International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:4, No:2, 2010

Hyers-Ulam Stability of Functional Equation $f(3x) = 4f(3x-3) + f(3x-6)$

Soon-Mo Jung

Abstract—The functional equation $f(3x) = 4f(3x-3) + f(3x-6)$ will be solved and its Hyers-Ulam stability will be also investigated in the class of functions $f : \mathbf{R} \to X$, where X is a real Banach space.

Keywords—Functional equation, Lucas sequence of the first kind, Hyers-Ulam stability.

I. Introduction

In 1940, Ulam gave a wide ranging talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of important unsolved problems (ref. [20]). Among those was the question concerning the stability of homomorphisms:

> Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given any $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

In the following year, Hyers affirmatively answered in his paper [8] the question of Ulam for the case where G_1 and G_2 are Banach spaces. Later, the result of Hyers has been generalized by Rassias (ref. [16]).

Let $(G_1, \cdot)$ be a groupoid and let $(G_2, +)$ be a groupoid with the metric d. The equation of homomorphism

$$f(x \cdot y) = f(x) + f(y)$$

is stable in the Hyers-Ulam sense (or has the Hyers-Ulam stability) if for every $\delta > 0$ there exists an $\varepsilon > 0$ such that for every function $h : G_1 \to G_2$ satisfying

$$d[h(x \cdot y), h(x) + h(y)] \le \varepsilon$$

for all $x, y \in G_1$ there exists a solution $g : G_1 \to G_2$ of the equation of homomorphism with

$$d[h(x), g(x)] \le \delta$$

for any $x \in G_1$ (see [15, Definition 1]).

This terminology is also applied to the case of other functional equations. It should be remarked that a lot of references concerning the stability of functional equations can be found in the books [3], [9], [12] (see also [1], [4], [5], [6], [7], [10], [11], [17], [18], [19]).

S.-M. Jung: Mathematics Section, College of Science and Technology, Hongik University, 339-701 Jochiwon, Republic of Korea
e-mail: smjung@hongik.ac.kr

The nth Fibonacci number will be denoted by F_n for $n \in \mathbf{N}$. It is well known that the Fibonacci numbers satisfy the equality $F_{3n} = 4F_{3n-3} + F_{3n-6}$ for all $n \ge 2$ (see [14, p. 89]). From this famous formula, the following functional equation

$$f(3x) = 4f(3x-3) + f(3x-6) \qquad (1)$$

may be derived.

In this paper, using the idea from [13], the functional equation (1) will be solved and its Hyers-Ulam stability will be investigated in the class of functions $f : \mathbf{R} \to X$, where X is a real Banach space.

Throughout this paper, the positive and the negative root of the equation $x^2 - 4x - 1 = 0$ will be denoted by a and b, respectively, i.e.,

$$a = 2 + \sqrt{5} \quad \text{and} \quad b = 2 - \sqrt{5}.$$

Moreover, the Lucas sequence of the first kind will be denoted by $\{U_n(4,-1)\}$ and an abbreviation U_n will be used instead of $U_n(4,-1)$, i.e., U_n is defined by

$$U_n = U_n(4,-1) = \frac{a^n - b^n}{a - b}$$

for all integers n. It is not difficult to see that

$$U_{n+2} = 4U_{n+1} + U_n \qquad (2)$$

for any integer n. For any $x \in \mathbf{R}$, $[x]$ stands for the largest integer that does not exceed x.

II. General Solution to Eq. (1)

Throughout this section, let X be a real vector space. The general solution of the functional equation (1) will be investigated.

Theorem 2.1. *Let X be a real vector space. A function $f : \mathbf{R} \to X$ is a solution of the functional equation* (1) *if and only if there exists a function $h : [-3,3) \to X$ such that*

$$f(x) = U_{[x/3]+1} h(x - 3[x/3]) + U_{[x/3]} h(x - 3[x/3] - 3). \qquad (3)$$

Proof. Since $a + b = 4$ and $ab = -1$, it follows from (1) that

$$\begin{cases} f(3x) - af(3x-3) = b\left[f(3x-3) - af(3x-6)\right], \\ f(3x) - bf(3x-3) = a\left[f(3x-3) - bf(3x-6)\right]. \end{cases}$$

If a function $g : \mathbf{R} \to X$ is defined by $g(x) = f(3x)$ for each $x \in \mathbf{R}$, then it follows from the above equalities that

$$\begin{cases} g(x) - ag(x-1) = b\left[g(x-1) - ag(x-2)\right], \\ g(x) - bg(x-1) = a\left[g(x-1) - bg(x-2)\right]. \end{cases} \tag{4}$$

By the mathematical induction, it can be proved that

$$\begin{cases} g(x) - ag(x-1) \\ \quad = b^n\left[g(x-n) - ag(x-n-1)\right], \\ g(x) - bg(x-1) \\ \quad = a^n\left[g(x-n) - bg(x-n-1)\right] \end{cases} \tag{5}$$

for all $x \in \mathbf{R}$ and $n \in \{0, 1, 2, \ldots\}$. Substitute $x + n$ $(n \geq 0)$ for x in (5) and divide the resulting equations by b^n resp. a^n, and then substitute $-m$ for n in the resulting equations to obtain the equations in (5) with m in place of n, where $m \in \{0, -1, -2, \ldots\}$. Therefore, the equations in (5) are true for all $x \in \mathbf{R}$ and $n \in \mathbf{Z}$.

Multiply the first and the second equation of (5) by b and a, respectively. And subtract the first resulting equation from the second one to obtain

$$g(x) = U_{n+1}g(x-n) + U_n g(x-n-1) \tag{6}$$

for any $x \in \mathbf{R}$ and $n \in \mathbf{Z}$.

Putting $n = [x]$ in (6) yields

$$g(x) = U_{[x]+1}g(x-[x]) + U_{[x]}g(x-[x]-1),$$

i.e., by the definition of g, it holds that

$$f(x) = U_{[x/3]+1}f(x-3[x/3]) + U_{[x/3]}f(x-3[x/3]-3)$$

for all $x \in \mathbf{R}$.

Since $0 \leq x - 3[x/3] < 3$ and $-3 \leq x - 3[x/3] - 3 < 0$, if a function $h : [-3, 3) \to X$ is defined by $h := f|_{[-3,3)}$, then f is a function of the form (3).

Now, assume that f is a function of the form (3), where $h : [-3, 3) \to X$ is an arbitrary function. Then, it follows from (3) that

$$f(3x) = U_{[x]+1}h(3x-3[x]) + U_{[x]}h(3x-3[x]-3),$$

$$f(3x-3) = U_{[x]}h(3x-3[x]) + U_{[x]-1}h(3x-3[x]-3),$$

$$f(3x-6) = U_{[x]-1}h(3x-3[x]) + U_{[x]-2}h(3x-3[x]-3)$$

for any $x \in \mathbf{R}$. Thus, by (2), it holds that

$$\begin{aligned} & f(3x) - 4f(3x-3) - f(3x-6) \\ &\quad = (U_{[x]+1} - 4U_{[x]} - U_{[x]-1})h(3x-3[x]) \\ &\quad\quad +(U_{[x]} - 4U_{[x]-1} - U_{[x]-2})h(3x-3[x]-3) \\ &\quad = 0, \end{aligned}$$

which completes the proof.

III. Hyers-Ulam Stability of Eq. (1)

In this section, a denotes the positive root of the equation $x^2 - 4x - 1 = 0$ and b is its negative root. The Hyers-Ulam stability of the functional equation (1) will be proved in the following theorem.

Theorem 3.1. *Let $(X, \|\cdot\|)$ be a real Banach space. If a function $f : \mathbf{R} \to X$ satisfies the inequality*

$$\|f(3x) - 4f(3x-3) - f(3x-6)\| \leq \varepsilon \tag{7}$$

for all $x \in \mathbf{R}$ and for some $\varepsilon \geq 0$, then there exists a unique solution function $F : \mathbf{R} \to X$ of (1) such that

$$\|f(x) - F(x)\| \leq \frac{5+\sqrt{5}}{20}\varepsilon \tag{8}$$

for all $x \in \mathbf{R}$.

Proof. First, define a function $g : \mathbf{R} \to X$ by $g(x) = f(3x)$ for all $x \in \mathbf{R}$. Analogously to the first equation of (4), it follows from (7) that

$$\|g(x) - ag(x-1) - b[g(x-1) - ag(x-2)]\| \leq \varepsilon$$

for each $x \in \mathbf{R}$. Replacing x with $x - k$ in the last inequality yields

$$\begin{aligned} & \|g(x-k) - ag(x-k-1) \\ &\quad\quad - b[g(x-k-1) - ag(x-k-2)]\| \\ &\quad \leq \varepsilon \end{aligned}$$

and further

$$\begin{aligned} & \big\|b^k[g(x-k) - ag(x-k-1)] \\ &\quad\quad - b^{k+1}[g(x-k-1) - ag(x-k-2)]\big\| \\ &\quad \leq |b|^k\varepsilon \end{aligned} \tag{9}$$

for all $x \in \mathbf{R}$ and $k \in \mathbf{Z}$. By (9), it obviously holds that

$$\begin{aligned} & \|g(x) - ag(x-1) - b^n[g(x-n) - ag(x-n-1)]\| \\ &\quad \leq \sum_{k=0}^{n-1}\|b^k[g(x-k) - ag(x-k-1)] \\ &\quad\quad\quad - b^{k+1}[g(x-k-1) - ag(x-k-2)]\| \\ &\quad \leq \sum_{k=0}^{n-1}|b|^k\varepsilon \end{aligned} \tag{10}$$

for $x \in \mathbf{R}$ and $n \in \mathbf{N}$.

For any $x \in \mathbf{R}$, (9) implies that the sequence $\{b^n[g(x-n) - ag(x-n-1)]\}$ is a Cauchy sequence. (Note that $|b| < 1$). Therefore, a function $G_1 : \mathbf{R} \to X$ can be defined by

$$G_1(x) = \lim_{n\to\infty} b^n[g(x-n) - ag(x-n-1)],$$

since X is complete. It follows from the definition of G_1 that

$$\begin{aligned}
&4G_1(x-1)+G_1(x-2)\\
&= 4b^{-1}\lim_{n\to\infty} b^{n+1}[g(x-(n+1))\\
&\qquad - ag(x-(n+1)-1)]\\
&\quad + b^{-2}\lim_{n\to\infty} b^{n+2}[g(x-(n+2))\\
&\qquad - ag(x-(n+2)-1)]\\
&= 4b^{-1}G_1(x)+b^{-2}G_1(x)\\
&= G_1(x)
\end{aligned} \tag{11}$$

for all $x \in \mathbf{R}$, since $b^2 = 4b+1$. If n goes to infinity, then (10) yields that

$$\|g(x)-ag(x-1)-G_1(x)\| \le \frac{3+\sqrt{5}}{4}\varepsilon \tag{12}$$

for every $x \in \mathbf{R}$.

On the other hand, it also follows from (7) that

$$\|g(x)-bg(x-1)-a[g(x-1)-bg(x-2)]\| \le \varepsilon$$

(see the second equation in (4)). Analogously to (9), replacing x by $x+k$ in the above inequality and then dividing by a^k both sides of the resulting inequality yield

$$\begin{aligned}
&\big\|a^{-k}[g(x+k)-bg(x+k-1)]\\
&\qquad - a^{-k+1}[g(x+k-1)-bg(x+k-2)]\big\|\\
&\le a^{-k}\varepsilon
\end{aligned} \tag{13}$$

for all $x \in \mathbf{R}$ and $k \in \mathbf{Z}$. It further follows from (13) that

$$\begin{aligned}
&\|a^{-n}[g(x+n)-bg(x+n-1)]-[g(x)-bg(x-1)]\|\\
&\le \sum_{k=1}^{n}\|a^{-k}[g(x+k)-bg(x+k-1)]\\
&\qquad - a^{-k+1}[g(x+k-1)-bg(x+k-2)]\|\\
&\le \sum_{k=1}^{n} a^{-k}\varepsilon
\end{aligned} \tag{14}$$

for $x \in \mathbf{R}$ and $n \in \mathbf{N}$.

On account of (13), the sequence $\{a^{-n}[g(x+n)-bg(x+n-1)]\}$ is a Cauchy sequence for any fixed $x \in \mathbf{R}$. Hence, a function $G_2 : \mathbf{R} \to X$ can be defined by

$$G_2(x) = \lim_{n\to\infty} a^{-n}[g(x+n)-bg(x+n-1)].$$

It follows from the definition of G_2 that

$$\begin{aligned}
&4G_2(x-1)+G_2(x-2)\\
&= 4a^{-1}\lim_{n\to\infty} a^{-(n-1)}[g(x+n-1)\\
&\qquad - bg(x+(n-1)-1)]\\
&\quad + a^{-2}\lim_{n\to\infty} a^{-(n-2)}[g(x+n-2)\\
&\qquad - bg(x+(n-2)-1)]\\
&= 4a^{-1}G_2(x)+a^{-2}G_2(x)\\
&= G_2(x)
\end{aligned} \tag{15}$$

for any $x \in \mathbf{R}$. By letting n go to infinity, (14) yields

$$\|G_2(x)-g(x)+bg(x-1)\| \le \frac{\sqrt{5}-1}{4}\varepsilon \tag{16}$$

for $x \in \mathbf{R}$.

From (12) and (16), it follows that

$$\begin{aligned}
&\left\|g(x)-\left[\frac{b}{b-a}G_1(x)-\frac{a}{b-a}G_2(x)\right]\right\|\\
&= \frac{1}{|b-a|}\|(b-a)g(x)-[bG_1(x)-aG_2(x)]\|\\
&\le \frac{1}{a-b}\|bg(x)-abg(x-1)-bG_1(x)\|\\
&\quad + \frac{1}{a-b}\|aG_2(x)-ag(x)+abg(x-1)\|\\
&\le \frac{5+\sqrt{5}}{20}\varepsilon
\end{aligned} \tag{17}$$

for all $x \in \mathbf{R}$. Now define a function $F : \mathbf{R} \to X$ by

$$F(x) = \frac{b}{b-a}G_1\Big(\frac{x}{3}\Big)-\frac{a}{b-a}G_2\Big(\frac{x}{3}\Big)$$

for all $x \in \mathbf{R}$. Then, it follows from (11) and (15) that

$$\begin{aligned}
&4F(3x-3)+F(3x-6)\\
&= \frac{4b}{b-a}G_1(x-1)-\frac{4a}{b-a}G_2(x-1)\\
&\quad + \frac{b}{b-a}G_1(x-2)-\frac{a}{b-a}G_2(x-2)\\
&= \frac{b}{b-a}[4G_1(x-1)+G_1(x-2)]\\
&\quad - \frac{a}{b-a}[4G_2(x-1)+G_2(x-2)]\\
&= \frac{b}{b-a}G_1(x)-\frac{a}{b-a}G_2(x)\\
&= F(3x)
\end{aligned}$$

for each $x \in \mathbf{R}$, i.e., F is a solution of (1). Moreover, the inequality (8) follows from (17).

The uniqueness of F will be proved. Assume that $F_1, F_2 : \mathbf{R} \to X$ are solutions of (1) and that there exist positive constants C_1 and C_2 with

$$\|f(x)-F_1(x)\| \le C_1 \quad \text{and} \quad \|f(x)-F_2(x)\| \le C_2 \tag{18}$$

for all $x \in \mathbf{R}$. According to Theorem 2.1, there exist functions $h_1, h_2 : [-3,3) \to X$ such that

$$\begin{aligned}
F_1(x) &= U_{[x/3]+1}h_1(x-3[x/3])\\
&\quad + U_{[x/3]}h_1(x-3[x/3]-3),\\
F_2(x) &= U_{[x/3]+1}h_2(x-3[x/3])\\
&\quad + U_{[x/3]}h_2(x-3[x/3]-3)
\end{aligned} \tag{19}$$

for any $x \in \mathbf{R}$, since F_1 and F_2 are solutions of (1).

Fix a t with $0 \le t < 3$. It then follows from (18) and (19) that

$$\begin{aligned}
&\|U_{n+1}[h_1(t)-h_2(t)]+U_n[h_1(t-3)-h_2(t-3)]\|\\
&= \|[U_{n+1}h_1(t)+U_nh_1(t-3)]\\
&\qquad - [U_{n+1}h_2(t)+U_nh_2(t-3)]\|\\
&= \|F_1(3n+t)-F_2(3n+t)\|\\
&\le \|F_1(3n+t)-f(3n+t)\|+\|f(3n+t)-F_2(3n+t)\|\\
&\le C_1+C_2
\end{aligned}$$

for each $n \in \mathbf{Z}$, i.e.,

$$\left\| \frac{a^{n+1} - b^{n+1}}{a-b}[h_1(t) - h_2(t)] + \frac{a^n - b^n}{a-b}[h_1(t-3) - h_2(t-3)] \right\| \leq C_1 + C_2 \tag{20}$$

for every $n \in \mathbf{Z}$. Dividing both sides by a^n yields that

$$\left\| \frac{a - (b/a)^n b}{a-b}[h_1(t) - h_2(t)] + \frac{1 - (b/a)^n}{a-b}[h_1(t-3) - h_2(t-3)] \right\| \leq \frac{C_1 + C_2}{a^n}.$$

Let $n \to \infty$ to get

$$a[h_1(t) - h_2(t)] + [h_1(t-3) - h_2(t-3)] = 0. \tag{21}$$

Analogously, divide both sides of (20) by $|b|^n$ and let $n \to -\infty$ to get

$$b[h_1(t) - h_2(t)] + [h_1(t-3) - h_2(t-3)] = 0. \tag{22}$$

From (21) and (22), it follows that

$$\begin{pmatrix} a & 1 \\ b & 1 \end{pmatrix} \begin{pmatrix} h_1(t) - h_2(t) \\ h_1(t-3) - h_2(t-3) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Because $a - b \neq 0$, it should hold that

$$h_1(t) - h_2(t) = h_1(t-3) - h_2(t-3) = 0$$

for any $t \in [0, 3)$, i.e., $h_1(t) = h_2(t)$ for all $t \in [-3, 3)$. Therefore, it is true that $F_1(x) = F_2(x)$ for any $x \in \mathbf{R}$.

Remark 1. The presented proof of uniqueness of F is due to an idea of Professor Changsun Choi. It should be remarked that the uniqueness of F can be obtained directly from [2, Proposition 1].

ACKNOWLEDGMENT

This work was supported by National Research Foundation of Korea Grant funded by the Korean Government (No. 2009-0071206).

REFERENCES

[1] J. Baker, J. Lawrence and F. Zorzitto, *The stability of the equation* $f(x+y) = f(x)f(y)$, Proc. Amer. Math. Soc. **74** (1979), 242–246.
[2] J. Brzdęk, D. Popa and B. Xu, *Hyers-Ulam stability for linear equations of higher orders*, Acta Math. Hungar. **120** (2008), 1–8.
[3] S. Czerwik, *Stability of Functional Equations of Ulam-Hyers-Rassias Type*, Hadronic Press, Palm Harbor, 2003.
[4] G. L. Forti, *Hyers-Ulam stability of functional equations in several variables*, Aequationes Math. **50** (1995), 143–190.
[5] Z. Gajda, *On stability of additive mappings*, Internat. J. Math. Math. Sci. **14** (1991), 431–434.
[6] P. Găvruţă, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.
[7] R. Ger and P. Šemrl, *The stability of the exponential equation*, Proc. Amer. Math. Soc. **124** (1996), 779–787.
[8] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. USA **27** (1941), 222–224.
[9] D. H. Hyers, G. Isac and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Boston, 1998.
[10] D. H. Hyers and Th. M. Rassias, *Approximate homomorphisms*, Aequationes Math. **44** (1992), 125–153.
[11] S.-M. Jung, *Hyers-Ulam-Rassias stability of functional equations*, Dynamic Sys. Appl. **6** (1997), 541–566.
[12] S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, 2001.
[13] S.-M. Jung, *Hyers-Ulam stability of Fibonacci functional equation*, Bull. Iranian Math. Soc., in press.
[14] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley & Sons, New York, 2001.
[15] Z. Moszner, *On the stability of functional equations*, Aequationes Math. **77** (2009), 33–88.
[16] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.
[17] Th. M. Rassias, *On the stability of functional equations and a problem of Ulam*, Acta Appl. Math. **62** (2000), 23–130.
[18] Th. M. Rassias, *On the stability of functional equations in Banach spaces*, J. Math. Anal. Appl. **251** (2000), 264–284.
[19] L. Székelyhidi, *On a theorem of Baker, Lawrence and Zorzitto*, Proc. Amer. Math. Soc. **84** (1982), 95-96.
[20] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience Publ., New York, 1960.

Soon-Mo Jung was born in Seochun, Chungchung-namdo, a typical farming village in Korea in 1957 where he spent his childhood. He received his B.S. in nuclear engineering from Seoul National University. Upon his graduation, he briefly worked for Hankook Heavy Industries and IBM Korea as a programmer. He received his 'Diplom' in number theory and Ph.D. in measure theory in February 1994 from Universität Stuttgart in Germany. His master's advisor was Professor Dr. B. Volkmann and his doctoral advisor was Professor Dr. D. Kahnert. Under Dr. Kahnert's guidance, he wrote his doctoral thesis in Housdorff dimensions and measures. He joined Hongik University as a mathematics professor in 1995. His research interest includes Hyers-Ulam stability problems of functional equations and differential equations and some topics in classical geometry.