

# Graphs with Metric Dimension Two- A Characterization

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**Abstract**—In this paper, we define distance partition of vertex set of a graph  $G$  with reference to a vertex in it and with the help of the same, a graph with metric dimension two (i.e.  $\beta(G) = 2$ ) is characterized. In the process, we develop a polynomial time algorithm that verifies if the metric dimension of a given graph  $G$  is two. The same algorithm explores all metric bases of graph  $G$  whenever  $\beta(G) = 2$ . We also find a bound for cardinality of any distance partite set with reference to a given vertex, whenever  $\beta(G) = 2$ . Also, in a graph  $G$  with  $\beta(G) = 2$ , a bound for cardinality of any distance partite set as well as a bound for number of vertices in any sub graph  $H$  of  $G$  is obtained in terms of diam  $H$ .

**Keywords**—Metric basis, Distance partition, Metric dimension.

## I. INTRODUCTION

EVERY network can be viewed as a graph in which the vertices represent the processors and an edge between any two vertices indicate the connection between the processors corresponding to the vertices. In Samir Khuller et al [8], navigations are studied in a graph-structured framework in which the navigating agent (the robot) moves from vertex to vertex of a graph space. The robot can locate itself by the presence of the distinct codes assigned for the vertices of the graph. There are several methods to associate a code for a vertex. For example, in [7], Paul F. Tsuchiya assigned the codes by decomposing the network into sub networks. The method of approach in [7] is random and the code associated depends only on the number of sub divisions. However, a mathematical approach for the assignment of distinct codes given by F. Harary et al [4] and further studied by various other authors [2, 6, 8-13], purely depends on the other invariants associated with the network namely, diameter, distance between two vertices etc. The codes generated by the methods given in the above references can easily be implemented to locate any vertex in the graph network. In this paper, we define distance partition of vertex set of a graph  $G$  with reference to a vertex in it and with the help of the same, we characterize graphs with metric dimension two (i.e.  $\beta(G) = 2$ ). In the process, we develop a polynomial time algorithm that verifies if the metric dimension of a given graph  $G$  is two. The same algorithm explores all metric bases of graph  $G$  whenever  $\beta(G) = 2$ .

We also find a bound for cardinality of any distance partite set with reference to a given vertex, whenever  $\beta(G) = 2$ .

Throughout this paper, we write  $G(V, E)$  or simply  $G$ , to denote a graph on a finite non empty set  $V$  of vertices and  $E$  of edges. All the graphs considered in this paper are simple, finite, undirected and connected. For all the other basic notations we refer to [1, 3, and 5]. For any two vertices  $u$  and  $v$ , the distance between  $u$  and  $v$  denoted by  $d(u, v)$ , is the length of the shortest path between them. For a given graph  $G$ , there are number of properties related to distance between two vertices and are widely studied by various authors.

## II. DEFINITIONS

**Definition 2.1:** Let  $G = (V, E)$  be a connected, undirected graph and  $u, v, w$  be vertices in  $V$ . A vertex  $v$  is said to resolve the vertices  $u$  and  $w$  if the distance of  $u$  from  $v$  is different from distance of  $w$  from  $v$ . A set  $S \subseteq V$  is a resolving set of  $V(G)$  if for each pair of distinct vertices  $u, w \in V$  there exists at least one  $v \in S$  such that  $v$  resolves  $u$  and  $w$ .

**Remark 2.2:** Note that  $V$  itself is a resolving set of  $V$ .

**Definition 2.3:** A resolving set  $T$  with minimum cardinality amongst all resolving sets is known as a metric basis of a graph  $G$ . Further, the cardinality of any metric basis is the metric dimension of  $G$  and is denoted by  $\beta(G)$ .

**Remark 2.4:** Clearly, metric basis exists for every given graph (by definition) and it need not be unique. For example, given a path graph  $G$  with pendant vertices  $u$  and  $v$ ,  $\{u\}$  and  $\{v\}$  are metric bases for  $G$ .

**Remark 2.5:** A metric basis of a graph  $G$  is minimal (set theoretic sense) among all resolving sets, but converse need not be true. For example, in case of  $u$  and  $v$  adjacent vertices in a path graph  $G$  and neither among  $u$  and  $v$  are pendant vertices, then  $\{u, v\}$  is a minimal resolving set but not a metric basis.

It is learnt further that a path graph has metric dimension one, a cyclic graph has metric dimension two and a complete graph on  $n$  vertices has metric dimension  $(n-1)$ .

**Definition 2.6:** Let  $G$  be a graph with vertex set  $V(G)$  and  $v$  be a vertex in it. Then  $\{V_0, V_1, V_2, \dots, V_k\}$  is called a

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**distance partition** of  $V(G)$  with reference to the vertex  $v$  if  $V_0 = \{v\}$  and  $V_i$  contains those vertices which are at distance  $i$  from  $v$  for  $0 < i \leq k$ , where  $k$  is the eccentricity of  $v$  in  $G$ . The sets  $V_0, V_1, V_2, \dots, V_k$  are called distance partite sets.

The result given in the following proposition was observed by Samir Khuller et al [8], and is an important tool in deriving several interesting results of the present paper.

**Proposition 2.7 (Samir khuller et al)**

*In a graph  $G(V, E)$ , consider any three vertices  $u, v$  and  $w$  such that  $(u, v) \in E$ . If  $d = d(u, w)$  then  $d(v, w)$  is one of  $d-1, d$  and  $d+1$ .*

In the following corollaries we consider a graph  $G$  with  $\beta(G) = 2$ , metric basis  $\{v_1, v_2\}$  and distance partite sets  $V_0, V_1, \dots, V_k$  with reference to  $v_1$ . Proof is immediate from the above proposition and the definition of  $\beta(G) = 2$ .

**Corollary 2.8:** *Given any vertex  $v \in V_i$  there exist at most three vertices in  $V_{i+1}$  adjacent to  $v$ , where  $0 \leq i \leq k-1$ . Similarly there exist at most three vertices in  $V_{i-1}$  adjacent to  $v$  when  $1 \leq i \leq k$ .*

**Corollary 2.9:** *Every pair of vertices  $w_1$  and  $w_2$  from different distance partite sets are resolved by at least  $v_1$  and when  $w_1$  and  $w_2$  are from same distance partite set then  $v_2$  resolves them.*

### III. PROPERTIES OF DISTANCE PARTITION

In this section we shall discuss some characteristics of the graph due to the properties of distance partition, proofs are straight forward.

Let  $v$  be a vertex in  $V(G)$  and  $\{V_0, V_1, V_2, \dots, V_k\}$  be the distance partition of  $V(G)$  with reference to the vertex  $v$ .

**Theorem 3.1:** *Every vertex in  $V_j$  is adjacent to atleast one vertex in  $V_{j-1}$  for every  $j$  with  $2 \leq j \leq k$  and every vertex in  $V_1$  is adjacent to  $v$ .* ♦

**Theorem 3.2:** *Let  $G$  be a graph and  $|G| = n$ . Then the following are equivalent:*

- There exists a  $v \in V(G)$ , such that  $|V_i| = 1$  for each of distance partite set  $V_i$  of  $G$  with reference to the vertex  $v$ .*
- $G$  is a path graph and  $v$  is a pendant vertex in it.*
- There exists  $v \in V(G)$  such that  $e(v) = n-1$ .*

*In fact, in the above there exist exactly two vertices in  $V(G)$  such that with reference to each of them the number of distinct distance partite sets of  $G$  is equal to  $n$*  ♦

**Theorem 3.3:** *Let  $G$  be a graph and  $k_v$  be the number of distance partite sets with reference to a vertex  $v \in V(G)$ . Then  $k_v = 2$  for every  $v \in V(G)$  if and only if  $G$  is a complete graph.* ♦

The following corollary is a result given by Samir Khuller et al [8] and is immediate from Theorem 3.2.

**Corollary 3.4:** *Metric dimension of a graph  $G$  is one if and only if  $G$  is a path graph.*

### IV. RESULTS

In this section, we establish some results pertaining to structure of a graph  $G$  with  $\beta(G) = 2$ .

Let  $G$  be a graph with  $\beta(G) = 2$  and  $\{v_1, v_2\}$  be a metric basis of  $G$ . Further, let  $\{V_0, V_1, V_2, \dots, V_k\}$  be the distance partition of  $G$  with reference to the vertex  $v_1$ .

The results of the theorem 4.3, 4.4 and 4.6 are due to Samir Khuller et al [8] and we give a simple alternative proof using the concept of distance partition.

**Theorem 4.1:** *For any vertex  $v \in V_j$  there exists a shortest path of length  $j$  between  $v_1$  and  $v$ . In fact, a shortest path from  $v_1$  to  $v$  contains exactly one vertex  $w_i \in V_i$  for  $1 \leq i \leq j$ , and the distance  $d(w_i, v) = j - i$ .*

**Theorem 4.2:** *If  $G$  is a graph with  $\beta(G) = 2$  and metric basis  $\{v_1, v_2\}$ , then there exists a unique shortest path between  $v_1$  and  $v_2$ .*

**Theorem 4.3:** *Let  $\{v_1, v_2\}$  be a metric basis of  $G$  with  $\beta(G) = 2$  then degree of both  $v_1$  and  $v_2$  are less than or equal to three.*

**Theorem 4.4:** *Let  $\{v_1, v_2\}$  be a metric basis of  $G$  where  $\beta(G) = 2$ . For any vertex  $v$  on the unique shortest path between  $v_1$  and  $v_2$ , there exists at most one vertex adjacent to it in the distance partite set with respect to  $v_1$  to which it belongs to. Further,  $v$  has exactly one vertex adjacent to it in the preceding distance partite set.*

**Theorem 4.5:** *Let  $\{v_1, v_2\}$  be a metric basis of  $G$ , where  $\beta(G) = 2$ . The Maximum degree of any vertex  $v$  on the unique shortest path between  $v_1$  and  $v_2$  is five. In the case of  $\deg(v) = 5$ , the adjacency of  $v$  is as shown in the following structure (Figure 1).*

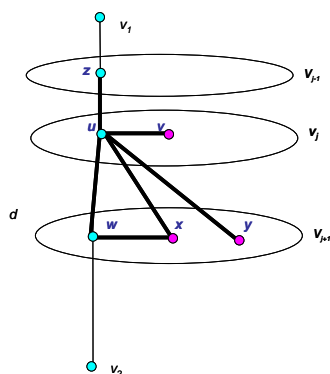


Fig. 1 Vertex  $u$  on the unique shortest path between  $v_1$  and  $v_2$  having degree five

**Proof:** Proof is immediate from Theorem 4.4 and corollary 2.8. ♦

**Theorem 4.6:** Let  $\{v_1, v_2\}$  be a metric basis of  $G$ , where  $\beta(G) = 2$ . Consider distance partite sets  $V_0, V_1, \dots, V_k$  with reference to  $v_1$ . Any connected component of the graph induced by a distance partite set is a path and in fact, degree of any vertex in the graph induced by the distance partite set is at most two.

The corollary 4.7 given below is due to Samir Khuller et al [8] and corollary 4.8 is due to Sooryanarayan B [13] and corollary 4.9 is due to Sooryanarayan B, Murali, K.S.Harinath [14] are immediate consequences of the Theorem 4.6.

**Corollary 4.7** A graph  $G$  with  $\beta(G) = 2$  cannot have  $K_5$ .

**Corollary 4.8:** Let  $\{v_1, v_2\}$  be a metric basis of  $G$  with  $\beta(G) = 2$ . Then for a triangle  $T$  in  $G$ , if any, all the vertices of  $T$  cannot be at the same distance from  $v_1$  or  $v_2$ .

**Proof:** Proof is immediate from Theorem 4.6. ♦

**Corollary 4.9:** For any graph  $G$  with  $\beta(G) = 2$ , the metric basis of  $G$  cannot have a vertex  $v$  of a sub graph  $K_4$  of  $G$ .

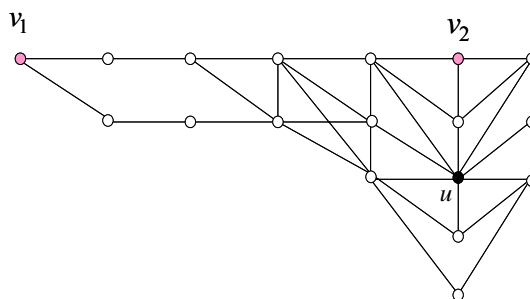
**Proof:** Let  $\{v_1, v_2\}$  be a metric basis of  $G$  and  $v_1 \in V(K_4)$ . Consider the distance partition of  $V(G)$  with reference to  $v_1$ . Then the distance partite set  $V_1$  has the other three vertices of  $K_4$  which induce a cycle, a contradiction to Theorem 4.6. ♦

**Definition :** The shortest path from vertex  $v_1$  to a vertex  $u_j$  of  $V_j$  is said to be downward extendable if there exists vertex  $u_{j+1}$  in  $V_{j+1}$  such that  $u_{j+1}$  is adjacent to  $u_j$ .

Note that the path  $v_1 \rightarrow \dots \rightarrow u_j \rightarrow u_{j+1}$  is a shortest path from  $v_1$  to  $u_{j+1}$ . Any path  $v_1 \rightarrow \dots \rightarrow u_j \rightarrow u_{j+1} \rightarrow \dots \rightarrow u_{j+t}$  is said to be a downward extension of a path  $v_1 \rightarrow \dots \rightarrow u_j$ . In the following theorem, we shall observe that a maximal downward extension of the unique shortest path  $v_1 \rightarrow \dots \rightarrow v_2$  is unique.

**Theorem 4.10:** Let  $\{v_1, v_2\}$  be a metric basis of  $G$  where  $\beta(G) = 2$ . The maximal downward extension of the unique shortest path  $v_1 \rightarrow v_2$  ( $v_2 \in V_j$ ) is unique and has at most one vertex  $u_{j+t}$  from  $V_{j+t}$  where  $0 \leq t \leq k - j$ .

**Theorem 4.11:** The maximum degree of any vertex in a graph  $G$  with  $\beta(G) = 2$  is eight and it is realizable.



**Remark 4.14:** From theorem 4.13 it is clear that neither  $K_5$  nor  $K_5 - \{e\}$  can be a sub graph of a graph with metric dimension two. so it is of natural curiosity that how further smaller sub graph of  $K_5$  can be excluded from being a sub graphs of a graph from the class of graphs with metric dimension two in the following we realize that  $K_5 - 2e$  could be a sub graph of some graph  $G$  with  $\beta(G) = 2$ .

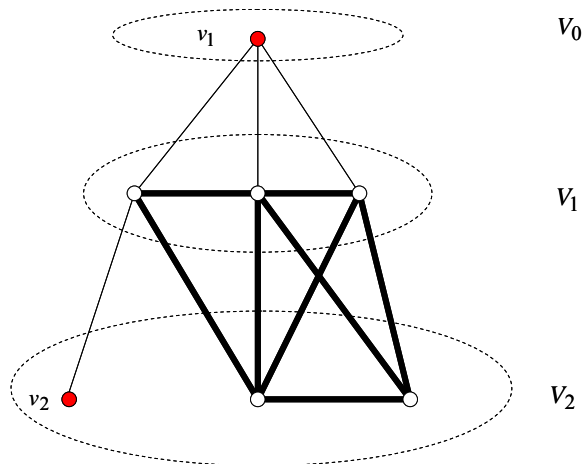


Fig. 4. A Graph  $G$  can have  $K_5 - 2e$  as a sub graph.

Similar to the above, we can prove the following theorem.

**Theorem 4.15:** If  $G$  is a graph with  $\beta(G) = 2$  then  $G$  cannot have  $K_{3,3}$  as a sub graph. ♦

**Theorem 4.16:** Let  $\{v_1, v_2\}$  be a metric basis of  $G$ , where  $\beta(G) = 2$ . Let  $e(v_1) = k$  and  $|V(G)| = n$ . Then eccentricity of the second resolving vertex  $v_2$  is greater than or equal to  $\left\lceil \frac{n-4}{k-1} \right\rceil$ ,  $k \neq 1$ , where  $\lceil x \rceil$  is the integer part of the number  $x$ .

**Remark 4.17:** In a graph  $G$  with  $\beta(G) = 2$  and metric basis  $\{v_1, v_2\}$ , if  $e(v_2) = s$  then any distance partite set  $V$  with respect to the vertex  $v_1$  other than the one with  $v_2$  in it contains at most  $s$  vertices (to be precise, not more than  $s+1-d(V, v_2)$ , where  $d(V, v_2)$  is shortest of distances between any vertex  $v$  of  $V$  and  $v_2$ , distance partite set that contains  $v_2$  may have  $s+1$  vertices.

**Proof:** Proof is immediate from Corollary 2.9 and Theorem 4.6. ♦

Though Theorem 4.21 and corollary 4.22 provide much stronger result, for general understanding of structure of  $G$  with  $\beta(G) = 2$ , we give following results.

**Theorem 4.18:** Let  $G$  be a graph with  $\beta(G) = 2$  and  $\{v_1, v_2\}$  be a metric basis of  $G$ . Let  $P$  be the Petersen graph. Then neither of  $v_1$  and  $v_2$  are in  $V(P)$ . Further, if eccentricity of any of  $v_1$  and  $v_2$  is not more than three, then  $P$  cannot be a sub graph of  $G$ .

**Proof:** Consider distance partite sets  $\{V_0, V_1, V_2, \dots, V_k\}$  with reference to  $v_1$ . If  $v_1 \in V(P)$ , then  $V_2$  consists of at least six vertices of  $V(P)$  which induces a cycle in  $V_2$  so a contradiction. Hence  $v_1 \notin V(P)$ . Similarly  $v_2 \notin V(P)$ .

Suppose that  $P$  is a sub graph of  $G$  and  $e(v_2) = 3$ . Now consider distance partite sets with reference to  $v_1$ . From the remark 4.17 at most one  $V_j$  which contains  $v_2$  may have four vertices and the remaining  $V_i$ 's have no more than three vertices. As  $v_1 \notin V(P)$  and diameter of  $P = 2$ ,  $V(P)$  is distributed among three  $V_j$ 's such that one having four vertices of  $V(P)$  and other two having three each. This implies  $v_2 \notin V(P)$ , a contradiction. ♦

**Theorem 4.19:** Let  $G$  be a graph with  $\beta(G) = 2$  then there is no connected sub graph  $H$  of  $G$  such that diameter of  $H < \sqrt{m-1}$ , where  $m$  is cardinality of  $V(H)$ .

**Proof:** Consider a metric basis  $\{v_1, v_2\}$  of  $G$ , where  $\beta(G) = 2$ , and distance partition  $\{V_0, V_1, \dots, V_k\}$  of  $V(G)$  with reference to one among the basis elements, say  $v_1$ . Let  $H$  be any connected sub graph of  $G$  with diameter of  $H$  equal to  $D$ . Any pairs of vertices, among vertices of  $H$  and in the same partite set, (say  $V_j$ ), are resolved by  $v_2$ . Since distance between any pair of vertices from  $\{v_h | v_h \in V(H) \cap V_j\}$  is not more than diameter  $D$  of  $H$ ,  $d(v_2, v_h)$  takes distinct values among  $d, d+1, \dots, d+D$ , where  $d = \min_{v \in H \cap V_j} \{d(v, v_2)\}$ . So, the cardinality of  $H \cap V_j$  is at most  $D+1$ . Further, as diameter  $H = D$ , the vertices of  $H$  could be distributed among at most  $D+1$  consecutive  $V_i$ 's. Hence the cardinality of  $H$  is at most  $(D+1)(D+1)$ . That is  $m \leq (D+1)^2$ , where  $m$  is cardinality of  $V(H)$ . Therefore  $\sqrt{m-1} \leq D$ . This proves the result. ♦

**Corollary 4.20:** The complete graph  $K_5$  or the Petersen graph  $P$  cannot be a sub graph of a graph  $G$  with  $\beta(G) = 2$ .

**Proof:** Proof is immediate from the Theorem 4.19 and  $K_5$  is of diam 1 with order 5, and Peterson graph is of diam 2 with order 10. ♦

**Lemma 4.21:** Let  $G$  be a graph with  $\beta(G) = 2$  and  $\{v_1, v_2\}$  be a metric basis of  $G$ . Further, let  $\{V_0, V_1, \dots, V_k\}$  be the

distance partition of  $V(G)$  with reference to the vertex  $v_1$ . Then every distance partite set can have at most two vertices more than the maximum possible cardinality of preceding distance partite set.

**Theorem 4.22:** Let  $G$  be a graph with  $\beta(G) = 2$  and  $\{v_1, v_2\}$  be a metric basis of  $G$ . Further, let  $\{V_0, V_1, \dots, V_k\}$  be the distance partition of  $V(G)$  with reference to one of the vertices in the metric basis. Then maximum number of vertices in any distance partite set, say  $V_i$ , for  $0 \leq i \leq k$  is  $(2i + 1)$ .

#### V. BOUND FOR NUMBER OF VERTICES IN A GRAPH $G$ WITH $\beta(G) = 2$

In the following a sharper bound for number of vertices in a graph  $G$  with  $\beta(G) = 2$  is given.

Let  $G$  be a graph with  $\beta(G) = 2$  and  $\{v_1, v_2\}$  be a metric basis of  $G$ . Further, let  $\{V_0, V_1, \dots, V_k\}$  be the distance partition of

$V(G)$  with reference to  $v_1$  and  $v_2 \in V_i$ . Let  $e(v_2) = s$ .  $V_i$  can have at most  $(s + 1)$  vertices only if  $(s + 1) \leq 2i + 1$ . Thus to find maximum number of vertices that  $G$  with  $\beta(G) = 2$  can have, we consider the following two cases.

**Case (i):**  $(s + 1) \leq 2i + 1$ , i.e.  $s \leq 2i$ .

Then  $V_i$  can have  $(s + 1)$  vertices. Let  $t$  be the smallest integer such that  $s - t + 1 \leq 2(i - t) + 1$  and  $s - t > 2(i - t - 1) + 1$  i.e.  $s \leq 2i - t$  and  $s > 2i - t - 1$ .

Then the distance partite sets  $V_i, V_{i-1}, \dots, V_{i-t}$  can have at most  $s + 1, s, \dots, s - t + 1$  vertices respectively. The distance partite sets  $\{V_0, V_1, \dots, V_{i-t-1}\}$ , can have at most  $1, 3, 5, \dots, 2(i - t - 1) + 1 = 2(i - t) - 1$  vertices respectively and finally the distance partite sets  $V_{i+1}, V_{i+2}, \dots, V_k$  can have at most  $s, s - 1, \dots, s - (k - i - 1)$  vertices respectively.

Thus the maximum number of vertices in a graph  $G$  with  $\beta(G) = 2$  is given by

$$\sum_{m=1}^{i-t} (2m-1) + \sum_{n=-1}^{t-1} (s-n) + \sum_{r=0}^{k-i-1} (s-r) \\ = \left[ (i-t)^2 \right] + \left[ (t+1)s - \frac{(t-1)t}{2} + 1 \right] + (k-i)s - \frac{(k-i-1)(k-i)}{2}$$

**Case(ii):**  $(s + 1) > 2i + 1$ , i.e.,  $s > 2i$ .

Then obviously,  $s > 2i - 1, 2i - 3, \dots, 3, 1$

Hence the distance partite sets  $V_0, V_1, V_2, \dots, V_i$  can have at most  $1, 3, \dots, 2i + 1$  vertices respectively.

Let  $t$  be the minimum positive integer such that  $s > 2i + 3t$  and  $s \leq 2i + 3t + 3$ . Then distance partite sets  $V_{i+1}, V_{i+2}, \dots, V_{i+t}$  can have at most  $2(i + 1) + 1, 2(i + 2) + 1, \dots, 2(i + t) + 1$  vertices respectively and the distance partite sets  $V_{i+t+1}, \dots, V_k$  can have at most

$s - t, s - t - 1, \dots, s - (k - i - 1)$  vertices respectively. Thus the maximum number of vertices in  $G$  with  $\beta(G) = 2$  is

$$\sum_{m=0}^{i+t} (2m+1) + \sum_{n=t}^{k-i-1} (s-m) \\ = (i+t+1)^2 + (k-i-t)s - \left( \frac{(k-1)k}{2} \right) + \frac{t(t-1)}{2}.$$

#### VI. CHARACTERIZATION OF GRAPHS WITH METRIC DIMENSION TWO

The following is the characterization of graphs with metric dimension two.

**Theorem 6.1:** Let  $G$  be a graph which is not a path with  $V(G) = \{v_1, v_2, \dots, v_n\}$  and  $\{V_{i0}, V_{i1}, \dots, V_{ik_i}\}$  be the distance partition of  $V(G)$  with reference to the vertex  $v_i$  where  $k_i$  is the eccentricity of  $v_i$ ,  $1 \leq i \leq n$ . The metric dimension of  $G$  is 2 if and only if there exist vertices  $v_i$  and  $v_j$  such that

$$|V_{ik} \cap V_{jl}| \leq 1 \text{ for every } k \text{ and } l$$

with  $1 \leq k \leq e(v_i)$  and  $1 \leq l \leq e(v_j)$ .

**Proof:** Given  $v_p$  and  $v_r$ ,  $|V_{p_q} \cap V_{r_s}| > 1$  for some  $p_q$  and  $r_s$  implies that there exists atleast two vertices, say  $u_1$  and  $u_2$  in  $V_{p_q} \cap V_{r_s}$  such that  $d(v_p, u_1) = d(v_p, u_2) = q$  and  $d(v_r, u_1) = d(v_r, u_2) = s$  and hence  $u_1$  and  $u_2$  are not resolved by both  $v_p$  and  $v_r$  so,  $|V_{p_q} \cap V_{r_s}| > 1$  for all  $p_q$  and  $r_s$  implies no pair of vertices  $v_p$  and  $v_r$  resolves  $V(G)$ , in other words  $\beta(G) \neq 2$ .

Conversely if there exist  $v_p$  and  $v_r$  such that  $|V_{p_q} \cap V_{r_s}| \leq 1$  for all  $p_q$  and  $r_s$ , then given any pair of vertices  $w_1$  and  $w_2$  from  $V(G)$  we have  $w_1 \in V_{p_{q_1}} \cap V_{r_{s_1}}$  and  $w_2 \in V_{p_{q_2}} \cap V_{r_{s_2}}$  where at least  $p_{q_1}$  is different from  $p_{q_2}$  or  $r_{s_1}$  is different from  $r_{s_2}$ . This implies that  $w_1$  and  $w_2$  are resolved by at least one of  $v_p$  and  $v_r$ . So  $\beta(G) \leq 2$  and in fact,  $\beta(G) = 2$  as  $G$  is not a path. ♦

#### VII. ALGORITHM TO CHECK WHETHER THE METRIC DIMENSION OF A GIVEN GRAPH $G$ IS TWO

The following algorithm follows from the Theorem 6.1.

**Step 1:** Input: distance matrix

Input is the distance matrix ordered according to vertices of a graph  $G$  which is not a path.

**Step 2** : Check if  $\beta(G) \neq 2$  from number of elements in  $V(G)$

If number of vertices in  $G$  i.e.,  $|V(G)| > (D-1)^2 + 8$  where  $D$  is the diameter of the graph  $G$  then  $\beta(G) \neq 2$  (Samir khuller et al) .

*Step 3: Selection of vertices for finding metric basis*

Select only those vertices in  $G$  with degree less than or equal to three (Samir khuller et al) .

*Step 4: Formation of distance partitions*

Form distance partition  $\{V_{i_0}, V_{i_1}, \dots, V_{i_k}\}$  of  $V(G)$  with reference to every vertex  $v_i$  having degree less than or equal to three  $1 \leq i \leq n$  .

*Step 5: Identify the pair of vertices for finding metric basis*

- i) Given a pair  $(u_1, u_2)$  if eccentricity of a vertex  $u_2$  is less than number of vertices at distance  $d(u_1, u_2)$  and vice versa then  $\{u_1, u_2\}$  cannot be a metric basis for  $G$  . Consider only a remaining pairs.
- ii) Among the pairs  $(u_i, u_j)$  remaining, consider only the pairs with unique shortest path between them.

*Step 6: Find intersection*

If there exists vertices  $v_i$  and  $v_j$  ( $i \neq j$ ) with  $|V_{i_k} \cap V_{j_l}| \leq 1$  for every  $k$  and  $l$  with  $1 \leq k \leq e(v_i)$  and  $1 \leq l \leq e(v_j)$  , then  $\{v_i, v_j\}$  is a metric basis for the graph  $G$  . Otherwise the metric dimension of  $G$  is not equal to two.

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VIII.COMPLEXITY

Let  $G$  be a graph with diameter  $D$  on  $n$  vertices. Every set in distance partition of  $V(G)$  with reference to a vertex  $v$  is to be compared with at most  $(n-1)D$  sets. Therefore totally there are  $((n-1)D)D$  comparisons for  $v$  . For the next vertex the number of comparisons needed is  $((n-2)D^2)$  . Similarly for the last vertex the number of comparisons needed is  $(n-(n-1))D^2$  . Therefore the total number of set comparisons required is

$$((n-1)D^2 + (n-2)D^2 + \dots + 1.D^2) = D^2 \left[ \frac{n(n-1)}{2} \right].$$

In every comparison of two sets there can be at most  $D^2$  comparisons of elements. Hence total number of element comparisons is  $D^2 \left[ \frac{n(n-1)}{2} \right] D^2$  .

Thus the complexity of the algorithm is  $= \left[ \frac{n(n-1)}{2} \right] D^4$  .