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Remarks on Energy Based Control of a Nonlinear, Underactuated, MIMO and Unstable Benchmark

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***Abstract*—In the last decade, energy based control theory has undergone a significant breakthrough in dealing with underactated mechanical systems with two successful and similar tools, controlled Lagrangians and controlled Hamiltanians (IDA-PBC). However, because of the complexity of these tools, successful case studies are lacking, in particular, MIMO cases. The seminal theoretical paper of controlled Lagrangians proposed by Bloch and his colleagues presented a benchmark example–a 4 d.o.f underactuated pendulum on a cart but a detailed and completed design is neglected. To compensate this ignorance, the note revisit their design idea by addressing explicit control functions for a similar device motivated by a vector thrust body hovering in the air. To the best of our knowledge, this system is the first MIMO, underactuated example that is stabilized by using energy based tools at the courtesy of the original design idea. Some observations are given based on computer simulation.

***Keywords*—Controlled Lagrangian, Energy Shaping, Spherical Inverted Pendulum, Controlled Hamiltonian.

I. Introduction

THe method of controlled Lagrangians (CL) [1] is a constructive approach to the derivation of stabilizing control laws for Lagrangian mechanical systems where the Lagrangian has the form of kinetic minus potential energy. The theory had its origins in [2], [3], [4] and was systematically introduced in [5], [1]. Various supplementary and additional results have appeared in the literature as well (e.g. [6], [7]). The method of controlled Lagrangians was developed in two salient phases: (i) the controlled Lagrangian method dealt with mechanical systems with symmetry and provided symmetry-preserving kinetic shaping and feedback-controlled dissipation for state-space stabilization in all variables but the symmetry variables [5]; (ii) the potential shaping complemented the kinetic shaping by breaking the symmetry and stabilizing the remaining state variables [1]. The key notion of the method of controlled Lagrangians was total energy shaping, which had advantages over the classical potential shaping methods [8]. Meanwhile, there had been a development of its Hamiltonian counterpart, which is called port-controlled Hamiltonian [9], [10]. The relation between these two methods was studied in [11], [12]. In principle, both methods are passivity based control tools and equivalent to each other in terms of simple mechanical systems.

Applying the method of controlled Lagrangian for the spherical inverted pendulum was presented in [4], [5], [1]. The design in [4], [5] did not solve the stabilization of full phase space because, on one hand, the tool with kinetic shaping alone could not regulate the translational position, on the other hand, the nutation angle was an additional symmetry which was out of control by the method [1]. We observed that the complete method in [1] cannot directly solve the regulation of the nutation angle (an additional symmetry). Due to this reason, a model in pure Cartesian coordinates was presented in [1]. The authors also showed in the same paper that the complete theory with both the kinetic energy and potential energy was applicable to the spherical inverted pendulum model in pure Cartesian coordinates by checking all matching conditions. Then, to complete the design, one was left to make other conditions satisfied in the asymptotic stability theorem. However, the rest is ignored in the paper and no simulation is presented.

The objective of this paper is to draw people's attention on a challenging nonlinear control system, which is MIMO, underactuated and unstable, that can be controlled by energy based control methods (to the best of our knowledge, it is the only successful example of its kind). To this end, we apply the tool in [1] for a more general model which is also in pure Cartesian coordinates. The device is motivated by several real-life applications, for example, an abstraction of a vector thrusted body hovering in a constant altitude. The metric tensor of the new model is invertible at the absence of the cart while the original one does not. Furthermore, some observations are given based on computer simulation. The work completes the original design [1]. Perhaps, more importantly, this tutorial-like note shows how this complex and powerful tool works for a challenging benchmark system. Although port-controlled Hamiltonian [9], [10] is considered to be more general than controlled Lagrangians, applying port-controlled Hamiltonian for such a system is still very hard where one must solve a set of second order partial differential equations. From this aspect, the case study is inspiring and aspiring.

The remaining of the paper is organized as follows: in Section II, we review the controlled Lagrangian method; we present a model of the spherical inverted pendulum in Section III; we complete the control design in Section IV; computer simulation is carried out in Section V; final observation is given in Section VI.

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[1]The nutation angle is with respect to the generalized coordinates: two translational variables and two angles–procession and nutation.

II. Preliminaries

A. The Notations

We consider a mechanical system with configuration space a n-dimensional manifold $Q = S \times G$ and let the configuration coordinates be denoted by $q = (x^\alpha, \theta^a) \in \mathrm{R}^n$, where coordinates $x^\alpha \in S$ with index α going from 1 to $n-r$ are called the shape variables and coordinates $\theta^a \in G$, with index a going from 1 to r are called the group variables and the corresponding S, G are called the shape space and the Abelian group respectively. We assume that the Lagrangian of the system does not depend on the variables θ^a but may depend on its derivative (i.e., the velocity), the group variables are fully actuated and the shape variables are unactuated. TQ denotes tangent bundle to Q and we have $(\dot{q}, q) \in TQ$ for $q \in Q$.

We use the tensor of type (r, s), that is, $T^{j_1,\ldots,j_r}_{i_1,\ldots,i_s}$ with r covectors and s vectors. As is standard practice, g_{ab} represents a matrix with index a going from 1 to m and b going from 1 to n, g^{ab} denotes the inverse of the matrix g_{ab} if $n = m$. We use comma to denote the partial derivative of a tensor as follows $\tau^b_{\delta,\alpha} = \frac{\partial \tau^b_\delta}{\partial \alpha}$. The summation convention over repeated indices is implied to the tensor product, for example, $\eta^a_\alpha \triangleq g_{\alpha c} g^{ca} = \sum_{i=1}^r g_{\alpha i} g^{ia}$ with repeated index c going from 1 to r, and $g_{a\beta}\dot{x}^a = \sum_{i=1}^r g_{i\beta}\dot{x}^i$ with repeated index a going from 1 to r.

In the case of the spherical inverted pendulum, x^α with α going from X to Y represents the angular positions and θ^a with a going from x to y represents the translational positions. We denote the metric tensor $g \triangleq g_{ij}$ with i, j going through x, y, X, Y respectively. If one is comfortable with matrix notations, we can represent $g_{ij} = \begin{pmatrix} g_{ab} & g_{a\alpha} \\ g_{\beta b} & g_{\beta\alpha} \end{pmatrix}$ where a, b going from x to y and α, β going from X to Y. We use English character a, b to represent that both indices go from x and y and a, b are independent of each other and the same argument applies to Greek character α, β. In addition, we denote a state x_e with subscript e the equilibrium.

B. Lagrangian and Controlled Lagrangian

Using the above notations, the Lagrangian $\mathcal{L} : TQ \to \mathrm{R}$ for the mechanical system is defined as

$$\begin{aligned} \mathcal{L}\left(x^\alpha, \dot{x}^\beta, \theta^a, \dot{\theta}^b\right) &= \frac{1}{2} g_{ij}\dot{q}^i\dot{q}^j - V(q) \\ &= \frac{1}{2} g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta + g_{\alpha a}\dot{x}^\alpha\dot{\theta}^a + \frac{1}{2} g_{ab}\dot{\theta}^a\dot{\theta}^b \\ &\quad - V(x^\alpha, \theta^a) \end{aligned} \tag{1}$$

where g_{ij} is the metric tensor, $\frac{1}{2} g_{ij}\dot{q}^i\dot{q}^j$ is the kinetic energy and $V(q)$ is the potential energy. The controlled Euler-Lagrange equations for the given Lagrangian (1) are

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} - \frac{\partial \mathcal{L}}{\partial x^\alpha} &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\theta}^a} - \frac{\partial V}{\partial \theta^a} &= u_a, \end{aligned} \tag{2}$$

where the controls u_a only act in the θ^a directions.

The modification of $\mathcal{L}$ involves changing the metric tensor g_{ij} that defines the kinetic energy $(1/2)g_{ij}\dot{q}^i\dot{q}^j$ and modifying the potential energy that breaks the symmetry in the group variables θ^a by introducing quantities $\tau, \sigma, \rho, \epsilon$. Thus, the controlled Lagrangian takes the form

$$\begin{aligned} \tilde{\mathcal{L}} \triangleq\ & \mathcal{L}\left(x^\alpha, \dot{x}^\beta, \theta^a, \dot{\theta}^a + \tau^a_\alpha \dot{x}^\alpha\right) + \frac{1}{2}\sigma g_{ab}\tau^a_\alpha\tau^b_\beta\dot{x}^\alpha\dot{x}^\beta \\ & + \frac{1}{2}(\rho - 1) g_{ab}\left(\dot{\theta}^a + g^{ac}g_{\alpha c}\dot{x}^\alpha + \tau^a_\alpha\dot{x}^\alpha\right) \\ & \times \left(\dot{\theta}^b + g^{bd}g_{\beta d}\dot{x}^\beta + \tau^b_\beta\dot{x}^\beta\right) - V_\epsilon(x^\alpha, \theta^a), \end{aligned} \tag{3}$$

where $\tilde{\mathcal{L}}$ denotes the controlled Lagrangian subject to some quantities $\tau, \sigma, \rho, \epsilon$, $V_\epsilon(\cdot,\cdot)$ is an arbitrary function to be defined which depends on the parameter ϵ. The controlled Lagrangian implies a new potential energy function $V'(x^\alpha, \theta^a) = V(x^\alpha, \theta^a) + V_\epsilon(x^\alpha, \theta^a)$. Quantities τ, σ will be defined by the matching conditions and the values σ, ρ, ϵ are determined by stability theorems which is reviewed next.

C. The Matching Theorem

The complete controlled Lagrangian method uses a modified kinetic energy (kinetic shaping) and a modified potential energy (potential shaping). The Euler-Lagrange equations corresponding to the controlled Lagrangian $\mathcal{L}$ will be our closed-loop equations. The new terms appearing in those equations corresponding to the directly controlled variables θ^a are interpreted as control inputs. The modifications to the Lagrangian are chosen so that no new terms appear in the equations corresponding to the variables that are not directly controlled. We refer to this procedure as *matching*.

We summarize the simplified matching conditions in [5], [1]:

SM-1: $\sigma_{ab} = \sigma g_{ab}$ for a constant σ (this defines σ_{ab});
SM-2: g_{ab} is independent of x^α;
SM-3: $\tau^b_\alpha = -(1/\sigma)g^{ab}g_{\alpha a}$ (this defines τ^b_α);
SM-4: $g_{\alpha a,\delta} = g_{\delta a,\alpha}$ (a second condition on the metric);
SM-5: $\frac{\partial^2 V}{\partial x^\alpha \partial \theta^a} g^{ad} g_{\beta d} = \frac{\partial^2 V}{\partial x^\beta \partial \theta^a} g^{ad} g_{\alpha d}$ (the condition for the existence of V_ϵ).

Theorem 2.1: (Matching theorem [1]) Under Assumptions (SM-1)-(SM-5), the Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{\mathcal{L}}}{\partial \dot{x}^\alpha} - \frac{\partial \tilde{\mathcal{L}}}{\partial x^\alpha} = 0\,, \quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{\mathcal{L}}}{\partial \dot{\theta}^a} - \frac{\partial \tilde{\mathcal{L}}}{\partial \theta^a} = 0, \tag{4}$$

for the controlled Lagrangian $\tilde{\mathcal{L}}$ coincide with the controlled Euler-Lagrange equations (2).

Actually, applying Theorem 2.1 defines a control law

$$u_a \triangleq u_a^{con} = \frac{d}{dt}(1/\sigma g_{\alpha a}\dot{x}^\alpha) + \frac{\rho - 1}{\rho}\frac{\partial V}{\partial \theta^a} - \frac{1}{\rho}\frac{\partial V_\epsilon}{\partial \theta^a} \tag{5}$$

where the acceleration terms and parameters σ, ρ, ϵ is chosen to satisfy the stability conditions next.

D. Asymptotic Stability Theorem

In the case that Theorem 2.1 is satisfied, the energy function associated with the closed-loop system, can be used as a Lyapunov function, that is, the energy function $\tilde{E}$ for the controlled Lagrangian $\tilde{\mathcal{L}}$. Then, the stability criteria based on the Lyapunov function can be established. To achieve asymptotic stability, a dissipative control u_a^{diss} is added, that is, we redefine the control law (5) as follows

$$u_a \triangleq u_a^{con} + \frac{1}{\rho} u_a^{diss}. \tag{6}$$

In this case, the Euler-Lagrange equation in terms of controlled Lagrangian are

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{\mathcal{L}}}{\partial \dot{\theta}^a} - \frac{\partial \tilde{\mathcal{L}}}{\partial \theta^a} &= \left(-\frac{1}{\sigma} + \frac{\rho-1}{\rho}\right) g^{ad} g_{\alpha d} u_a^{diss} \\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{\mathcal{L}}}{\partial \dot{\theta}^a} - \frac{\partial \tilde{\mathcal{L}}}{\partial \theta^a} &= u_a^{diss} \end{aligned} \tag{7}$$

where the additional term u_a^{diss} in (7) does not affect the matching conditions.

Here, we review a result which is used in our case. To this end, we need two extra assumptions which replace the assumption **SM-5** and introduce a new coordinate chart.

A new coordinate chart for Q is defined as follows:

$$(x^\alpha, \eta^a) \triangleq (x^\alpha, \theta^a + h^a(x^\alpha)), \tag{8}$$

where the function $h : U \to g$ for an open subset U in S is the solution of the first order partial differential equation $\frac{\partial h^a}{\partial x^\alpha} = \left(\frac{\rho-1}{\rho} - \frac{1}{\sigma}\right) g^{ac} g_{\alpha c}$ with $h^a(x_e) = 0$.

Two extra assumptions are:

SM-5': The potential $V(x^\alpha, \theta^a)$ is of the form $V(x^\alpha, \theta^a) = V_1(x^\alpha) + V_2(\theta^a)$ where V_1 has a maximum at $(x^\alpha) = (x_e^\alpha)$ ((SM-5') is a particular case of (SM-5)).

SM-6: The matrix $(g_{a\alpha}(x_e^\alpha))$ is one-to-one (injective).

In the new coordinates (x^α, η^a), $V(x^\alpha, \theta^a) = V_1(x^\alpha) + V_2(\theta^a))$ becomes $V(x^\alpha, \eta^a) = V_1(x^\alpha) + V_2(y^a - h^a(x^\alpha))$. Then, the solution V_ϵ is given by

$$V_\epsilon(x^\alpha, \theta^a) \triangleq V_\epsilon(x^\alpha, \eta^a) = -V_2(y^a - h^a(x^\alpha)) + \tilde{V}_\epsilon(\eta^a), \tag{9}$$

where $\tilde{V}_\epsilon(\eta^a)$ is an arbitrary function and the total modified potential energy function is given by

$$V'_\epsilon(x^\alpha, \eta^a) \triangleq V(x^\alpha, \theta^a) + V_\epsilon(x^\alpha, \eta^a) = V_1(x^\alpha) + \tilde{V}_\epsilon(\eta^a), \tag{10}$$

We express the kinetic energy as follows

$$\tilde{K} = \frac{1}{2} A_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta + \frac{1}{2}\rho g_{ab} \dot{\zeta}^a \dot{\zeta}^b \tag{11}$$

where $\dot{\zeta}^a = \dot{y}^a + (1/\rho) g^{ab} g_{\alpha b} \dot{x}^\alpha$ and $A_{\alpha\beta} = g_{\alpha\beta} - (1 - 1/\sigma) g_{\alpha d} g^{da} g_{a\beta}$. The controlled energy, $\tilde{E}$, is written in new coordinates as

$$\tilde{E} = \tilde{K} + V_1(x^\alpha) + \tilde{V}_\epsilon(\eta^a). \tag{12}$$

In the new coordinates $(x^\alpha, \dot{x}^\alpha, \eta^a, \dot{\eta}^a)$, the controlled Lagrangian takes the form

$$\begin{aligned} \tilde{\mathcal{L}} \triangleq \frac{1}{2}&\left(g_{\alpha\beta} - \left(\frac{\rho-1}{\rho} - \frac{1}{\sigma}\right) g^{ab} g_{\alpha a} g_{\beta b}\right) \dot{x}^\alpha \dot{x}^\beta \\ &+ g_{\alpha a} \dot{x}^\alpha \dot{\eta}^a + \frac{1}{2}\rho g_{ab} \dot{\eta}^a \dot{\eta}^b - V_1(x^\alpha) - \tilde{V}_\epsilon(\eta^a) \end{aligned} \tag{13}$$

and the Euler-Lagrange equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{\mathcal{L}}}{\partial \dot{x}^\alpha} - \frac{\partial \tilde{\mathcal{L}}}{\partial x^\alpha} = 0\,, \quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{\mathcal{L}}}{\partial \dot{\eta}^a} - \frac{\partial \tilde{\mathcal{L}}}{\partial \eta^a} = u_a^{diss}. \tag{14}$$

LaSalle's invariance principle gives the asymptotic stability of the equilibrium as follows.

Theorem 2.2: (Asymptotic Stabilization-Specific Case [1]): Assume that conditions (SM-1)-(SM-4), (SM-5') and (SM-6) hold. Let (x_e^α) be the maximum point of V_1 of interest. Then, there is an explicit feedback control such that $(x_e^\alpha, \theta_e^a, 0, 0)$ becomes an asymptotically stable equilibrium such that

$$\frac{d}{dt}\tilde{E} = c_a^b g_{bd} \dot{\eta}^a \dot{\eta}^b \geq 0 \tag{15}$$

and the total control law (6) is written as follows

$$\begin{aligned} u_a = -\kappa &\left(g_{\beta a,\gamma} - g_{\delta a} A^{\delta\alpha}\left(g_{\alpha\beta,\gamma} - \frac{1}{2} g_{\beta\gamma,\alpha}\right.\right. \\ &\left.\left. -(1+\kappa) g_{\alpha d} g^{da} g_{\beta a,\gamma}\right)\right) \dot{x}^\beta \dot{x}^\gamma + \kappa g_{\delta a} A^{\delta\alpha} \frac{\partial V}{\partial x^\alpha} \\ &+ \kappa g_{\delta a} A^{\delta\alpha} \frac{1}{\rho} g_{\alpha d} g^{db}\left(-\frac{\partial V'}{\partial \theta^b} + u_b^{diss}\right) \\ &+ \frac{\rho-1}{\rho}\frac{\partial V}{\partial \theta^a} - \frac{1}{\rho}\frac{\partial V_\epsilon}{\partial \theta^a} + \frac{1}{\rho} u_a^{diss} \end{aligned} \tag{16}$$

where $\kappa \triangleq -1/\sigma$, $A_{\alpha\beta} \triangleq g_{\alpha\beta} - (1+\kappa) g_{\alpha d} g^{da} g_{\beta a}$, $u_a^{diss} = c_a^d g_{bd}\left(\frac{\rho-1}{\rho} - \kappa\right) g^{ac} g_{\alpha c} \dot{x}^\alpha$ with c_a^d a positive definite matrix and parameters to satisfy the following three conditions:

1) $V_\epsilon(\eta^a)$ should be chosen to have a maximum at $\eta_e^a = \theta_e^a$;
2) $\rho < 0$;
3) $\kappa > \max\left\{\lambda \middle| \det(g_{\alpha\beta} - \lambda g_{\alpha a} g^{ab} g_{b\beta})|_{x^\alpha = x_e^\alpha} = 0\right\} - 1$.

III. Modelling in Pure Cartesian Coordinates

Successfully applying the controlled Lagrangian method to the spherical inverted pendulum depends on which model is used. We observe that the method of controlled Lagrangians can not derive the full control law to the model in the coordinates: nutation and procession angles (ϕ, θ) in [4], [5] but can derive a control law for the model in pure Cartesian coordinates considered in [1]. However, the model in [1] has a problem that the inverse of metric tensor g_{ij} does not exist if the mass of cart is not incorporated. This motives us to derive a new model in pure Cartesian coordinates.

We consider the spherical inverted pendulum (see Fig. 1) be a rigid body. Here, we assume that our pendulum is a pole with the uniform mass density other than the bob with mass on the top of the massless pole. Let the 4-dimentional configuration space $Q = S \times G$. We denote the Cartesian coordinates $(x, y) \in G$ be the local coordinates for translational coordinates of the pendulum and assume that there are two independent controls (F_x, F_y) that can move the pendulum

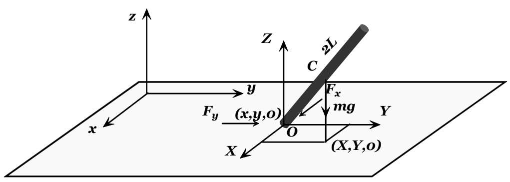

Fig. 1. The configuration of the spherical inverted pendulum

in x and y directions. We take Cartesian coordinates (X, Y) as the coordinates in S where (X, Y) is the projections of the center of mass on the horizontal plane under the local chart with the origin attached to the bottom of the pendulum as was suggested in [1]. Thus, $q = (x, y, X, Y) \in Q$ is the vector of the generalized coordinates. As shown in Fig. 2, an infinitesimal section with the length $\mathrm{d}l$ along the pole is regarded as an particle with volume $1 \cdot \mathrm{d}l$, the mass of the infinitesimal section (resp. the particle) is $\frac{m}{2L}\mathrm{d}l$ where l is the length of the particle to the pivot and the velocity vector of the particle is $V_l = \left(\dot{x} + \frac{l\dot{X}}{L}, \dot{y} + \frac{l\dot{Y}}{L}, \frac{l}{L}\frac{X\dot{X}+Y\dot{Y}}{\sqrt{L^2-X^2-Y^2}}\right)$.

Then, the kinetic energy of the pendulum can be expressed as the sum of the kinetic energy of all particles along the pole and we redefine the kinetic energy in [1] as follows

$$
\begin{aligned}
T &= \frac{1}{2}\int_0^{2L} \frac{m}{2L} < V_l, V_l > \mathrm{d}l \\
&= \frac{1}{2}\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{X} \\ \dot{Y} \end{pmatrix}^T \begin{pmatrix} m & 0 & m & 0 \\ 0 & m & 0 & m \\ m & 0 & \frac{4m}{3}\frac{L^2-Y^2}{L^2-X^2-Y^2} & \frac{4m}{3}\frac{XY}{L^2-X^2-Y^2} \\ 0 & m & \frac{4m}{3}\frac{XY}{L^2-X^2-Y^2} & \frac{4m}{3}\frac{L^2-X^2}{L^2-X^2-Y^2} \end{pmatrix} \\
&\quad \times \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{X} \\ \dot{Y} \end{pmatrix} \\
&= \frac{1}{2} g_{ij}\dot{q}^i\dot{q}^j \,,
\end{aligned}
$$

where g_{ij} is the metric tensor.

The total potential energy is given by [2]

$$V \stackrel{\triangle}{=} mg(\sqrt{L^2 - X^2 - Y^2} - L). \tag{17}$$

We define the Lagrangian of the pendulum $\mathcal{L} : TQ \mapsto Q$

$$\mathcal{L} = K(\dot{x}, \dot{y}, X, Y, \dot{X}, \dot{Y}) - V(X, Y), \tag{18}$$

which is independent of (x, y), the cyclic variables.

Then, applying Euler-Lagrange equations (2) to (18) gives the equations of dynamics, x^α with index α going from X to Y, θ^a with index a going from x to y and u_a with index α going from x to y, that is, $(u_x, u_y) \stackrel{\triangle}{=} (F_x, F_y)$.

[2]Noting that we do not use the same potential energy as [1] and the potential energy in our formulation is non-positive in order to make the closed loop energy function a Lyapunov candidate.

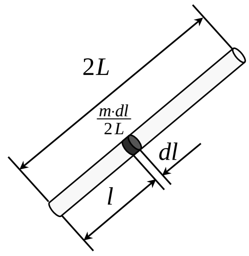

Fig. 2. A partical in the spherical inverted pendulum

IV. Control Design and Domain of Attraction

A. Outline

In this section, we apply Theorem 2.2 to the spherical inverted pendulum to obtain the control law. In addition, we estimate the domain of attraction for the closed loop system. To this end, we proceed as follows:

- Step 1: We check that all matching conditions in Theorem 2.2 are satisfied by defining the controlled Lagrangian $\tilde{\mathcal{L}}$ in (3) with respect to the Lagrangian (18) of the spherical inverted pendulum [3];
- Step 2: We modify the potential energy function such that the new potential energy function has a maximum at the upper equilibrium of the pendulum and after completing step 1-2, we are ready to compute the control law;
- Step 3: We compute the control law according to the general control formula (16) and choose the parameters in the obtained control law to satisfy the remaining conditions in Theorem 2.2 such that the asymptotic stability of the closed loop system is achieved;
- Step 4: We provide a technical lemma to give an estimate of the domain of attraction. In fact, for any given compact subset of the upper space of the pendulum, the asymptotic stability can be achieved by adjusting a parameter in the controller.

B. Step 1: Defining Controlled Lagrangian and checking matching conditions

For the configuration coordinates $q = (\theta^a, x^\alpha)$ with index a going from x to y and index α going from X to Y, we let $\theta^a = (x, y)$, $x^\alpha = (X, Y)$. As we can seen from the kinetic energy $T = \frac{1}{2}g_{ij}\dot{q}^i\dot{q}^j$ in (17), we read the sub-matrices of the metric tensor g_{ij} as follows $g_{ab} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$, $g_{\alpha a} = g_{a\beta} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$, $g_{\alpha\beta} = \begin{pmatrix} \frac{4m}{3}\frac{L^2-Y^2}{L^2-X^2-Y^2} & \frac{4m}{3}\frac{XY}{L^2-X^2-Y^2} \\ \frac{4m}{3}\frac{XY}{L^2-X^2-Y^2} & \frac{4m}{3}\frac{L^2-X^2}{L^2-X^2-Y^2} \end{pmatrix}$. So, we can define the controlled Lagrangian as the formula in (3).

[3]The matching conditions were checked in [1] for the simplified model in pure Cartesian coordinates.

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We check that all matching conditions (SM-1)-(SM-4), (SM-5') and (SM-6) in Theorem 2.2 are satisfied:

(SM-1) is satisfied if we define $\sigma_{ab} \overset{\triangle}{=} \sigma g_{ab} = \sigma m \delta_{ab}$ where σ is constant and δ_{ab} is the Kronecker $\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$.

(SM-2) is satisfied because g_{ab} is constant matrix, which is independent of (X, Y).

(SM-3) is satisfied if we define $\tau_\alpha^b \overset{\triangle}{=} -(1/\sigma) g^{ab} g_{\alpha a}$ such that $\tau_X^x = \tau_Y^y = -(1/\sigma)$, $\tau_Y^x = \tau_X^y = 0$.

(SM-4) is satisfied because $g_{Xx,Y} = \frac{\partial m}{\partial Y} = 0$, $g_{Yx,X} = \frac{\partial 0}{\partial X} = 0$, $g_{Xy,Y} = \frac{\partial 0}{\partial Y} = 0$ and $g_{Yy,X} = \frac{\partial m}{\partial X} = 0$.

(SM-5') is satisfied because $V = V_1(x^\alpha) + 0 = mg(\sqrt{L^2 - X^2 - Y^2} - L)$ and V_1 has a maximum at the equilibrium $(X, Y) = (0, 0)$.

(SM-6) is satisfied since the mapping $g_{\alpha a}(x_e^\alpha) = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}_{(X,Y)=(0,0)}$ evaluated at the equilibrium is injective.

C. Step 2: Defining the Modified Potential Energy Function

Since all matching conditions (SM-1)-(SM-4), (SM-5') and (SM-6) in Theorem 2.2 are satisfied, according to the procedure, we must modify the potential energy function, which is referred to as the symmetry breaking.

Firstly, to modify the potential energy, we introduce the new coordinate chart. The solutions of the partial differential equations $\frac{\partial h^x}{\partial X} = \frac{\rho - 1}{\rho} - \frac{1}{\sigma}$, $\frac{\partial h^x}{\partial Y} = 0$, $\frac{\partial h^y}{\partial Y} = \frac{\rho - 1}{\rho} - \frac{1}{\sigma}$ and $\frac{\partial h^y}{\partial X} = 0$ with boundary conditions $h^x|_{(X,Y)=(0,0)} = 0$ and $h^y|_{(X,Y)=(0,0)} = 0$ in this case are trivial, which are $h^x = \left(\frac{\rho-1}{\rho} - \frac{1}{\sigma}\right) X$, $h^y = \left(\frac{\rho-1}{\rho} - \frac{1}{\sigma}\right) Y$. We define the new coordinate chart $(\eta^a, x^\alpha) = (\eta^x, \eta^y, X, Y)$ according to (8), where $\eta^x = x + \left(\frac{\rho-1}{\rho} - \frac{1}{\sigma}\right) X$, $\eta^y = y + \left(\frac{\rho-1}{\rho} - \frac{1}{\sigma}\right) Y$.

Next, we define the potential V'_ϵ for the controlled Lagrangian. To this end, we define a negative definite function as $\tilde{V}_\epsilon(\eta^a) \overset{\triangle}{=} -\epsilon mg \left((\eta^x)^2 + (\eta^y)^2\right)$ which has a maximum at the equilibrium $(\eta^x, \eta^y) = (0, 0)$ when $\epsilon > 0$. As shown in (10), the potential V'_ϵ for the controlled Lagrangian in the new coordinates is given by

$$V'_\epsilon \overset{\triangle}{=} mg(\sqrt{L^2 - X^2 - Y^2} - L) - \epsilon mg \left((\eta^x)^2 + (\eta^y)^2\right). \tag{19}$$

D. Step 3: Computing the control law

According to the general formula (16), we obtain the control law as

$$u_x \overset{\triangle}{=} \tilde{F}_x(q, \dot{q}, \kappa, \rho, \epsilon),\; u_y \overset{\triangle}{=} \tilde{F}_y(q, \dot{q}, \kappa, \rho, \epsilon) \tag{20}$$

where κ, ρ, ϵ are design parameters to be defined which is given in appendix. To make the closed loop system asymptotically stable, we determine κ, ρ and ϵ by applying Theorem 2.2. We choose $\epsilon > 0$ such that the appended potential energy function $\tilde{V}_\epsilon$ is negative definite. We also check that the following conditions are satisfied.

(1) $\tilde{V}_\epsilon(y^a)$ has a maximum at the equilibrium $(\theta_e^a) = (x_e, y_e) = (0, 0)$ because the equilibrium $(x_e^\alpha) \in Q/G$, i.e., $(X, Y) = (0, 0)$, are the maximum point of V_1 according to (SM-5') and the constructed function $\tilde{V}_\epsilon(y^a)$ have a maximum at the equilibrium $(y_e^a) = (0, 0)$.

(2) We assign ρ to be a negative real number.

(3) We assign κ a positive real number such that $\kappa > \max\left\{\lambda \in \mathrm{R} \mid \det(g_{\alpha\beta} - \lambda g_{\alpha a} g^{ab} g_{b\beta})|_{x^\alpha = x_e^\alpha} = 0\right\} - 1$, that is, $\kappa > \max\left\{\lambda \in \mathrm{R} \mid \det\begin{pmatrix} (4/3 - \lambda)m & 0 \\ 0 & (4/3 - \lambda)m \end{pmatrix} = 0\right\} - 1$ and obtain $\kappa > 1/3$. Consequently, we have $\kappa > 1/3$, $\rho < 0$ and $\epsilon > 0$.

E. Step 4: Estimating the Domain of Attraction

The function $h : U \to R$ is valid for $U = \{(X, Y) \in R^2 | \sqrt{X^2 + Y^2} < L\}$ which corresponds to the upper space as the case that the pendulum is above the horizontal plane. We use $R^2 \times U \subset Q$ as a domain of a local chart on Q and the new local chart on TQ is given as: $(x, y, X, Y, \dot{x}, \dot{y}, \dot{X}, \dot{Y}) \in (\mathrm{R}^2 \times U) \times \mathrm{R}^4$. Likewise, for the new local chart (x^α, η^a) on $R^2 \times U \subset Q$, its corresponding local chart on TQ is given as: $(x, y, X, Y, \dot{x}, \dot{y}, \dot{X}, \dot{Y}) \mapsto (\eta^x, \eta^y, X, Y, \dot{\eta}^x, \dot{\eta}^y, \dot{X}, \dot{Y}) \in (\mathrm{R}^2 \times U) \times \mathrm{R}^4$.

By (12), the energy function in the new chart is written as

$$\tilde{E} = \frac{1}{2}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix}^T A_{\alpha\beta} \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} + \frac{1}{2} m\rho \left((\dot{\eta}^x)^2 + (\dot{\eta}^y)^2\right) - mg(L - \sqrt{L^2 - X^2 - Y^2}) - \epsilon mg \left((\eta^x)^2 + (\eta^y)^2\right). \tag{21}$$

where $A_{\alpha\beta} = \begin{pmatrix} \alpha_1 - m(1+\kappa) & \alpha_2 \\ \alpha_2 & \alpha_3 - m(1+\kappa) \end{pmatrix}$ with $\alpha_1 = \frac{4m}{3} \frac{L^2 - Y^2}{L^2 - X^2 - Y^2}$, $\alpha_2 = \frac{4m}{3} \frac{XY}{L^2 - X^2 - Y^2}$ and $\alpha_3 = \frac{4m}{3} \frac{L^2 - X^2}{L^2 - X^2 - Y^2}$

Then, we can conclude the next result,

Corollary 4.1: Suppose that all conditions in Theorem 2.2 are satisfied. For any compact subset $\tilde{U} \in U$, there exists a $\kappa^* > 1/3$ such that for $\kappa > \kappa^*$, the controlled energy (21) is negative definite and the corresponding Lyapunov function $\mathcal{V} = -\tilde{E}$ is positive definite for all states

$$(\eta^x, \eta^y, X, Y, \dot{\eta}^x, \dot{\eta}^y, \dot{X}, \dot{Y}) \in (R^2 \times \tilde{U}) \times \mathrm{R}^4.$$

Furthermore, let the set $\Omega_c = \{(x^\alpha, \eta^a, \dot{x}^\alpha, \dot{\eta}^a) \in TQ | \mathcal{V} \leq c\} \subset (\mathrm{R}^2 \times \tilde{U}) \times \mathrm{R}^4$ be a positively invariant set for some $c \in R$. Let the set $\mathcal{E} = \{(x^\alpha, \eta^a, \dot{x}^\alpha, \dot{\eta}^a) \in \Omega_c | \dot{\eta}^a = 0$ or $\frac{d}{dt}\tilde{E} = 0\}$ and $\mathcal{M}$ is the largest invariant subset of $\mathcal{E}$. Then, Ω_c is an estimate of domain of attraction and $\mathcal{M} = (x_e^a, \dot{x}_e^a, \eta_e^\alpha, \dot{\eta}_e^\alpha) = (0, 0, 0, 0)$.

Proof: The proof is carried out in two steps: at step one, we find κ^* such that for $\kappa > \kappa^*$, the energy function is negative definite; at step two, we show that the set Ω_c is an estimate of domain of attraction and our argument is based on the result for the general case in [1] where the LaSalle's invariance principle is used to establish the stability.

At the first stage, we try to make the energy (21) negative definite and zero at zero.

We check that:

(1) $-mg(L - \sqrt{L^2 - X^2 - Y^2}) \leq 0$ for $(X, Y) \in U$ where the equality holds if and only if $X = Y = 0$;

(2) $-\epsilon mg \left((\eta^x)^2 + (\eta^y)^2\right) \leq 0$ for $(\eta^x, \eta^y) \in \mathrm{R}^2$ where the equality holds if and only if $\eta^x = \eta^y = 0$;

(3) $\frac{1}{2} m\rho \left((\dot{\eta}^x)^2 + (\dot{\eta}^y)^2\right) \leq 0$ where the equality holds if and only if $\dot{\eta}^x = \dot{\eta}^y = 0$.

Then, to make the energy function (21) negative definite, we must render the first quadratic term in (21) negative definite with respect to $(\dot{X},\dot{Y})$ by adjusting the parameter κ. This implies the symmetric matrix $A_{\alpha\beta}$ is negative definite, that is, $\alpha_1-m(1+\kappa)<0$ and $(\alpha_1-m(1+\kappa))(\alpha_3-m(1+\kappa))-\alpha_2^2<0$ are satisfied. To this end, we let κ satisfy

$$\begin{aligned}&\kappa>\frac{1}{3},\quad \kappa>\frac{\|\alpha_1\|_\infty}{m}-1\\&\kappa>2m\left(\|\alpha_1\|_\infty+\|\alpha_3\|_\infty+\left(\|\alpha_1\|_\infty^2+4\|\alpha_2\|_\infty^2+\right.\right.\\&\quad\left.\left.\|\alpha_1\|_\infty\|\alpha_3\|_\infty+\|\alpha_3\|_\infty^2\right)^{1/2}\right)-1\end{aligned}\tag{22}$$

simultaneously where we introduce infinity norm $\|\cdot\|_\infty$ such that $\alpha_i\le\|\alpha_i\|_\infty$, $i=1,2,3$ for all $(X,Y)\in\tilde{U}\subset U$. We define $\kappa^*=\inf_{\kappa\in R^+}\{\kappa \text{ satisfies } (22)\}$.

For $\kappa>\kappa^*$, the controlled energy (21) is negative definite with a maximum at $(0,0,0,0,0,0,0,0)\in(\tilde{U}\times\mathrm{R}^2)\times\mathrm{R}^4$. We conclude that the corresponding Lyapunov function $\mathcal{V}$ is positive definite at a domain $(R^2\times\tilde{U})\times\mathrm{R}^4$. Seeing from (22), we conclude that as $\kappa^*\to\infty$ implies $\|\alpha_i\|_\infty\to\infty$ and $\sqrt{X^2+Y^2}\to L$, the set $\tilde{U}$ expands to U.

At the second stage, we show that Ω_c is an estimate of domain of attraction by applying LaSalle's invariance principle. Here, we relax the conditions in [1] for general cases. Specifically, we do not shrink Ω_c as the domain of attraction.

In the last step, $\mathcal{V}$ is positive definite in a domain $R^2\times\tilde{U}\times R^4$. By Theorem 2.2, the time derivative of the Lyapunov function satisfy $\frac{d\mathcal{V}}{dt}\le 0$. So, Ω_c is a positively invariant set such that $(x^\alpha(0),\eta^a(0),\dot{x}^\alpha(0),\dot{\eta}^a)(0)\in\Omega_c$ implies $(x^\alpha(t),\eta^a(t),\dot{x}^\alpha(t),\dot{\eta}^a(t))\in\Omega_c$ for $t\ge 0$.

The set $\mathcal{E}$ is a subset of Ω_c where $\frac{d\mathcal{V}}{dt}=0$. As $\mathcal{M}$ is the largest invariant subset of $\mathcal{E}$, we suppose $z(t)=(x^\alpha(t),\eta^a(t),\dot{x}^\alpha(t),\dot{\eta}^a(t))\in\mathcal{M}$ for all $t\ge 0$ and then, in $\mathcal{M}$, we have $\eta^a(t)=\eta^a(0)=\eta^a_e=0$, $\dot{\eta}^a(t)=0$, $g^{ac}g_{\alpha c}\dot{x}^\alpha=0$ (i.e., $\dot{x}^\alpha=0$) for all $t\ge 0$, where we use some results: equations (40) and (43) in [1, page 1563]. So, we have $z(t)=(x^\alpha(t),0,0,0)\in\mathcal{M}$ for all $t\ge 0$. Substituting these conditions into Euler-Lagrange equations (14) for x^α variables, we know that $z(t)=(x^\alpha(t),\eta^a_e(t),0,0)\in\mathcal{M}$ complies with the following equation (the general form is given by equation (45) in [1, page 1563]):

$$\begin{pmatrix}\ddot{X}\\\ddot{Y}\end{pmatrix}=\begin{pmatrix}\frac{3gX\sqrt{L^2-X^2-Y^2}}{4L^2}\\\frac{3gY\sqrt{L^2-X^2-Y^2}}{4L^2}\end{pmatrix}.\tag{23}$$

In [1], the authors shrink Ω_c to study the linearized dynamics of the general form which includes (23) to conclude a general stability result. Here, we directly investigate the nonlinear dynamics (23). Since $\frac{3g\sqrt{L^2-X^2-Y^2}}{4L^2}>0$ in U, there is only one equilibrium $(x^\alpha,\dot{x}^\alpha)=(X_e,Y_e,\dot{X}_e,\dot{Y}_e)=(0,0,0,0)$ of the dynamics (23) such that any trajectory $(X(t),Y(t),\dot{X}(t),\dot{Y})(t)$ starting in $U\times R^2$ will escape from $U\times R^2$ except when the trajectory is the equilibrium. Thus, we have the invariant equilibrium $z(t)=(0,0,0,0)\in\mathcal{M}$. The above argument implies that the largest invariant set in $\mathcal{E}$ is the origin: $\mathcal{M}\stackrel{\triangle}{=}(x^\alpha_e,\dot{x}^\beta_e,\eta^a_e,\dot{\eta}^b_e)=(0,0,0,0)$.

Then, we conclude that any states starting in Ω_c approach an invariant set $\mathcal{M}$ which contains only the origin as $t\to\infty$. ■

Remark 1: It is easy to check that the result in Corollary 4.1 in the new chart (x^α,η^a) implies a similar result in original chart (x^α,θ^a) because the mapping $T:(x^\alpha,\theta^a)\to(x^\alpha,\eta^a)$ is invertible in the whole upper space. For $\kappa>\kappa^*$, the only restriction on initial conditions is the restriction on shape variables (X,Y), i.e., $(X,Y)\in\tilde{U}\subset U$. Therefore, for all initial conditions of other states in R^6 and $(X,Y)\in\tilde{U}$, we can find a domain Ω_c with some $c\in R$ which contains those initial conditions such that all trajectories do not leave Ω_c for all $t\ge 0$. As seen from the first step of the proof, as $\tilde{U}$ expands to U, we need $\kappa^*\to\infty$ to make the energy negative definite.

V. Computer Simulation

To ease the visualization of the projections in the moving frame, we give the total projection length in the horizontal plane, that is, $2r=2\sqrt{X^2+Y^2}$. Let the pendulum length $2L=0.6\,(m)$ and $m=0.35\,(kg)$ and $g=9.8\,(N/s^2)$.

By trials and errors, we start with all values of parameters 1, then change those values with increasing some values or decreasing some values and finally select the design parameters as $\kappa=100$, $\rho=-0.02$, $\epsilon=1\times10^{-4}$, and $c^x_x=c^y_y=0.01$, $c^y_x=c^x_y=0$.

Remark 2: Admittedly, one has the freedom to tune the parameters in the control function (20) such as κ, ρ, ϵ, c^x_x, c^y_y, c^y_x and c^x_y. The design process is, however, not systematic and the design parameters are difficult to optimize. Many of our choices lead to very bad performance. For example, with an increase in $\epsilon=1\times10^{-3}$ and other design parameters as before, the trajectory oscillates heavily before converging to the origin (see Figure 3).

Case 1: Let the exogenous disturbance and unmodelled dynamics be zero. Figure 4 shows the simulation result with the initial values

$$(x,\dot{x},y,\dot{y},X,\dot{X},Y,\dot{Y})=(20,2,-20,2,0.1,0.1,-0.1,0.1).$$

which indicates a large domain of attraction.

Analytically, there exist a compact set $\Omega_c\subset U$, the domain of attraction for the given parameters. However, it is unclear how the domain of attraction increases with those design parameters. Here, we approximately estimate some projections of the domain of attraction associated with the nominal controlled system based on the (quantitative) simulation study. To reduce the complexity of analysis, let $(\dot{x}(0),\dot{y}(0),\dot{X}(0),\dot{Y}(0))=(0,0,0,0)$ be initial conditions for the rates. Figure 5 shows the projections in two scenarios: first, let $(x(0),y(0))=0$ and all initial angles inside the outer layer converge to the origin and diverge outside the outer layer; second, let $\sqrt{x(0)^2+y(0)^2}=570(m)$ (this implies many cases for $(x(0),y(0))$) and only initial angles inside the inner layer, a very small neighborhood about the origin, converge to the origin. The result indicates that the method of controlled Lagrangain yields some bounded domain of attraction (maybe large).

Case 2: Introduce an exogenous input to the system (2) such that its right hand side becomes $(-C_X\dot{X},\,-C_Y\dot{Y},\,u_x-C_x\dot{x},u_y-C_y\dot{y})$, where $C_x=C_y=10^{-4}\,(N\cdot s/m)$ and $C_X=C_Y=5\times10^{-4}\,(N\cdot s/m)$. Figure 6 shows the simulation result with the initial values

$$(x,\dot{x},y,\dot{y},X,\dot{X},Y,\dot{Y})=(2,0,2,0,0,0,0,0).$$

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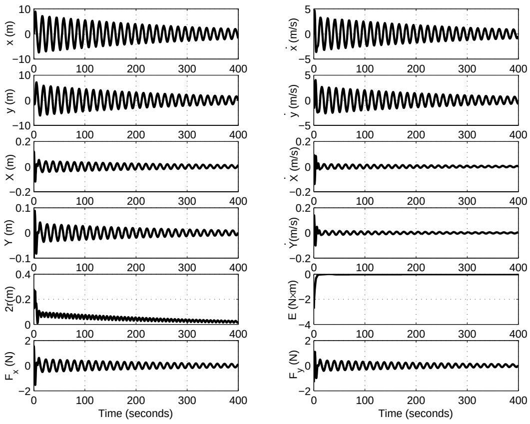

Fig. 3. The oscillatory trajectory results from inappropriate design parameters

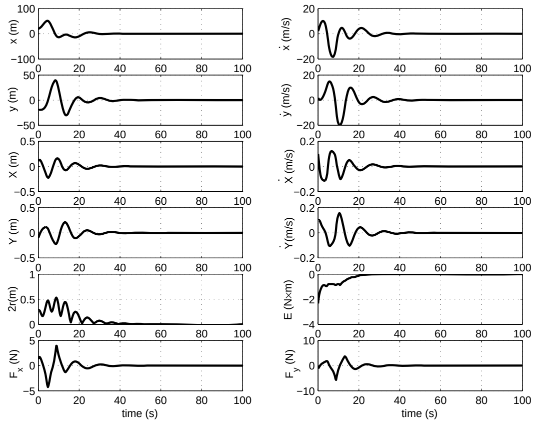

Fig. 4. Simulation results in *Case 1*

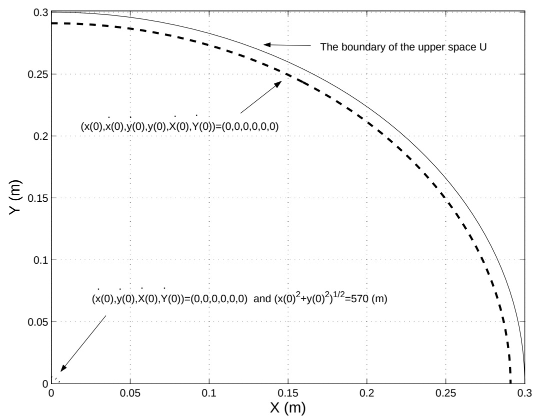

Fig. 5. The estimate of some projections of D.O.A based on simulations

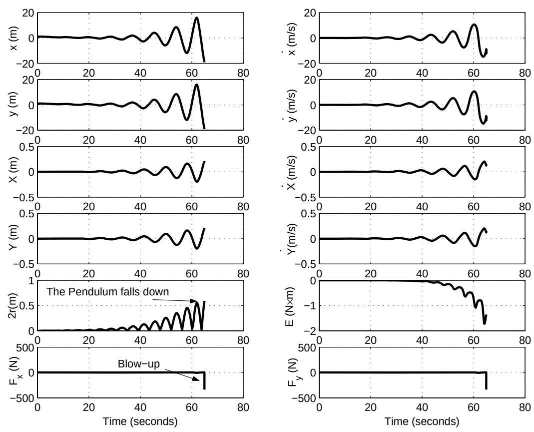

Fig. 6. Instability caused by friction only in Case 2

Eventually, the pendulum falls over. The controlled Lagrangian design yields relatively poor robustness for this set of design parameters.

However, our claims in the domain of attraction and the robustness are based on a simulation study and should be interpreted tentatively since we have not explored all degrees of freedom in the simulations. A better alternative would be to analytically analyze robustness but the the best of our knowledge this problem remains open in the literature.

Remark 3: Controlled Lagrangians and controlled Hamiltonians solve the matching conditions for an open loop system without physical damping. It has been shown that physical damping can affect stability in the closed loop because whenever the kinetic energy is modified, physical damping terms do not always enter as dissipation with respect to the closed energy function [13]. Some ongoing research is dedicated to make controlled Lagrangians and controlled Hamiltonians more robust to physical damping [13], [14].

Remark 4: The approach is also summarized in [15] together with several other design approaches in a flavor of comparing the performance based on computer simulation.

VI. Conclusion

We derive an explicit controller for the spherical inverted pendulum which was initially proposed in [1] via the method of controlled Lagrangians in the same paper. Simulation results show that the closed loop system yields a large domain of attraction. However, the performance is very sensitive to the design parameters. For the design parameters that we have used, the closed loop system may yield poor robustness. Therefore, it is desirable to systematically address the tuning rules and make the closed loop system more robust in future work.

Appendix A
Control Function

The explicit formula of the full control law (20) is given in next page where we define the constant $\kappa_\rho \triangleq \frac{\rho-1}{\rho} + \kappa$ to shorten the expression.

$$u_x = \frac{m\kappa}{L^2(1-3\kappa)+3(X^2+Y^2)(1+\kappa)} \left(-3gX\sqrt{L^2-X^2-Y^2} - \frac{4(Y^2-L^2)X\dot{X}^2+4(X^2-L^2)X\dot{Y}^2-8X^2Y\dot{X}\dot{Y}}{L^2-X^2-Y^2} \right.$$
$$+3\left(\frac{L^2-X^2}{\rho}+\frac{3Y^2(1+\kappa)}{\rho(1-3\kappa)}\right)\left(2\epsilon g\left(x+\kappa_\rho X\right)+c_x^x\left(\dot{x}+\kappa_\rho\dot{X}\right)+c_x^y\left(\dot{y}+\kappa_\rho\dot{Y}\right)\right)+\frac{12XY}{\rho(1-3\kappa)}\times$$
$$\left.\left(2\epsilon g\left(y+\kappa_\rho Y\right)+c_y^x\left(\dot{x}+\kappa_\rho\dot{X}\right)+c_y^y\left(\dot{y}+\kappa_\rho\dot{Y}\right)\right)\right)+\frac{m}{\rho}\left(2\epsilon g\left(x+\kappa_\rho X\right)+c_x^x\left(\dot{x}+\kappa_\rho\dot{X}\right)+c_x^y\left(\dot{y}+\kappa_\rho\dot{Y}\right)\right).$$

$$u_y = \frac{m\kappa}{L^2(1-3\kappa)+3(Y^2+X^2)(1+\kappa)} \left(-3gY\sqrt{L^2-Y^2-X^2} - \frac{4(X^2-L^2)Y\dot{Y}^2+4(Y^2-L^2)Y\dot{X}^2-8Y^2X\dot{Y}\dot{X}}{L^2-Y^2-X^2} \right.$$
$$+3\left(\frac{L^2-Y^2}{\rho}+\frac{3X^2(1+\kappa)}{\rho(1-3\kappa)}\right)\left(2\epsilon g\left(y+\kappa_\rho Y\right)+c_y^y\left(\dot{y}+\kappa_\rho\dot{Y}\right)+c_y^x\left(\dot{x}+\kappa_\rho\dot{X}\right)\right)+\frac{12YX}{\rho(1-3\kappa)}\times$$
$$\left.\left(2\epsilon g\left(x+\kappa_\rho X\right)+c_x^y\left(\dot{y}+\kappa_\rho\dot{Y}\right)+c_x^x\left(\dot{x}+\kappa_\rho\dot{X}\right)\right)\right)+\frac{m}{\rho}\left(2\epsilon g\left(y+\kappa_\rho Y\right)+c_y^y\left(\dot{y}+\kappa_\rho\dot{Y}\right)+c_y^x\left(\dot{x}+\kappa_\rho\dot{X}\right)\right).$$

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