Application of He's amplitude frequency formulation for a nonlinear oscillator with fractional potential

Meng Hu and Lili Wang

Abstract—In this paper, He's amplitude frequency formulation is used to obtain a periodic solution for a nonlinear oscillator with fractional potential. By calculation and computer simulations, compared with the exact solution shows that the result obtained is of high accuracy.

Keywords—He's amplitude frequency formulation; Periodic solution; Nonlinear oscillator; Fractional potential.

I. INTRODUCTION

IN this paper, we shall consider the following nonlinear oscillator with fractional potential

$$u'' + au + bu^{2n+1} + cu^{\frac{1}{2n+1}} = 0, \ u(0) = A, \ u'(0) = 0, \quad (1)$$

where a, b, c are constants, and $n \in N^+$.

If we take $n = 1$ in equation (1), then equation (1) reduced to a class of nonlinear oscillator[1]

$$u'' + au + bu^3 + cu^{\frac{1}{3}} = 0.$$

If we take $a = 1, c = 0, n = 1$ in equation (1), then equation (1) reduced to the well-known Duffing equation

$$u'' + u + bu^3 = 0. \quad (2)$$

In recent years, with the ever-increasing development of nonlinear science, various kinds of analytical methods and numerical methods have been used to handle the problem and other nonlinear problems, such as the Exp-function method [2-4], the variational iteration method [5,6], parameter-expansion method[7], and the homotopy perturbation method [8-10], etc. Hereby, we will apply He's frequency amplitude formulation [11,12] to solve the problem.

II. HE'S FREQUENCY AMPLITUDE FORMULATION

In order to use He's amplitude frequency formulation, we choose two trial functions $u_1(t) = A \cos t$ and $u_2(t) = A \cos \omega t$, which are, respectively, the solutions of the following linear equations:

$$u'' + \omega_1^2 u = 0, \quad \omega_1^2 = 1,$$
$$u'' + \omega_2^2 u = 0, \quad \omega_2^2 = \omega^2,$$

Meng Hu and Lili Wang are with the Department of Mathematics, Anyang Normal University, Anyang, Henan 455000, People's Republic of China.
E-mail address: humeng2001@126.com.

where ω is assumed to be the frequency of the nonlinear oscillator equation (1). Substituting $u_1(t)$ and $u_2(t)$ into equation (1), we obtain, respectively, the following residuals

$$
\begin{aligned}
R_1(t) &= -A \cos t + aA \cos t + bA^{2n+1} \cos^{2n+1} t \\
&\quad + cA^{\frac{1}{2n+1}} \cos^{\frac{1}{2n+1}} t, \quad (3) \\
R_2(t) &= -A\omega^2 \cos \omega t + aA \cos \omega t + bA^{2n+1} \cos^{2n+1} \omega t \\
&\quad + cA^{\frac{1}{2n+1}} \cos^{\frac{1}{2n+1}} \omega t. \quad (4)
\end{aligned}
$$

He's amplitude frequency formulation reads[11,12]

$$\omega^2 = \frac{\omega_1^2 R_2(t_2) - \omega_2^2 R_1(t_1)}{R_2(t_2) - R_1(t_1)}, \quad (5)$$

where t_1 and t_2 are location points. Generally, setting

$$t_1 = \frac{T_1}{12}, \quad t_2 = \frac{T_2}{12},$$

where T_1 and T_2 are periods of the trial functions $u_1(t) = A \cos t$ and $u_2(t) = A \cos \omega t$, respectively, i.e. $T_1 = 2\pi$ and $T_2 = 2\pi/\omega$.

From(3),(4),(5), by direct calculates, yields

$$\omega^2 = a + bA^{2n} \left(\frac{3}{4}\right)^n + cA^{-\frac{2n}{2n+1}} \left(\frac{3}{4}\right)^{-\frac{n}{2n+1}},$$

then

$$\omega = \left(a + bA^{2n} \left(\frac{3}{4}\right)^n + cA^{-\frac{2n}{2n+1}} \left(\frac{3}{4}\right)^{-\frac{n}{2n+1}}\right)^{\frac{1}{2}}. \quad (6)$$

We, therefore, obtain the following periodic solution

$$u(t) = A \cos \left[\left(a + bA^{2n} \left(\frac{3}{4}\right)^n + cA^{-\frac{2n}{2n+1}} \left(\frac{3}{4}\right)^{-\frac{n}{2n+1}}\right)^{\frac{1}{2}} t\right].$$

To illustrate the accuracy of the obtained results, we give two examples as follows:

In case $n = 1, a = b = 0$, equation (1) becomes

$$u'' + cu^{\frac{1}{3}} = 0,$$

its frequency reads $\omega = c^{\frac{1}{2}} A^{-\frac{1}{3}} \left(\frac{3}{4}\right)^{-\frac{1}{6}} = 1.0491 c^{\frac{1}{2}} A^{-\frac{1}{3}}$, its exact frequency[1] is $\omega_{ex} = 1.0705 c^{\frac{1}{2}} A^{-\frac{1}{3}}$. Therefore its accuracy reaches 0.0204.

In case $n = 1, a = c = 0$, equation (1) becomes

$$u'' + bu^3 = 0,$$

its frequency reads $\omega = \left(\frac{3}{4}\right)^{\frac{1}{2}} Ab^{\frac{1}{2}} = 0.866 Ab^{\frac{1}{2}}$, its exact frequency is $\omega_{ex} = 0.8472 Ab^{\frac{1}{2}}$. Therefore its accuracy reaches 0.0222.

III. Numerical Simulations

In this section, we will present some numerical results at different values.

Dashed line: exact solution, continuous line: approximate solution.

As examples, Fig.1, Fig.2 and Fig.3 illustrate excellent agreement of the obtained result with the exact one.

IV. Conclusions

In this work, the nonlinear equations are efficiently handled by He's frequency formulation. It has been proved to be a powerful mathematical tool for searching exact solutions for nonlinear equations without requirement of perturbation or nonlinearities. The analytical approximation obtained by this new method is valid for the whole solution domain with high accuracy.

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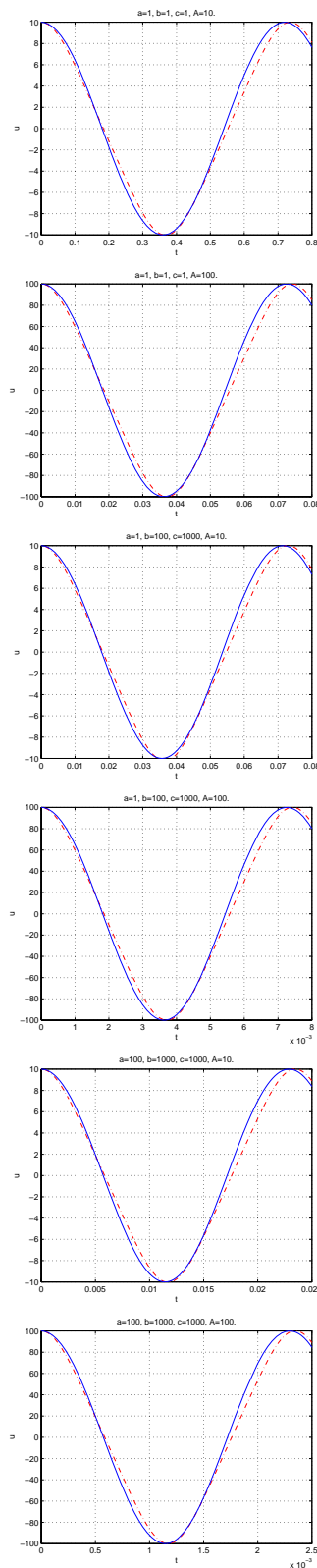
Fig.1. In case n=1, comparison of exact solution of equation(1) with approximate solution $u = A \cos \omega t$ under different values of a, b, c and A, where ω is defined by equation(6).

Fig.2. In case n=3, comparison of exact solution of equation(1) with approximate solution $u = A\cos\omega t$ under different values of a, b, c and A, where ω is defined by equation(6).
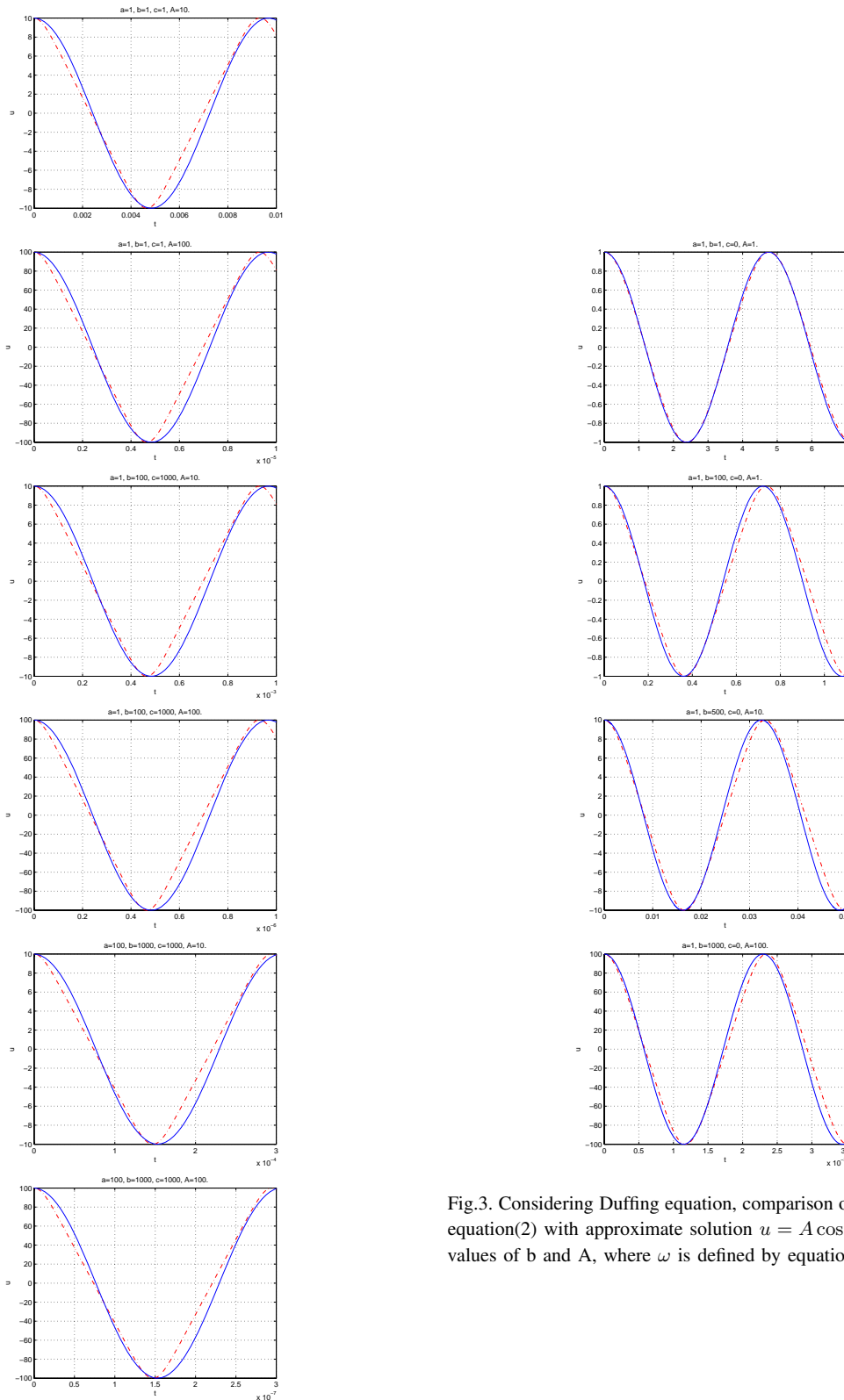

Fig.3. Considering Duffing equation, comparison of exact solution of equation(2) with approximate solution $u = A\cos\omega t$ under different values of b and A, where ω is defined by equation(6).