partial}{\partial\psi}\log_e\left(\frac{q}{B}\right)\right] = 0 \tag{17}$$

Regarding temporarily η, ω_α and q as known functions of φ and ψ the system (13) and (14) is quasi-linear hyperbolic with characteristics parallel to the φ and ψ axes which maps the physical flow field into an infinite strip in the (φ,ψ) plane. Bearing in mind the freedom available in the stream wise variation of φ and the cross stream variation of ψ, suitable values of A can be prescribed along one φ characteristic and those of B can be prescribed along one ψ characteristic.

Regarding similarly A and B as known functions of φ and ψ equation (17) is linear elliptic and although boundary conditions for it will depend on the particular application, the Dirichlet choice involves the prescription of q over the flow field. Numerical coupling of the two schemes yields the solution in the design plane.

VII. Physical Coordinates

From elementary geometric considerations and definitions given previously

$$\begin{aligned} dz &= e^{i\vartheta}(ds + idn) \\ &= \frac{e^{i\vartheta}}{q}\left(Bd\varphi + iAd\psi\right) \end{aligned} \tag{18}$$

Thus $qds = Bd\varphi$ and $qdn = Ad\psi$. So that when ϑ, q, A and B are known in the (φ,ψ) plane the physical coordinates x and y can be calculated. Alternatives to equations (15), (16) and (17) which are more convenient in some applications can be obtained using the values of

$$\frac{\partial z}{\partial\varphi} \text{ and } \frac{\partial z}{\partial\psi}$$

given by equation (18), so that

$$\frac{\partial x}{\partial\varphi} = \frac{B}{q}\cos\vartheta, \qquad \frac{\partial y}{\partial\varphi} = \frac{B}{q}\sin\vartheta$$

$$\frac{\partial x}{\partial\psi} = -\frac{A}{q}\sin\vartheta, \quad \frac{\partial y}{\partial\psi} = \frac{A}{q}\cos\vartheta.$$

so that

$$\frac{\partial x}{\partial\varphi} = \frac{B}{A}\frac{\partial y}{\partial\psi} \tag{19}$$

and $$\frac{\partial x}{\partial\psi} = -\frac{A}{B}\frac{\partial y}{\partial\varphi} \tag{20}$$

hence eliminating x in (19) and (20) yields

$$\frac{\partial}{\partial\varphi}\left(\frac{A}{B}\frac{\partial y}{\partial\varphi}\right) + \frac{\partial}{\partial\psi}\left(\frac{B}{A}\frac{\partial y}{\partial\psi}\right) = 0 \tag{21}$$

Equation (21) may be used to replace equation (17) in the design system previously described and for use in equations (13) and (14)

$$\frac{1}{q^2} = \frac{1}{A^2}\left(\frac{\partial y}{\partial\psi}\right)^2 + \frac{1}{B^2}\left(\frac{\partial y}{\partial\varphi}\right)^2$$

this time completion of the physical coordinates is provided from equations (19) and (20) by

$$dx = \frac{B}{A}\frac{\partial y}{\partial\psi}d\varphi - \frac{A}{B}\frac{\partial y}{\partial\varphi}d\psi$$

The Dirichlet boundary condition involves the prescription of y on the boundaries of the design plane, whilst the Neumann the prescription of $\frac{\partial y}{\partial\varphi}$ or $\frac{\partial y}{\partial\psi}$ (depending on which bounding surface is being considered). The technique can easily be extended to cope with the so-called Cauchy and Robin boundary conditions. An analytic treatment of equation (21) can be found in [2], [4] and [5].

VIII. The Numerical Algorithm in the Design Plane

Rewriting the partial differential equation that y satisfies as:

$$\frac{\partial}{\partial\phi}\left(a\frac{\partial y}{\partial\phi}\right) + \frac{\partial}{\partial\psi}\left(b\frac{\partial y}{\partial\psi}\right) = c$$

where $a \equiv a(\varphi,\psi,y)$, $b \equiv b(\varphi,\psi,y)$ and $c \equiv c(\varphi,\psi,y)$, where a, b and c are function of y, φ and ψ. For problems posed in the design plane c=0, the a and b will vary depending on whether the flow field is irrotational or swirl free etc. Writing in finite difference form using central differences (with c≠0) gives:

$$\frac{\partial}{\partial\phi}\left(a\frac{\partial y}{\partial\phi}\right)_{i,j} = \frac{1}{2(\Delta\phi)^2}\begin{bmatrix}(a_{i+1,j}+a_{i,j})y_{i+1,j} - 4a_{i,j}y_{i,j} \\ +(a_{i-1,j}+a_{i,j})y_{i-1,j}\end{bmatrix}$$

and

$$\frac{\partial}{\partial\psi}\left(b\frac{\partial y}{\partial\psi}\right)_{i,j} = \frac{1}{2(\Delta\psi)^2}\begin{bmatrix}(b_{i,j+1}+b_{i,j})y_{i,j+1} - 4b_{i,j}y_{i,j} \\ +(b_{i,j-1}+b_{i,j})y_{i,j-1}\end{bmatrix}$$

Thus at the point $(i\Delta\varphi, j\Delta\psi)$ (to be denoted by (i, j) from now on in this paper), the equation is represented by a computational molecule as:

$$+N_{i,j}y_{i,j-1}$$
$$W_{i,j}y_{i-1,j} - C_{i,j}y_{i,j} + E_{i,j}y_{i+1,j} \quad R_{i,j} \qquad = \qquad (22)$$
$$+S_{i,j}y_{i,j+1}$$

Where the N, S, E and W and R may be identified as:

$$W_{i,j} = (\Delta\psi)^2(a_{i,j} + a_{i-1,j})$$
$$E_{i,j} = (\Delta\psi)^2(a_{i+1,j} + a_{i,j})$$
$$N_{i,j} = (\Delta\varphi)^2(b_{i,j-1} + b_{i,j})$$
$$S_{i,j} = (\Delta\varphi)^2(b_{i,j+1} + b_{i,j})$$
$$C_{i,j} = 4((\Delta\varphi)^2 a_{i,j} + (\Delta\psi)^2 b_{i,j}))$$
$$R_{i,j} = 2(\Delta\varphi)^2(\Delta\psi)^2 {}_{i,j}$$

IX. THE DIFFERENCE EQUATIONS

Equation (22) applies for i=1 to M; j=1 to N on a uniform mesh as described in [5], with special consideration at j=1 and j=N, so that with Dirichlet boundary conditions, say for j=N

$$+N_{i,N}y_{i,N-1}$$
$$W_{i,N}y_{i-1,N} - C_{i,N}y_{i,N} + E_{i,N}y_{i+1,N} \quad R_{i,j} \quad S_{i,N}y_{i,N+1} =$$

with $y_{i,N+1}$ prescribed as the Dirichlet data for $0 \le i \le M$. For j=2 to N-1

$$+N_{i,j}y_{i,j-1}$$
$$W_{i,j}y_{i-1,j} - C_{i,j}y_{i,j} + E_{i,j}y_{i+1,j} \quad R_{i,j} \qquad =$$
$$+S_{i,j}y_{i,j+1}$$

and for j=1

$$W_{i,1}y_{i-1,1} - C_{i,1}y_{i,1} + E_{i,1}y_{i+1,1} \quad R_{i,1} \quad N_{i,1}y_{i,0} =$$
$$+S_{i,1}y_{i,2}$$

similarly $y_{i,0}$ prescribed as the Dirichlet data for $0 \le i \le M$.

X. VECTOR FORM OF THE DIFFERENCE EQUATIONS

The above equations can be written more conveniently in matrix-vector form as:

$$\begin{bmatrix} W_{i,1} & 0 & 0 & . & . \\ 0 & W_{i,2} & 0 & & \\ & 0 & W_{i,3} & . & . \\ & & & . & . \\ & & & & W_{i,N} \end{bmatrix} \begin{matrix} y_{i-1,1} \\ y_{i-1,2} \\ . \\ . \\ y_{i-1,N} \end{matrix} + \begin{bmatrix} -C_{i,1} & S_{i,1} & 0 & . & . \\ N_{i,2} & -C_{i,2} & S_{i,2} & 0 & . \\ & N_{i,3} & -C_{i,3} & . & . \\ & & & . & . \\ & & & & -C_{i,N} \end{bmatrix} \begin{bmatrix} y_{i,1} \\ y_{i,2} \\ . \\ . \\ y_{i,N} \end{bmatrix} + \begin{bmatrix} E_{i,1} & 0 & 0 & . & . \\ 0 & E_{i,2} & 0 & & \\ & 0 & E_{i,3} & . & . \\ & & & . & . \\ & & & & E_{i,N} \end{bmatrix} \begin{matrix} y_{i+1,1} \\ y_{i+1,2} \\ . \\ . \\ y_{i+1,N} \end{matrix}$$

$$= \begin{bmatrix} R_{i,1} - N_{i,1}y_{i,0} \\ R_{i,2} \\ . \\ . \\ R_{i,N} - S_{i,N}y_{i,N+1} \end{bmatrix} = R^{(i)}, say. \qquad (23)$$

XI. DIRECT SOLUTION OF THE DIFFERENCE EQUATIONS

The matrix-vector equation (equation (23)) can be written as

$$W^{(i)}\underline{Y}^{(i-1)} + A^{(i)}\underline{Y}^{(i)} + E^{(i)}\underline{Y}^{(i+1)} = \underline{R}^{(i)} \qquad (24)$$

With diagonal matrices $W^{(i)}$ and $E^{(i)}$ and tridiagonal matrix $A^{(i)}$ all of order (NXN), and column vectors $\underline{Y}^{(i)}$ and $\underline{R}^{(i)}$ of order N. To solve the vector recurrence relation a speculation is made that the $\underline{Y}^{(i-1)}$ vector can be related linearly to the $\underline{Y}^{(i)}$ vector as follows:

$$\underline{Y}^{(i-1)} = B^{(i)}\underline{Y}^{(i)} + \underline{K}^{(i)} \qquad (25)$$

where the $B^{(i)}$ and the $\underline{K}^{(i)}$ are at present unknown matrices and column vectors respectively. Substituting (25) into (24) gives

$$(W^{(i)}B^{(i)} + A^{(i)})\underline{Y}^{(i)} = \underline{R}^{(i)} - W^{(i)}\ \underline{K}^{(1)} - E^{(i)}\underline{Y}^{(i+1)}$$

$$\Rightarrow \underline{Y}^{(i)} = -(W^{(i)}B^{(i)} + A^{(i)})^{-1}\ E^{(i)}\underline{Y}^{(i+1)} + (W^{(i)}B^{(i)} + A^{(i)})^{-1}\ (\underline{R}^{(i)} - W^{(i)}\ \underline{K}^{(i)})$$

but $\underline{Y}^{(i)} = B^{(i+1)}\ \underline{Y}^{(i+1)} + \underline{K}^{(i+1)}$

Thus equating coefficients implies

$$B^{(i+1)} = -(W^{(i)}B^{(i)} + A^{(i)})^{-1}\ E^{(i)} \tag{26}$$

and

$$\underline{K}^{(i+1)} = (W^{(i)}B^{(i)} + A^{(i)})^{-1}\ (\underline{R}^{(i)} - W^{(i)}\ \underline{K}^{(i)})$$

For i=0 this gives,

$$\underline{Y}^{(0)} = B^{(1)}\ \underline{Y}^{(1)} + \underline{K}^{(1)} \tag{27}$$

To determine the $\underline{K}^{(1)}$, if the first iterate $B^{(1)} = 0$ then $\underline{K}^{(1)} = \underline{Y}^{(0)}$

The matrix and vector sequences are now defined by equations (26) and (27) for i=1 to M. The $\underline{Y}^{(i)}$ vectors are now calculated starting from right to left (as $\underline{Y}^{(M+1)}$ is known) using $\underline{Y}^{(M)} = B^{(M+1)}\ \underline{Y}^{(M+1)} + \underline{K}^{(M+1)}$

The diagonal matrices $W^{(i)}$ and $E^{(i)}$ have elements

$W^{(i)} = W_{i,j}$ and $E^{(i)} = E_{i,j}$

The tridiagonal matrix A has entries

$A_{j,j} = -C_{i,j}$ j = 1 to N
$A_{j,j+1} = S_{i,j}$, $A_{j,+1,j} = N_{i,j+1}$, j = 1 to N-1

XII. THE BOUNDARY CONDITIONS

Initially the Neumann boundary condition will be analysed. In this case the vector of unknown y values is extended to include the j=0 row (for the top boundary) and j=N+1 for the bottom boundary, (as shown in [6]). The difference scheme is now applied over this extended set i.e. the scheme is centered on the point j=0, (and j=N+1 for the bottom boundary). Considering for the moment only having a Neumann condition on the top boundary, the centering the scheme on j=0 will involve the value of y at j=-1, this term is expressed in terms of the value of y at j=1 using the known normal derivative, such that:

$$\frac{y_{i,-1} - y_{i,1}}{2\Delta\varphi} \approx \left(\frac{\partial y}{\partial \varphi}\right)_{i,0} = known\ \exp ression$$

so at the mesh point (i,0) (i=1,2,….M) the finite difference scheme gives

$$W_{i,0}y_{i-1,0} - C_{i,0}y_{i,0} + E_{i,0}y_{i+1,0} \quad R_{i,0} \quad N_{i,0}y_{i,\ 1} = \quad - \quad + S_{i,0}y_{i,1}$$

Applying the boundary condition gives

$$W_{i,0}y_{i-1,0} - C_{i,0}y_{i,0} + E_{i,0}y_{i+1,0} \quad R_{i,0} \quad 2 \quad \psi N_{\overline{i,0}}\left(\frac{\partial y}{\partial \psi}\right)_{i,0} \quad - \quad + (S_{i,0} + N_{i,0})y_{i,1}$$

Using

$$\left(\frac{\partial y}{\partial \psi}\right)_{i,0} = \sqrt{A_{i,0}^2\left(\frac{1}{q_{i,0}^2} - \frac{1}{B_{i,0}^2}\left(\frac{\partial y}{\partial \varphi}\right)_{i,0}^2\right)}$$

The normal derivative is now known in terms of the prescribed speed which in this case is along the top boundary. The matrix-vector equations become

$$\begin{bmatrix} W_{i,0} & 0 & 0 & . & . \\ 0 & W_{i,1} & 0 & & \\ & 0 & W_{i,2} & 0 & . \\ & & & . & . \\ & & & & W_{i,N} \end{bmatrix} \begin{matrix} y_{i-1,0} \\ y_{i-1,1} \\ . \\ . \\ y_{i-1,N} \end{matrix} \ .+$$

$$\begin{bmatrix} -C_{i,0} & (S_{i,0} + N_{i,0}) & 0 & . & . \\ N_{i,1} & -C_{i,1} & S_{i,1} & 0 & . \\ & N_{i,2} & -C_{i,2} & . & . \\ & & & . & . \\ & & & & -C_{i,N} \end{bmatrix} \begin{bmatrix} y_{i,0} \\ y_{i,1} \\ . \\ . \\ y_{i,N} \end{bmatrix} +$$

$$\begin{bmatrix} E_{i,0} & 0 & 0 & . & . \\ 0 & E_{i,1} & 0 & & \\ & 0 & E_{i,2} & . & . \\ & & & . & . \\ & & & & E_{i,N} \end{bmatrix} \begin{matrix} y_{i+1,0} \\ y_{i+1,1} \\ . \\ . \\ y_{i+1,N} \end{matrix}$$

$$= \begin{bmatrix} R_{i,0} - 2\Delta\psi N_{i,0}\left(\frac{\partial y}{\partial \psi}\right)_{i,0} \\ R_{i,1} \\ . \\ . \\ R_{i,N} - S_{i,N}y_{i,N+1} \end{bmatrix}$$

Similar analysis can be performed if the bottom boundary is to have a Neumann boundary condition as described in [5]. The technique can also be applied to the case of Robin boundary conditions.

XIII. Axisymmetric Flow in the Absence of Body Forces: Prescription of the Axial and Swirl Velocity Components

Here numerical solutions to inviscid axisymmetric flow with constant vorticity and a swirl velocity will be derived. The axial velocity component $u_x(y)$ at inlet will be chosen to be of the form:

$$u_x(y) = ay^2 + by \quad c \quad +$$

where that a, b and c are constants and chosen so that this parabolic inlet axial velocity profile is chosen such that

$$u_x(y_1) = u_1$$
$$u_x(y_2) = u_2$$
$$u_x(y_3) = u_3$$

where the y_i, u_i, $i = 1,2,3$ are prescribed (known) radii and axial velocity values respectively.

XIV. Theoretical/Mathematical Justification of Choosing a Parabolic Axial Velocity Profile as Oppossed to Another with Similar Characteristics

For simplicity the variation of velocity $u_x(y)$ will be considered in a cylindrical pipe of inner radius Y. Considering the flow of a cylindrical element of fluid coaxial with the pipe of length L then the net force on the pipe due to the static pressure is:

$$\underline{F} = \pi y^2 (p_1 - p_2)$$

where the p_1 and p_2 are the inlet and exit values of the pressure p. If the element does not accelerate then this force is equal to the viscous retarding force on the element which is given by

$$\underline{F} = 2\pi y L \tau$$

where $\tau = \mu \dfrac{du_x}{dy}$ with μ equal to the coefficient of dynamic viscosity, equating these two forces gives

$$\frac{du_x}{dy} = -\left(\frac{p_1 - p_2}{2\mu L}\right) y \quad \Rightarrow \int_{u_x}^{0} du_x'$$
$$= -\left(\frac{p_1 - p_2}{2\mu L}\right) \int_{y}^{Y} y' dy'$$

where of course the no-slip hypothesis has been incorporated at the wall of the pipe. Thus

$$u_x(y) = \left(\frac{p_1 - p_2}{4\mu L}\right)(Y^2 - y^2) \quad A \quad ay^2 \text{, say} \qquad =$$

which is clearly of degree two and hence parabolic. Thus even though there exist many functions in mathematics that resemble the parabola (in parts) it must be a quadratic expression that is chosen to model viscous behavior. For the general case it can be shown that the constants a, b and c are given by:

$$a = \frac{(u_1 - u_2)(y_2 - y_3) \quad (u_2 \quad u_3)(y_1 \; - y_2)}{(y_1^2 - y_2^2)(y_2 - y_3) \quad (y_2^2 \quad y_3^2)(y_1 \; - y_2)} \quad - \quad -$$

$$b = \frac{(u_2 - u_3)(y_1^2 - y_2^2) \quad (u_1 \quad u_2)(y_2^2 \; - y_3^2)}{(y_1^2 - y_2^2)(y_2 - y_3) \quad (y_2^2 \quad y_3^2)(y_1 \; - y_2)} \quad - \quad -$$

and c now following from the a, b above and $u_x(y_1) = u_1$.The swirl velocity $u_\alpha(y)$, will be of the form $u_\alpha(y) = ky + \dfrac{l}{y}$ where the k and l are constants with ky representing solid body rotation and l/y the so-called free vortex term respectively.

XV. The Flow Equations in the Physical Plane (y, α, x)

Adopting cylindrical polar coordinates with y being the radial coordinate, α the circumferential and x the axial coordinate, defining velocity components u_y, u_α and u_x with corresponding vorticity components ω_y , ω_α, ω_x in the
direction of increasing y, α and x respectively, then the equation of motion with unit density becomes:

$$\frac{D\underline{u}}{Dt} = -\underline{\nabla p} \qquad (28)$$

Where $\dfrac{D}{Dt}$ is the material derivative. Equation (28) can be written using well known vector identities as:

$$\frac{\partial u_y}{\partial t} + u_x \frac{\partial u_y}{\partial x} + u_y \frac{u_y}{y} \quad \frac{u_\alpha^2}{y} \quad \frac{\partial p \partial}{y \partial} \quad - \quad = \quad - \quad \partial$$

$$\frac{\partial u_\alpha}{\partial t} + u_x \frac{\partial u_\alpha}{\partial x} + u_y \frac{u}{y} \quad \frac{u_\alpha u_y}{y} \quad 0 \frac{\partial}{\partial} \quad \alpha - \qquad (29) =$$

$$\frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{u_x}{y} \quad \frac{\partial p}{x} \quad \frac{\partial}{\partial} \quad = \quad - \partial$$

Furthermore

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u}.\underline{\nabla})\underline{u} \quad \_.\underline{p} \quad - \quad \nabla$$

can be written (once again using an appropriate vector identity as)

$$\frac{\partial \underline{u}}{\partial t}+(\underline{\omega}\wedge\underline{u})=-\underline{\nabla}(\underline{p}+\frac{1}{2}q^2).$$

Thus for steady flow Crocco's form of the equation of motion is obtained, i.e.

$$(\underline{u}\wedge\underline{\omega})=\underline{\nabla}H \qquad (30)$$

where H is the total head defined by $H=p+\frac{1}{2}q^2$.

Calculating the cross product on the left hand side of equation (30), gives

$$\frac{\partial H}{\partial y}=u_\alpha\omega_x-u_x\omega_\alpha$$

$$0=u_x\omega_y-u_y\omega_x \qquad (31)$$

$$\frac{\partial H}{\partial x}=u_y\omega_\alpha-u_\alpha\omega_x$$

In addition for axisymmetric flow the vorticity vector $\underline{\omega}$ becomes

$$\underline{\omega}=\underline{\nabla}\wedge\underline{u}=-\left\{\frac{\partial u_\alpha}{\partial x}\right\}\underline{y}+\left\{\frac{\partial u_y}{\partial x}-\frac{\partial u_x}{\partial y}\right\}\underline{\alpha}+\left\{\frac{1}{y}\frac{\partial(yu_\alpha)}{\partial y}\right\}\underline{x} \qquad (32)$$

The equation of continuity becomes

$$\underline{\nabla}.\underline{u}=\frac{\partial(yu_x)}{\partial x}+\frac{\partial(yu_y)}{\partial y}=0$$

XVI. The Design Plane Counterparts

In order to compute numerical solutions in the design plane, expressions are required for the terms A, B and ω_α, thus

$$\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}=-\frac{1}{y}\left(u_x\frac{\partial y}{\partial x}+u_y\frac{\partial y}{\partial y}\right)$$

$$=-q\frac{\partial}{\partial s}(\log(y))$$

or

$$\eta=-\frac{q^2}{B}\frac{\partial}{\partial\varphi}(\log(y))$$

but

$$\eta=\frac{q^2}{B}\frac{\partial}{\partial\varphi}(\log(A))$$

thus $Ay=f(\psi)$, that is $\frac{\partial\psi}{\partial n}=\frac{yq}{f(\psi)}$. The arbitrary function $f(\psi)$ represents the freedom in the cross stream distribution of ψ and choosing $f(\psi)$ to be unity everywhere ψ can be identified as the usual Stokes stream function given by

$$\frac{\partial\psi}{\partial x}=-yu_y;\frac{\partial\psi}{\partial y}=yu_x$$

Equation (32), (circumferential component) gives

$$0=u_x\frac{\partial(yu_\alpha)}{\partial x}+u_y\frac{\partial(yu_\alpha)}{\partial y}$$

Referring to the meridional plane of fig 16.1, it may be deduced that

$$u_x=q\frac{\partial x}{\partial s};u_y=q\frac{\partial y}{\partial s}$$

$$\Rightarrow\frac{\partial}{\partial s}(yu_\alpha)=0$$

$$\therefore yu_\alpha=C(\psi)$$

where $q=\frac{ds}{dt}$. In terms of $C(\psi)$ the vorticity vector (equation (32)) becomes

$$\underline{\omega}=\underline{\nabla}\wedge\underline{u}=\left\{-\frac{1}{y}\frac{\partial C}{\partial x}\right\}\underline{y}+\left\{\frac{\partial u_y}{\partial x}-\frac{\partial u_x}{\partial y}\right\}\underline{\alpha}+\left\{\frac{1}{y}\frac{\partial C}{\partial y}\right\}\underline{x}$$

$$=\omega_y\underline{y}+\omega_\alpha\underline{\alpha}+\omega_x\underline{x}, \text{ by definition.}$$

An expression for ω_α is required as this appears in the expression for B, so using the radial component of equation (31) gives

$$\omega_\alpha=\frac{u_x}{u_\alpha}\left(\frac{1}{y}\frac{\partial C}{\partial y}\right)-\frac{1}{u_x}\frac{\partial H}{\partial y}$$

using the Stokes' stream function this becomes

$$\omega_\alpha=\frac{C(\psi)}{y}\left(\frac{dC}{d\psi}\right)-y\frac{dH}{d\psi}$$

which is the required expression to be used in calculation of B according to definition (14). If far upstream the flow is assumed to be cylindrical so that all quantities are independent of x, then with unit density the equation of motion and the Stokes' Stream function give

$$u_y=0;\frac{\partial p}{\partial x}=0;\frac{\partial p}{\partial y}=\frac{u_\alpha^2}{y};\frac{\partial\psi}{\partial x}=0;\frac{\partial\psi}{\partial y}=yu_x$$

giving

$$\omega_\alpha=\frac{C(\psi)}{y}\left(\frac{dC}{d\psi}\right)-\frac{y}{2}\frac{d}{d\psi}(u_x^2+u_\alpha^2)-\frac{u_\alpha^2}{u_xy}$$

With $u_x(y)=ay^2+by+c$ and $u_\alpha(y)=ky+\frac{l}{y}$ as previously defined. Once $\frac{dH}{d\psi}$ has been calculated upstream it takes this value throughout the (φ,ψ) plane since as is self evident the expression is independent of φ. This last expression for ω_α is required in the calculation of B and numerical coupling with equation (21) gives the numerical solution in the design plane.

XVII. Downstream Conditions

Downstream a cylindrical flow condition as discussed below will be prescribed. Defining the pressure function $H(\psi)$ and the function $C(\psi)$ as

$$H(\psi)=\frac{1}{2}(u_x^2+u_\alpha^2)+\frac{p}{\rho} \text{ and } C(\psi)=yu_\alpha$$

for cylindrical flow radial equilibrium (from equation (29) radial component gives

$$\frac{1}{\rho}\frac{dp}{dy}=\frac{u_\alpha^2}{y}$$

Integrating gives

$$\frac{1}{\rho}(p-p_{y-inner})=\int_{y-inner}\frac{u_\alpha^2}{y}dy=\int_{y-inner}\frac{C^2(\psi)}{y^3}dy$$

Which gives H(ψ) as

$$H(\psi)=\frac{1}{2}(u_x^2+u_\alpha^2)\quad\frac{p_{y-inner}}{\rho}\quad+\int_{y-inner}\frac{C^2(\psi)}{y^3}dy\quad+$$

Now

$$\int_{y-inner}\frac{C^2(\psi)}{y^3}dy=-\frac{1}{2}\int_{y-inner}C^2d(1/y^2)$$

$$=-\frac{1}{2}\left[\frac{C^2}{y^2}-\left(\frac{C^2}{y^2}\right)_{y-inner}\right]+\frac{1}{2}\int_{y-inner}\frac{1}{y^2}\frac{dC^2}{dy}dy$$

Therefore

$$H(\psi)=\frac{1}{2}u_x^2+\frac{p_{y-inner}}{\rho}+\frac{1}{2}(u_\alpha^2)_{y-inner}$$
$$+\int_{\psi=0}\frac{1}{y^2}\frac{dC^2}{d\psi}d\psi$$

Suppose $u_{x,1}=u_{x,1}(\psi)$ and $u_{\alpha,1}=u_{\alpha,1}(\psi)$, where the subscript 1 denotes upstream conditions, then $u_{x,2}=u_{x,2}(\psi)$ and $u_{\alpha,2}=u_{\alpha,2}(\psi)$ are required as functions of ψ, where the subscript 2 similarly denoting downstream conditions, so that

$$\frac{1}{2}u_{x,2}^2=H(\psi)-\frac{p_{2,inner}}{\rho}\quad\frac{1}{2}(u_{\alpha,2}^2)_{inner}\;-$$
$$-\frac{1}{2}\int_{\psi=0}\frac{1}{y_1^2}\frac{dC^2}{d\psi}d\psi \tag{33}$$

and $\int_{\psi=0}\frac{d\psi}{u_{x,2}}d\psi=\frac{1}{2}\left(y_2^2-y_{2,inner}^2\right)$

Furthermore $C(\psi)=y_1u_{\alpha,1}=y_2u_{\alpha,2}$ and equation (33) now gives

$$\frac{1}{2}u_{x,2}^2=\frac{1}{2}u_{x,1}^2+\frac{p_{1,inner}}{\rho}\quad\frac{p_{2,inner}}{\rho}\quad-\quad+$$
$$\frac{1}{2}((u_{\alpha,1}^2)_{inner}-(u_{\alpha,2}^2)_{inner})+\frac{1}{2}\int_{\psi=0}\left(\frac{1}{y_1^2}\quad\frac{1}{y_2^2}\right)d(C^2)\quad-$$

$$or\; u_{x,2}^2=u_{x,1}^2\quad K\quad\int_{\psi=0}\left(\frac{1}{y_1^2}+\frac{1}{y_2^2}\right)d(C^2)\quad- \tag{34}$$

where

$$K=2\left(\frac{p_{1,inner}}{\rho}-\frac{p_{2,inner}}{\rho}\right)+(u_{\alpha,1}^2)_{inner}-(u_{\alpha,2}^2)_{inner} \text{ and}$$

$$y_2^2=y_{2,inner}^2+2\int_{\psi=0}\frac{d\psi}{u_{x,2}} \tag{35}$$

with $u_{x,2}$ in this case given by (34).

XVIII. Calculation Procedure

The calculation of the downstream radii $y_2(\psi)$ follow from equation (35) with $u_{x,2}$ given by equation (34), which can be written as

$$u_{x,2}^2=g(\psi)+K, \text{ where} \tag{36}$$

$$g(\psi)=u_{x,1}^2+\int_{\psi=0}\left(\frac{1}{y_1^2}\quad\frac{1}{y_2^2}\right)\frac{d(C^2)}{d\psi}d\psi$$

In order to calculate the $(n+1)^{th}$ iterate it is known that:

$$\frac{\partial}{\partial K}(y_{2,outer}^2)=2\int_{\psi=0}\frac{\partial}{\partial K}\left(\frac{d\psi}{\sqrt{g(\psi)+K}}\right)$$
$$=-\int_{\psi=0}^{\Psi}\frac{d\psi}{\left(u_{x,2}^3\right)^{(n)}}$$

but

$$\left(\frac{\partial}{\partial K}(y_{2,outer}^2\right)^{(n)}=\frac{\left(y_{2,outer}^2\right)^{(n+1)}-(y_{2,outer}^2)^{(n)}}{K^{(n+1)}-K^{(n)}} \tag{37}$$

from which as can be seen from equation (37) the $K^{(n)}$ must be calculated iteratively with $K^{(0)}=0$. Once the $K^{(n+1)}$ has been calculated it is introduced into equation (36), giving rise to a new $(u_{x,2}^2)^{(n+1)}$ which in turn gives a new $(y_{x,2}^2)^{(n+1)}$ from equation (35) and the process repeated until some convergence criteria is satisfied.

XIX. Prescription of Wall Geometries

In this paper the Dirichlet boundary conditions will be prescribed on the wall boundaries so that it is the radii values, y that are given as a function of φ on the boundaries. The function chosen to give a y distribution is based on the hyperbolic tangent, choosing $y(\varphi) = C\tanh(a\varphi + b) + k$ where C, a, b and k are constants, applying the conditions that $y = y_u$ at $\varphi = 0$ and $y = y_d$ at $\varphi = \Phi$ taking $a\Phi + b \approx 3$ (arbitrary) and b=-3, so that $\tanh(a\varphi + b) \approx 1$ and $\tanh(b) \approx -1$, then it follows that:

$$y(\varphi) = \left(\frac{y_d - y_u}{2}\right)\tanh(a\varphi + b) + \left(\frac{y_d + y_u}{2}\right) \qquad (38)$$

Replacing φ by x in equation (38) gives a $y(x)$ distribution. The inner radius is prescribed to be equal to unity in this paper (arbitrary). The geometries produced are shown in figs 19.1.1, 19.1.2 and 19.1.3 respectively.

XX. Conclusions

As shown, geometries have been produced subject to given upstream and downstream conditions with prescribed Dirichlet boundary conditions. In this case vorticity at inlet has been specified by defining the axial velocity to be of the form $u_x(y) = ay^2 + by + c$ with swirl velocity given by $u_\alpha(y) = ky + \frac{l}{y}$, where the k and l are constants, defining the so-called free and forced vortex whirl respectively. The downstream conditions were such that: cylindrical flow was present. Dirichlet boundary conditions were prescribed however the case with Neumann conditions can be accommodated using the algorithm, in addition so can the case with Robin boundary condition. Further examples of the algorithm with a combination of boundary condition is given in [6]. It was found that at most eight iterations were required to achieve an acceptable level of convergence, with the technique accelerated using Aitken's Method.

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XXI. Figures

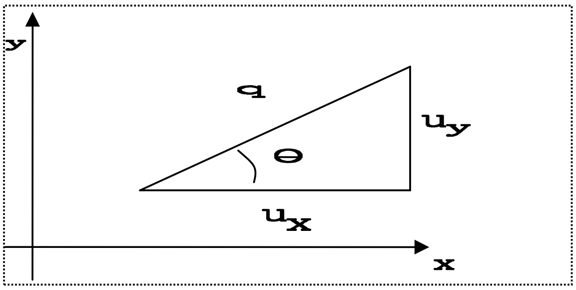

Fig. 16.1. The meridional plane

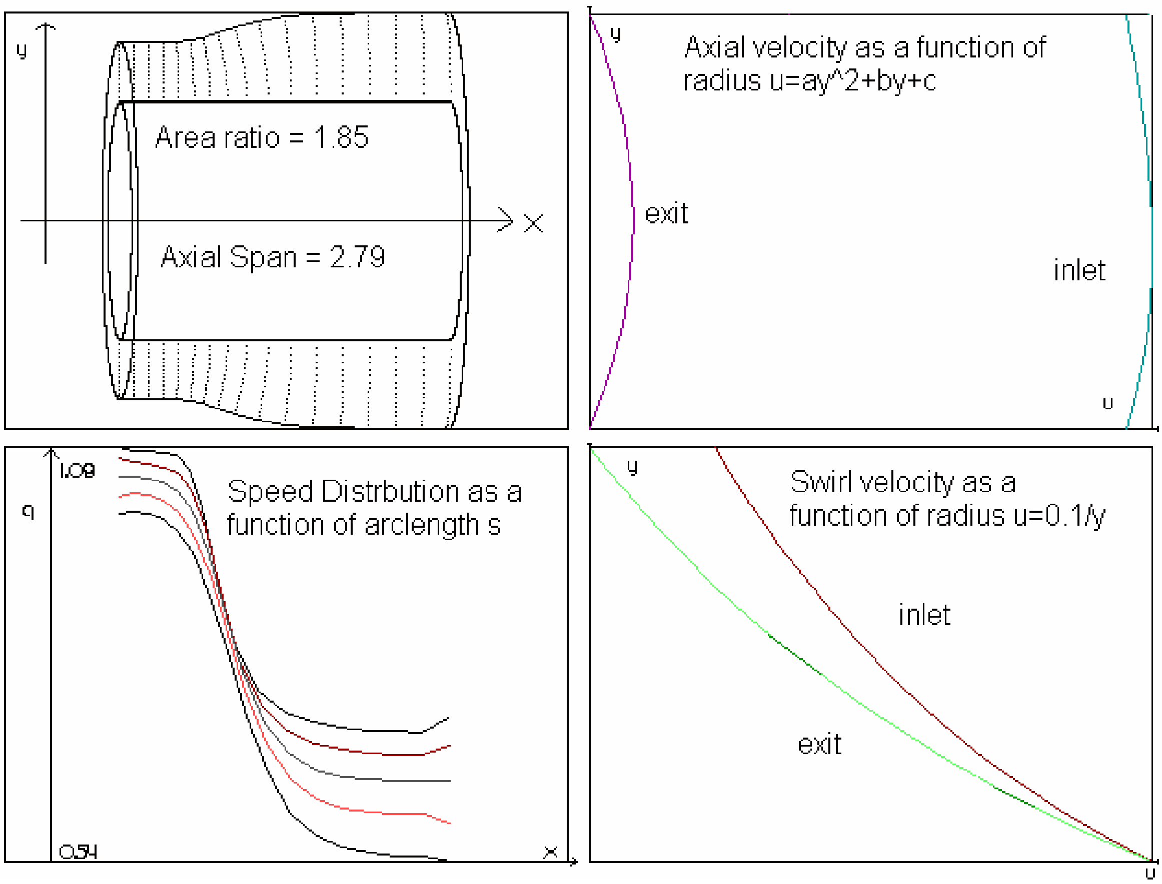

Fig. 19.1.1 The geometry and speed distribution (along the top boundary) produced given a swirl velocity $u_\alpha = 0.1/y$ and an axial velocity at inlet given by $u_x(y) = ay^2 + by \quad c$. +

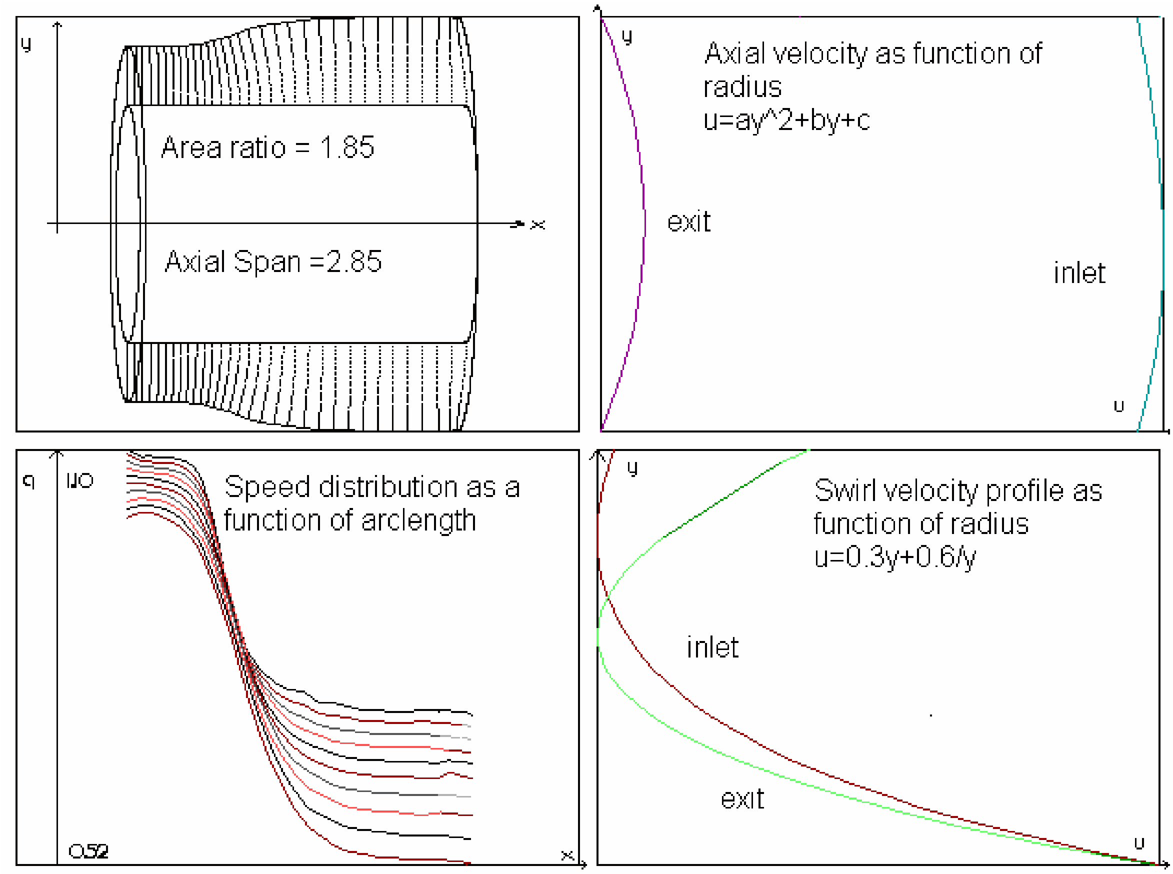

Fig 19.1.2. The geometry and speed distribution (along the top boundary) produced given a Swirl velocity $u_\alpha = 0.3y + 0.6/y$ and an axial velocity at inlet given by $u_x(y) = ay^2 + by \quad c$. +

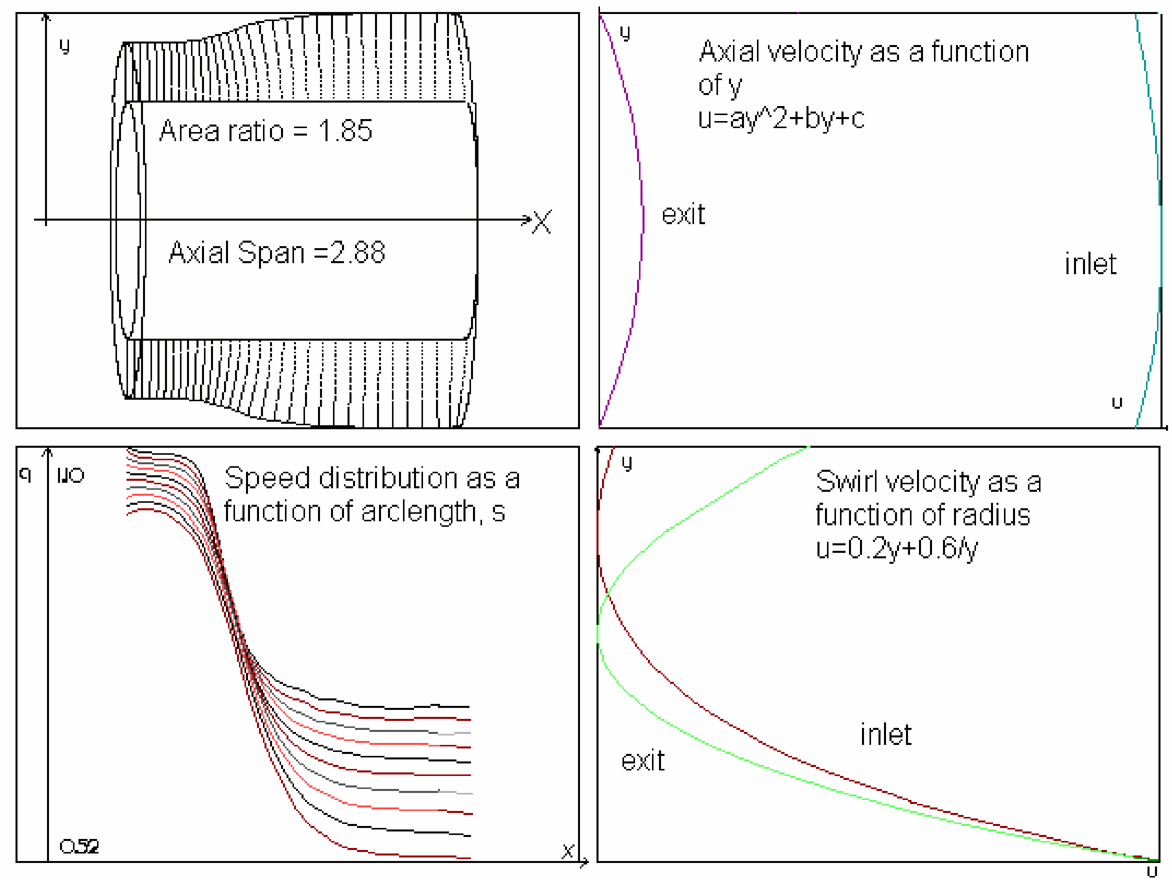

Fig 19.1.3. The geometry and speed distribution (along the top boundary) produced given a Swirl velocity $u_{\alpha} = 0.2y + 0.6/y$ and an axial velocity at inlet given by $u_x(y) = ay^2 + by \quad c$. +