Efficiency of Different GLR Test-statistics for Spatial Signal Detection

Olesya Bolkhovskaya, and Alexander Maltsev

Abstract—In this work the characteristics of spatial signal detection from an antenna array in various sample cases are investigated. Cases for a various number of available prior information about the received signal and the background noise are considered. The spatial difference between a signal and noise is only used. The performance characteristics and detecting curves are presented. All test-statistics are obtained on the basis of the generalized likelihood ratio (GLR). The received results are correct for a short and long sample.

Keywords—GLR test-statistic, detection task, generalized likelihood ratio, antenna array, detection curves, performance characteristics.

I. INTRODUCTION

Multichannel adaptive signal detection is interesting with a practical point. It is more and more widely using the multielement antenna array in the modern radio systems, hydrolocations and communications. The solving of this problem is based on GLRT method application [1]. In different applications is a large variety of tasks depending on the known a priori information. In the most cases of solved tasks the temporary structure of the useful signal, the direction of its arrival a form of a wavefront set (for example pane wave) is supposed known [2] - [6]. In contrary of these works in the present paper it is used only difference spatial properties of the signal and noise for detection. In the present work the detailed comparison of test-statistics is carried out, starting from a basic case (in this case a hindrance is the noise independence for each element of the antenna and signal is everything what is not noise) to the detection case the signal is completely coherent, but has an unknown wavefront set.

II. DETECTION ALGORITHMS

Consider a p-element receiving antenna array with arbitrary locations of sensors. The p-dimensional input signal $\vec{z}$ is assumed to be complex random Gaussian vector and N samples of the signal $\vec{z}_1, \vec{z}_2, ..., \vec{z}_N$ are statistically independent and identically distributed zero-mean random vectors with spatial covariance matrix Σ. By reviewing of non-singular case, let us suppose that the sample size N is greater than the number of the antenna elements $(N > p)$. Consider enough general case when the useful signal is spatially correlated, and gaussian

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noise is spatially and temporarily white. Then the detection problem formulation is consolidated to a classical statistical two-alternative:

$$H_0 : \Sigma = \Sigma_0, H_1 : \Sigma \neq \Sigma_0, \qquad (1)$$

where H_0 is the null hypothesis about only noise presence, and H_1 is the alternative hypothesis about additive mix of noise and useful signal presence.

The decision about signal presence or absence is accepted by a way of comparison of the likelihood ratio Λ with some threshold value Λ_{th}. If the likelihood ratio doesn't exceed a threshold, i.e.

$$\Lambda < \Lambda_{th} \qquad (2)$$

the decision about signal available is made. According to Neumann-Pearson's criterion threshold value of a statistic Λ_{th} in (2) is defined from a condition of the probability of the right detection at the given false alarm probability. The introduction of noise characteristics occurs with the help of definition of covariance matrix Σ_{0i} [7], [8]. The description of the signal is carried out by means of the spatial covariance matrices Σ_{1i}. Using concrete type of the covariance matrix is defined by a quantity of available prior information.

In the first considered case information about noise independence of its counting in various antenna array elements only available. The covariance matrix Σ_0 for such noise look like

$$\Sigma_{01} = \begin{pmatrix} \sigma_{11}^2 & 0 & \dots & 0 \\ 0 & \sigma_{22}^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_{pp}^2 \end{pmatrix} \qquad (3)$$

and correspond to a null hypothesis of H_{01}.

In the second case in addition to the previous information of the noise independence there is information about its homogeneity (identical power in the different elements of the antenna array: $\sigma_{ii}^2 = \sigma^2 = const$) is added. Then covariance matrix Σ_0 is equal to

$$\Sigma_{02} = \sigma^2 \mathbf{I} \qquad (4)$$

The appropriate null hypothesis designate H_{02}. If, in addition to this information, the noise power σ^2 is also known (for example, it is measured during the signal absence or

transducers are preliminary calibrated, so that $\sigma_{ii}^2 = \sigma^2 = 1$), then the noise covariance matrix $\mathbf{\Sigma}_0$ will be identity:

$$\mathbf{\Sigma}_{03} = \mathbf{I} \qquad (5)$$

The appropriate null hypothesis is denoted by H_{03}. GLR test-statistic for the (1) is written as

$$\Lambda = \frac{\max\limits_{\mathbf{\Sigma} \in \omega} L(\vec{\mathbf{0}}, \mathbf{\Sigma})}{\max\limits_{\mathbf{\Sigma} \in \Omega} L(\vec{\mathbf{0}}, \mathbf{\Sigma})}, \qquad (6)$$

where

$$L(\vec{\mathbf{0}}, \mathbf{\Sigma}) = \frac{1}{(det(\mathbf{\Sigma}))^N \pi^{pN}} e^{-\sum\limits_{\alpha=1}^{N} \vec{\mathbf{z}}_\alpha^\dagger \mathbf{\Sigma}^{-1} \vec{\mathbf{z}}_\alpha} \qquad (7)$$

is the likelihood function for complex variables, ω is the parameter subspace matching to the null hypothesis H_0 in the full parameter space Ω The detailed conclusion of the GLR test-statistic for H_{01} (3) is given by work [9] where the following expression is received ($\mathbf{A} = \frac{1}{N}\mathbf{\Sigma}$):

$$\Lambda_1 = \frac{|A|^N}{\prod\limits_{i=1}^{p} (a_{ii})^N}. \qquad (8)$$

The GLR test-statistics Λ_2 and Λ_3 were received in [7] and can be represented as:

$$\Lambda_2 = \frac{|\hat{\mathbf{\Sigma}}_\Omega|^N}{(\frac{Sp\mathbf{A}}{p})^{pN}(\frac{1}{N})^{pN}} = \frac{|\mathbf{A}|^N}{(\frac{Sp\mathbf{A}}{p})^{pN}} \qquad (9)$$

$$\Lambda_3 = \frac{(\pi)^{-pN} e^{-\sum\limits_{\alpha=1}^{N} \vec{\mathbf{z}}^{(\alpha)} \vec{\mathbf{z}}^{(\alpha)\dagger}}}{(\pi)^{-pN}|\frac{1}{N}\mathbf{A}|^{-N} \mathbf{e}^{-\mathbf{pN}}} = (\frac{e}{N})^{pN} |\mathbf{A}|^{\mathbf{N}} \mathbf{e}^{-\mathbf{spA}} \qquad (10)$$

For completeness of comparison it is considered test-statistic, which based on the GLR (6) for the case of a null hypothesis (5), but with additional prior information about the complete coherence of the receiveded useful signal [10]. The components of a narrowband spatially coherent signal received by the antenna array elements are completely correlated and differ only in amplitude and a phase. Therefore, vector of the useful signal $\vec{S}(t)$ in this case can be written as

$$\vec{S}(t) = a(t)\vec{S}. \qquad (11)$$

Here $a(t)$ - is the complex amplitude (Gaussian complex signal with a zero mean, power ν, and a fazor vector $\vec{S}$ defines the phase shift and amplitudes distribution between signals, accepted of the antenna array elements. Without loss of generality it is considered that the vector $\vec{S}$ submits to the following normalization:

$$\vec{S}^\dagger \vec{S} = p, \qquad (12)$$

where p is the number of antenna array elements. For such normalization ν is meaningful to average power of an external

TABLE I
TEST-STATISTICS

H_0	H_0 physical meaning	GLR test-statistic
H_{01}	independent noise	$\dfrac{|A|^N}{\prod\limits_{i=1}^{p}(a_{ii})^N}$
H_{02}	independent homogeneous noise	$\dfrac{|A|^N}{(\frac{spA}{p})^{pN}}$
H_{03}	independent homogeneous noise with identity covariance matrix	$(\frac{e}{N})^{pN}|\mathbf{A}|^{\mathbf{N}}\mathbf{e}^{-\mathbf{spA}}$
H_{04}	independent homogeneous noise with identity covariance matrixr	$e^N(\hat{\lambda}_1^{-1} e^{\hat{\lambda}_1})^{-N}$

signal on the array elements (each element on the average accepts signal in power ν). In this case covariance matrix for the vector $\vec{z}$ which consists of the additive sum of a signal $\vec{S}(t)$ and Gaussian noise of unit power could be written in the following form:

$$\mathbf{\Sigma}_\omega = \mathbf{I} + \nu \vec{\mathbf{S}} \vec{\mathbf{S}}^\dagger \qquad (13)$$

For such a model of the useful signal and noise it is possible to show that numerator a denominator of the generalized likelihood ratio (6), will be defined by expression

$$L(\vec{\mathbf{0}}, \mathbf{\Sigma}) = (\pi)^{-pN} e^{-\sum\limits_{\alpha=1}^{N} \vec{\mathbf{z}}^{(\alpha)} \vec{\mathbf{z}}^{(\alpha)\dagger}} \qquad (14)$$

$$\max\limits_{\mathbf{\Sigma} \in \Omega} L(\vec{\mathbf{0}}, \mathbf{\Sigma}) = \pi^{-pN} e^{-N(Sp\hat{\mathbf{\Sigma}} - \hat{\lambda}_1)} \hat{\lambda}_1^{-N} e^{-N}, \qquad (15)$$

where $\hat{\lambda}_1$ is the maximum eigenvalue of the sample covariance matrix $\hat{\mathbf{\Sigma}}$. Thus, the likelihood ratio for this hypothesis looks like:

$$\Lambda_4 = e^N(\hat{\lambda}_1^{-1} e^{\hat{\lambda}_1})^{-N} \qquad (16)$$

and called the max-λ test [10]. Problem of finding of threshold values for this test it was solved in work [10].

The received results can be generalized into the table I:

The physical meaning of hypotheses H_{03} and H_{04} is identical. The difference between this two case consists in hypotheses H_{1i}. In the hypothesis H_{13} the useful signal has a covariance matrix, distinct from the identity matrix, and in the hypothesis H_{13} the useful signal is complitely spatially coherent with unknown wave front.

In order to evaluate Λ_{th_i} in (2) for a given false alarm probability P_{FA} it is necessary to know the PDF or the cumulative distribution function CDF of the random variable Λ_i then the null hypothesis is true. While the PDF of the GLR test-statistic Λ is unknown (for the first three statistics), the exact analytic form for the statistical moments of arbitrary order for the test-statistic Λ_i can be found by using the complex Wishart distribution for the matrix $\mathbf{A}$ given in [11]
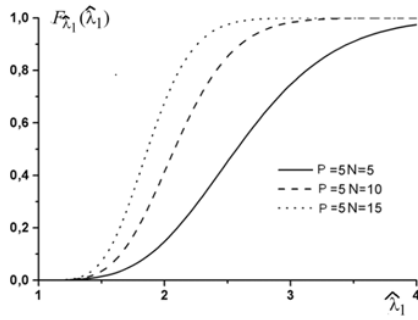
Fig. 1.   CDFs of the test-statistic V_4

$$W(\mathbf{A}, \boldsymbol{\Sigma}, N) = (det(\mathbf{A}))^{N-p} K(\boldsymbol{\Sigma}) e^{-tr(\boldsymbol{\Sigma}^{-1}\mathbf{A})}, \qquad (17)$$

where $K(\boldsymbol{\Sigma}) = 1/\pi^{\frac{1}{2}p(p-1)} \Gamma(n) \dots \Gamma(N-p+1)(det(\boldsymbol{\Sigma}))^N$.

If the null hypothesis H_0 is valid, the matrix $\mathbf{A}$ has $W(\mathbf{A}, \boldsymbol{\Sigma}_0, N)$ distribution. Analytical expressions for the test-statistics Λ_i (table 1) allow to consider their $V_i = \Lambda i^{1/N}$ function as such conversion is monotonic. The exact expression for the GLR test-statistics $V_1 - V_3$ moments were found in works [7], [9]. This result allows to determine PDF $W_{V_i}(x)$ of each random variable of V_i as an expansion in a series by the orthogonal polynomials total system. In this case the Jacobi polynomials were used. Such approximation of PDF give the possibility to find the threshold values for this test-statistics.

The hypothesis H_{04} is tested by comparing with a certain threshold of the maximum eigenvalue $\hat{\lambda}$ of a sample covariance matrix $\hat{\boldsymbol{\Sigma}}$. CDF of the maximal noise eigenvalue corresponding to a null hypothesis of H_{04} was received in work [10] and looks like:

$$F_{\hat{\lambda}_1}(\hat{\lambda}_1) = |\frac{\gamma(N-p+i-j-1, \hat{\lambda}_1 N)}{\Gamma(N-p+i)\Gamma(j)}|, \qquad (18)$$

In the fig. 1 CDF for various values of dimension of an antenna array p and number of samples of N are presented.

Thus threshold value for a maximum eigenvalue estimation is a solution equation

$$P_{FA} = 1 - F_{\hat{\lambda}_1, H_0}(hh), \qquad (19)$$

where P_{FA} the given false alarm probability level.

III. SIMULATION RESULTS

The above considered detection algorithms were simulated numerically. According to suggested null hypotheses of H_{01} and H_{02} noise was simulated as unhomogeneous with co-variance matrix (3) and as homogeneous with covariance matrix (4) correspondingly. In the homogeneous noise case the powers in the antenna array elements were identical, in the homogeneous noise case this powers were different and the power averaged over all antenna elements coincided with that of homogeneous noise in the one element. In all considered cases, in homogeneous and unhomogeneous, the trace of a

covariance matrix of noise remained equal to number of elements of an antenna arrray.

The useful signal were simulated as a plane wave, completely or patially coherent. In the patially coherent model a plane wave with a fluctuating arrival angle used. Obviously, the useful signal in this case can be written in the following form:

$$\mathbf{S(t)} = \mathbf{a(t)S} = (\mathbf{e^{i\theta_0}}, \mathbf{e^{i(2\pi\delta \sin((\theta(t))+\theta_0)}},$$
$$\dots, e^{i(2\pi\delta(p-1)\sin((\theta(t))+\theta_0)}). \qquad (20)$$

Here $a(t)$ is the complex Gaussian amplitude of a signal, $\mathbf{S}$ is the vector fazor of the useful signal wave front , θ_0 is an initial phase of a signal in the first antenna element, δ is the ratio of the distance between elements of the antenna array to a wavelength of a source signal, $\theta(t)$ is a variable angle of incidence of the wave concerning a normal. It was supposed that the law of an incidence angle change variation of a wave represents the normal process with independent increments. Choice of parameters for partially coherent signal provided its significant spatial incoherence on the receiving antenna array aperture. In this experiment the linear equidistant 5-element antenna array was simulated. The distance between elements of array was equal to half wavelength.

Comparing of the application efficiency for all four statistics V_1, V_2, V_3, V_4 for the useful signal detection was carried out. On the base of method [9] and using the specified P_{FA} the threshold values for used GLR test-statistics were calculated and curve detection (probability of the right detection as functions of the signal-to-noise ratio SNR in one antenna element) were constructed.

In the fig. 2 detection curves for completely coherent signal on a background a homogeneous noise for all statistics of V_1-V_4 are represented. In this case noise power in elements of the antenna array was assumed to be identical and equal to unity.

Probability of the false alarms in this case and in all subsequent experiments was equal to $P_{FA} = 0.05$. Fig. 3 shows similar curves for detection of the partially coherent signal. From the provided results it is seen that most efficient in the detection completely coherent signal case (see fig. 2) is using statistics V_4 because conditions of its application in this case completely correspond to considered model of the useful signal and noise. In the case of the partially coherent signal (fig. 3), it is possible to make conclusion that the max-λ-test characteristics are weakly influenced arrival angle fluctuations (linear fluctuations of the wave front) of the signal.

In fig. 4 - 5 detection curves is received for the detection task of the useful signal on a background of a unhomogeneous noise from the different nonuniformity degree - feeblly (fig.4) and middle (fig.5).

From comparing of detection curves shown in fig. 4-5 it is visible that with increasing degree of noise unhomogenity the most effective becomes statistics V_1, as from all four researched GLR test-statistics only for it a null hypothesis completely corresponded to conditions of carried-out simulation. In the middle unhomogeneous noise casee the efficiencies of V_1 and V_4 are practically compared. Already in case of middle unhomogeneous noise application the statistics V_2
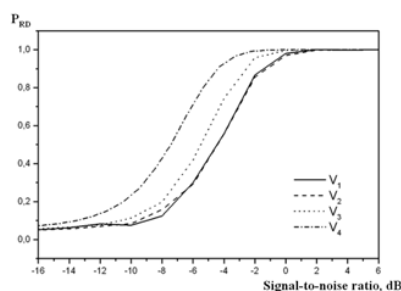
Fig. 2. Detection curves for V_1,V_2,V_3,V_4 for completely coherentl signal on a background a homogeneous noise
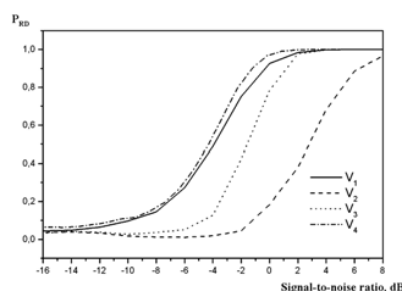

Fig. 5. Detection curves for V_1,V_2,V_3,V_4 for completely coherentl signal on a background a middle unhomogeneous noise
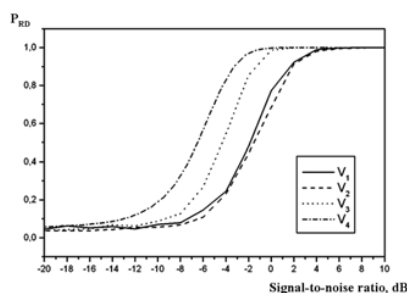

Fig. 3. Detection curves for V_1,V_2,V_3,V_4 for partial coherentl signal on a background a homogeneous noise
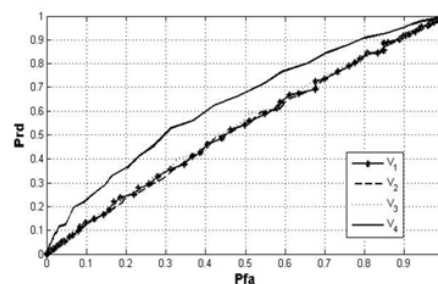

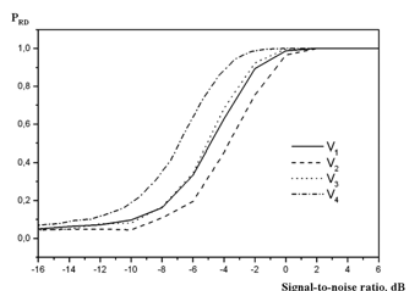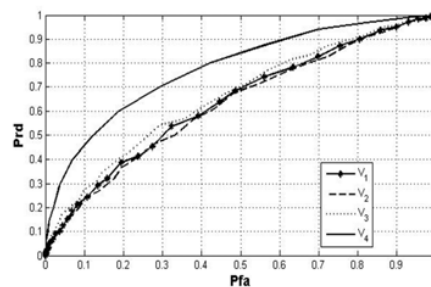Fig. 6. Performance characteristics for V_1 - V_4 for p=4,N=8, homogeneous noise

Fig. 7. Performance characteristics for V_1 - V_4 for p=4,N=32, homogeneous noise

Fig. 4. Detection curves for V_1,V_2,V_3,V_4 for completely coherentl signal on a background a feeblly unhomogeneous homogeneous noise

and V_3 becomes ineffective (see fig. 5), and for strongly unhomogeneous noise by the almost unacceptable.

The noise immunity and the application possibility for the detection algorithms for the considered noise and useful signals was studied by the performance characteristics of antenna array (probability of right detection as function of the false alarm probability) for the given signal-to-noise ratio SNR (i.e. the ratio of the useful-signal power to the noise power in one antenna element). In this experiment the linear equidistant 4-element antenna array was simulated. The distance between elements was equal to half wavelength again.

In fig. 6-8 performance characteristics for detection of a completely coherent signal (the plane wave) on the homogeneous noise background by means of statistics of $V_1 - V_4$ are

presented for the different sample size.

Doubling sample size with $N = 8$ to $N = 16$ (in the short sample case) increase the probability of the right detection P_{rd} approximately for 40% for V_1, V_3, V_4 statistics (for 9% for V_2) while doubling sample size with $N = 64$ to $N = 128$ (in the large sample size) increase value of the P_{rd} approximately for 80% (for V_1 and V_4) to 200% (for V_2 and V_3).

Analogous performance characteristics were constructed for a case when noise becomes rather strongly unhomogeneous (fig.9-11). In this case improvement of the performance characteristics of the statistics V_4 occurs very sluggishly, while the statistics V_1 shows very good results, and also is much more simply and fast calculated.

Doubling sample size with $N = 8$ to $N = 16$ (in the short sample case) increase P_{rd} approximately for 130% for V_1 and only for 9% for V_4. Doubling sample size with $N = 64$ to
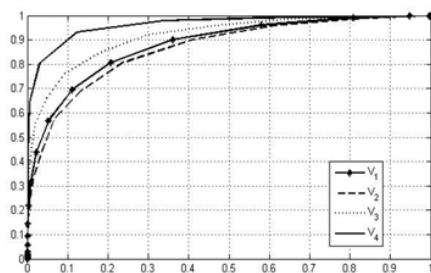
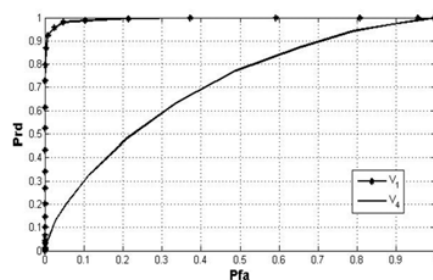Fig. 8. Performance characteristics for V_1 - V_4 for p=4,N=128, homogeneous noise

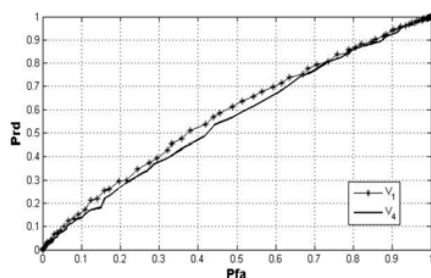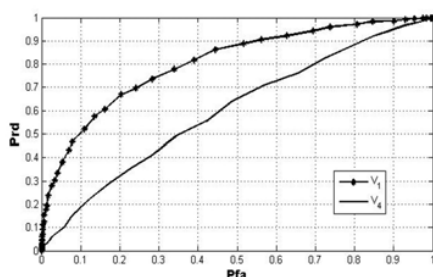Fig. 11. Performance characteristics for V_1 and V_4 for p=4,N=128, unhomogeneous noise

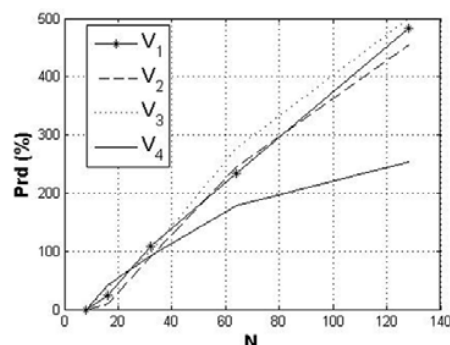Fig. 9. Performance characteristics for V_1 and V_4 for p=4,N=8, unhomogeneous noise

Fig. 12. The probability of the right detections for V_1,V_2,V_3,V_4 for completely coherent signal on a background a homogeneous noise for different sample size
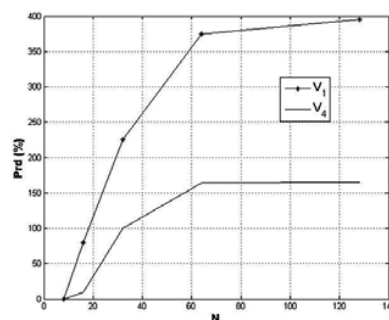

Fig. 10. Performance characteristics for V_1 and V_4 for p=4,N=32, unhomogeneous noise

Fig. 13. The probability of the right detections for V_1 and V_4 for completely coherent signal on a background a unhomogeneous noise for different sample size

$N = 128$ (in the large sample case) increase probability P_{rd} approximately for 20% for V_1 and only for 1% for V_4. It is necessary to consider that the percent of increasing for the V_1 test-statistic P_{rd} is limited by fact that already at $N = 64$ value of $P_{rd} = 0.95$.

In fig. 12-13 it is visible, how many percent the probability of the right detection P_{rd} increase with a sample size increasing in the homogeneous and inhomogeneous noise cases. Most quickly the P_{rd} grows for a statistics V_1 in case of the inhomogeneous noise. Thus the algorithm based on a GLR test-statistics V_1 is applicable and in the short sampl case, and in case of the unhomogeneous noise.

IV. CONCLUSION

In this paper comparison of usability of the different signal processing algorithms received on a base of GLR for detection space signals in the different samples case is carried out. The most efficient and robust (insensitive to unhomogeneity)

statistics are found in each case. The performance characteristics of test-statistic V_1 are robust to unhomogeneity and can be used for the detection spatially coherent and partially coherent signals on the background of unhomogeneous noise in the short sample case. The increase in a sample size for V_1 twice leads to increase in probability of the right detection approximately for 25% and for 80% , and increase in a sample in 4 times - for 100% and for 225% for homogeneous and unhomogeneous noises respectively. Therefore at the practical tasks it is recommended to find a compromise between time which can be spent for calculation and demanded probability of the right detection.

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