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Existence of multiple positive periodic solutions to n species nonautonomous Lotka-Volterra cooperative systems with harvesting terms

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Abstract—In this paper, the existence of 2^n positive periodic solutions for n species non-autonomous Lotka-Volterra cooperative systems with harvesting terms is established by using Mawhin's continuation theorem of coincidence degree theory and matrix inequality. An example is given to illustrate the effectiveness of our results.

Keywords—Multiple positive periodic solutions; Nonautonomous Lotka-Volterra cooperative system; Coincidence degree; Harvesting term.

I. Introduction

THE n species Lotaka-Volterra cooperative model with harvesting terms is described as follows ([1,2]):

$$\dot{x}_i(t) = x_i(t)\bigg(a_i - b_i x_i(t) + \sum_{j=1, j\neq i}^{n} c_{ij} x_j(t)\bigg) - h_i,$$

$i = 1, 2, \ldots, n$, where $x_i(t)(i = 1, 2, \ldots, n)$ is the densities functions of the ith species; a_i and b_i are all positive constant and denote the intrinsic growth rate, death rate, respectively; $c_{ij} > 0$ stand for the cooperative rate between the ith species and the jth species; $h_i(i = 1, 2, \ldots, n)$ is the ith species harvesting terms standing for the harvests. Since realistic models require taking into account the effect of changing environment we will consider the following nonautonomous model

$$\begin{aligned}\dot{x}_i(t) &= x_i(t)\bigg(a_i(t) - b_i(t)x_i(t) + \sum_{j=1, j\neq i}^{n} c_{ij}(t) x_j(t)\bigg) \\ &\quad - h_i(t),\ i = 1, 2, \ldots, n. \qquad (1)\end{aligned}$$

In addition, the effects of a periodically varying environment are important for evolutionary theory as the selective forces on systems in a fluctuating environment differ from those in a stable environment. Therefore, the assumptions of periodicity of the parameters are a way of incorporating the periodicity of the environment (e.g, seasonal effects of weather, food supplies, mating habits, etc), which leads us to assume that $a_i(t), b_i(t), c_{ij}(t)$ and $h_i(t)(i, j = 1, 2, \ldots, n)$ are all positive continuous ω-periodic functions.

A very basic and important problem in the study of a population growth model with a periodic environment is the global existence and stability of a positive periodic solution, which plays a similar role as a globally stable equilibrium does in an autonomous model. Also, only a few results concerning the existence of positive periodic solutions to system (1) can be found in the literature. This motivates us to investigate the existence of a positive periodic or multiple positive periodic solutions for system (1). In fact, it is more likely for some biological species to take on multiple periodic change regulations and have multiple local stable periodic phenomena. Therefore it is essential for us to investigate the existence of multiple positive periodic solutions for population models. Our main purpose of this paper is by using Mawhin's continuation theorem of coincidence degree theory [3], to establish the existence of 2^n positive periodic solutions for system (1). For the work concerning the multiple existence of periodic solutions of periodic population models which was done using coincidence degree theory, we refer to [4-10].

The organization of the rest of this paper is as follows. In Section 2, by employing the continuation theorem of coincidence degree theory and matrix inequality, we establish the existence of 2^n positive periodic solutions of system (1). In Section 3, an example is given to illustrate the effectiveness of our results.

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II. Existence of 2^n positive periodic solutions

In this section, by using Mawhin's continuation theorem and linear inequality, we shall show the existence of positive periodic solutions of (1). To do so, we need to make some preparations.

Let X and Z be real normed vector spaces. Let $L : \operatorname{Dom} L \subset X \to Z$ be a linear mapping and $N : X \times [0, 1] \to Z$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if dim Ker L = codim Im $L < \infty$ and Im L is closed in Z. If L is a Fredholm mapping of index zero, then there exists continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I - Q)$, and $X = \operatorname{Ker} L \bigoplus \operatorname{Ker} P, Z = \operatorname{Im} L \bigoplus \operatorname{Im} Q$. It follows that $L|_{\operatorname{Dom} L\cap \operatorname{Ker} P} : (I - P)X \to \operatorname{Im} L$ is invertible and its inverse is denoted by K_P. If Ω is a bounded open subset of X, the mapping N is called L-compact on $\bar{\Omega} \times [0, 1]$, if $QN(\bar{\Omega}\times[0, 1])$ is bounded and $K_P(I - Q)N : \bar{\Omega}\times[0, 1] \to X$ is compact. Because Im Q is isomorphic to Ker L, there exists an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$.

The Mawhin's continuous theorem [3, p.40] is given as follows:

Lemma 1. ([3]) *Let L be a Fredholm mapping of index zero and let N be L-compact on $\bar{\Omega} \times [0, 1]$. Assume*

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(a) *for each* $\lambda \in (0,1)$, *every solution* x *of* $Lx = \lambda N(x,\lambda)$ *is such that* $x \notin \partial\Omega \cap \operatorname{Dom} L$;
(b) $QN(x,0)x \neq 0$ *for each* $x \in \partial\Omega \cap \operatorname{Ker} L$;
(c) $\deg(JQN(x,0), \Omega \cap \operatorname{Ker} L, 0) \neq 0$.

Then $Lx = N(x,1)$ *has at least one solution in* $\overline{\Omega} \cap \operatorname{Dom} L$.

In this paper, since we need some related properties of M-matrix we introduce them as follows.

Definition 1. ([11]) *If a real matrix* $A = (a_{ij})_{n\times n}$ *satisfies the following conditions* (i) *and* (ii):

(i) $a_{ii} > 0, i = 1,2,\ldots,n, a_{ij} \le 0, i \neq j, i,j = 1,2,\ldots,n$,
(ii) *A is a positive-definite matrix,*

then A is called a M-matrix.

Lemma 2. ([11]) *If matrix* $A = (a_{ij})_{n\times n}$ *is a M-matrix, then* A^{-1} *exists and its every element is nonnegative.*

For the sake of convenience, we denote by $f^l = \min_{t\in[0,\omega]} f(t), f^M = \max_{t\in[0,\omega]} f(t), \bar{f} = \frac{1}{\omega}\int_0^\omega f(t)\,\mathrm{d}t$, respectively, here $f(t)$ is a continuous ω-periodic function. In this paper, matrix $A = (a_{ij}) \ge 0$ means that each elements $a_{ij} \ge 0$.

For simplicity, we need to introduce some notations as follows.

$$D = \begin{pmatrix} b_1^l & -c_{12}^M & \cdots & -c_{1n}^M \\ -c_{21}^M & b_2^l & \cdots & -c_{2n}^M \\ \vdots & \vdots & \vdots & \vdots \\ -c_{n1}^M & -c_{n2}^M & \cdots & b_n^l \end{pmatrix}_{n\times n},$$

$$D^{-1}\begin{pmatrix} a_1^M \\ a_2^M \\ \vdots \\ a_n^M \end{pmatrix} = \begin{pmatrix} H_1^+ \\ H_2^+ \\ \vdots \\ H_n^+ \end{pmatrix}_{n\times 1},$$

$$l_i^\pm = \frac{a_i^l \pm \sqrt{(a_i^l)^2 - 4b_i^M h_i^M}}{2b_i^M},$$

$$L_i^\pm = \frac{a_i^M \pm \sqrt{(a_i^M)^2 - 4b_i^l h_i^l}}{2b_i^l},$$

$$G_i^- = \frac{h_i^l}{a_i^M + \sum_{j=1,j\neq i}^n c_{ij}^M H_j^+} = \frac{h_i^l}{b_i^l H_i^+}, \ \ i = 1,2,\ldots,n.$$

Throughout this paper, we need the following assumptions.

(H_1) $a_i^l > 2\sqrt{b_i^M h_i^M}, i = 1,2,\ldots,n$;
(H_2) Matrix D is a positive-definite matrix.

Lemma 3. *Suppose that matrix* $A = (a_{ij})_{n\times n}$ *is a M-matrix, then* $AX < B$ *implies* $X < A^{-1}B$.

Proof: In fact, there exists a positive vector $\varepsilon_0 = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)^T \in R^n$ such that $AX - B + \varepsilon_0 = (0,0,\ldots,0)^T$ which imply that $X - A^{-1}B + A^{-1}\varepsilon_0 = (0,0,\ldots,0)^T$. According to Lemma 2.2, there has at least one positive element in the every row of A^{-1}, which imply $A^{-1}\varepsilon_0 > (0,0,\ldots,0)^T$. Thus, we obtain $X < A^{-1}B$. ■

Lemma 4. *Assume that* (H_1) *and* (H_2) *hold, then we have the following inequalities:*

$$G_i^- < L_i^- < l_i^- < l_i^+ < L_i^+ < H_i^+, i = 1,2,\ldots,n.$$

Proof: In fact,

$$\begin{aligned} G_i^- &= \frac{h_i^l}{a_i^M + \sum_{j=1,j\neq i}^n c_{ij}^M H_j^M} = \frac{h_i^l}{b_i^l H_i^+} < \frac{h_i^l}{a_i^M} \\ &< \frac{h_i^l}{a_i^l} < \frac{2h_i^l}{a_i^M + \sqrt{(a_i^M)^2 - 4b_i^l h_i^l}} = L_i^-, \end{aligned}$$

$$\begin{aligned} L_i^- &= \frac{2h_i^l}{a_i^M + \sqrt{(a_i^M)^2 - 4b_i^l h_i^l}} \\ &< \frac{2h_i^M}{a_i^l + \sqrt{(a_i^l)^2 - 4b_i^M h_i^M}} = l_i^- < l_i^+, \end{aligned}$$

$$\begin{aligned} l_i^+ &= \frac{a_i^l + \sqrt{(a_i^l)^2 - 4b_i^M h_i^M}}{2b_i^M} \\ &< \frac{a_i^M + \sqrt{(a_i^M)^2 - 4b_i^l h_i^l}}{2b_i^l} = L_i^+, \end{aligned}$$

$$\begin{aligned} L_i^+ &= \frac{a_i^M + \sqrt{(a_i^M)^2 - 4b_i^l h_i^l}}{2b_i^l} < \frac{a_i^M}{b_i^l} \\ &= \frac{b_i^l H_i^+ - \sum_{j=1,j\neq i}^n c_{ij}^M H_j^+}{b_i^l} < H_i^+. \end{aligned}$$

■

Theorem 1. *Assume that* (H_1) *and* (H_2) *hold. Then system* (1) *has at least* 2^n *positive ω-periodic solutions.*

Proof: By making the substitution

$$x_i(t) = \exp\{u_i(t)\}, \ i = 1,2,\ldots,n \tag{2}$$

system (1) can be reformulated as

$$\begin{aligned} \dot{u}_i(t) &= a_i(t) - b_i(t)e^{u_i(t)} + \sum_{j=1,j\neq i}^n c_{ij}(t)e^{u_j(t)} \\ &\quad -h_i(t)e^{-u_i(t)}, \ i = 1,2,\ldots,n. \end{aligned} \tag{3}$$

Let

$$\begin{aligned} & X = Z \\ = & \left\{u = (u_1,u_2,\ldots,u_n)^T \in C(R,R^n) : u(t+\omega) = u(t)\right\} \end{aligned}$$

and define

$$\|u\| = \sum_{i=1}^n \max_{t\in[0,\omega]} |u_i(t)|, \quad u \in X \text{ or } Z.$$

Equipped with the above norm $\|\cdot\|$, X and Z are Banach spaces. Let

$$N(u,\lambda) = \begin{pmatrix} a_1(t)-b_1(t)e^{u_1(t)} \\ +\lambda\sum_{j=2}^{n} c_{1j}(t)e^{u_j(t)}-h_1(t)e^{-u_1(t)} \\ \vdots \\ a_i(t)-b_i(t)e^{u_i(t)} \\ +\lambda\sum_{j=1,j\neq i}^{n} c_{ij}(t)e^{u_j(t)}-h_i(t)e^{-u_i(t)} \\ \vdots \\ a_n(t)-b_n(t)e^{u_n(t)} \\ +\lambda\sum_{j=1}^{n-1} c_{nj}(t)e^{u_j(t)}-h_n(t)e^{-u_n(t)} \end{pmatrix}_{n\times 1}, \quad u\in X$$

$Lu=\dot{u}=\frac{du(t)}{dt}$. We put $Pu=\frac{1}{\omega}\int_0^\omega u(t)dt,\ u\in X;\ Qz=\frac{1}{\omega}\int_0^\omega z(t)dt,\ z\in Z$. Thus it follows that Ker $L=R^n$, Im $L=\{z\in Z:\int_0^\omega z(t)dt=0\}$ is closed in Z, $\dim\mathrm{Ker}\ L=n=\mathrm{codim}\ \mathrm{Im}\ L$, and P,Q are continuous projectors such that

$$\mathrm{Im}\ P=\mathrm{Ker}\ L,\quad \mathrm{Ker}\ Q=\mathrm{Im}\ L=\mathrm{Im}\ (I-Q).$$

Hence, L is a Fredholm mapping of index zero. Furthermore, the generalized inverse (to L) $K_P:\mathrm{Im}\ L\to\mathrm{Ker}\ P\bigcap\mathrm{Dom}\ L$ is given by

$$K_P(z)=\int_0^t z(s)ds-\frac{1}{\omega}\int_0^\omega\int_0^s z(s)\mathrm{d}s.$$

Then

$$QN(u,\lambda)=\begin{pmatrix}\frac{1}{\omega}\int_0^\omega F_1(s,\lambda)\mathrm{d}s\\ \vdots\\ \frac{1}{\omega}\int_0^\omega F_i(s,\lambda)\mathrm{d}s\\ \vdots\\ \frac{1}{\omega}\int_0^\omega F_n(s,\lambda)\mathrm{d}s\end{pmatrix}_{n\times 1}$$

and

$$K_p(I-Q)N(u,\lambda) = \begin{pmatrix}\int_0^t F_1(s,\lambda)\mathrm{d}s-\frac{1}{\omega}\int_0^\omega\int_0^t F_1(s,\lambda)\mathrm{d}s\mathrm{d}t\\ +(\frac{1}{2}-\frac{t}{\omega})\int_0^\omega F_1(s,\lambda)\mathrm{d}s\\ \vdots\\ \int_0^t F_i(s,\lambda)\mathrm{d}s-\frac{1}{\omega}\int_0^\omega\int_0^t F_i(s,\lambda)\mathrm{d}s\mathrm{d}t\\ +(\frac{1}{2}-\frac{t}{\omega})\int_0^\omega F_i(s,\lambda)\mathrm{d}s\\ \vdots\\ \int_0^t F_n(s,\lambda)\mathrm{d}s-\frac{1}{\omega}\int_0^\omega\int_0^t F_n(s,\lambda)\mathrm{d}s\mathrm{d}t\\ +(\frac{1}{2}-\frac{t}{\omega})\int_0^\omega F_n(s,\lambda)\mathrm{d}s\end{pmatrix}_{n\times 1}$$

where

$$\begin{aligned}F_i(s,\lambda) &= a_i(s)-b_i(s)e^{u_i(s)}+\lambda\sum_{j=1,j\neq i}^{n}c_{ij}(s)e^{u_j(s)}\\ &\quad -h_i(s)e^{-u_i(s)},\ i=1,2,\ldots,n.\end{aligned}$$

Obviously, QN and $K_P(I-Q)N$ are continuous. Similar to the proof of Theorem 2.1 in [12], it is not difficult to show that $K_P(I-Q)N(\overline{\Omega})$ is compact for any open bounded set $\Omega\subset X$ by using the Arzela-Ascoli theorem. Moreover, $QN(\overline{\Omega})$ is clearly bounded. Thus, N is L-compact on $\overline{\Omega}$ with any open bounded set $\Omega\subset X$.

In order to use Lemma 1, we have to find at least 2^n appropriate open bounded subsets in X. Considering the operator equation $Lu=\lambda N(u,\lambda),\lambda\in(0,1)$, we have

$$\begin{aligned}\dot{u}_i(t) &= \lambda\bigg(a_i(t)-b_i(t)e^{u_i(t)}+\lambda\sum_{j=1,j\neq i}^{n}c_{ij}(t)e^{u_j(t)}\\ &\quad -h_i(t)e^{-u_j(t)}\bigg),i=1,2,\ldots,n.\end{aligned}\tag{4}$$

Assume that $u\in X$ is an ω-periodic solution of system (3) for some $\lambda\in(0,1)$. Then there exist $\xi_i,\eta_i\in[0,\omega]$ such that $u_i(\xi_i)=\max_{t\in[0,\omega]}u_i(t), u_i(\eta_i)=\min_{t\in[0,\omega]}u_i(t), i=1,2,\ldots,n$. It is clear that $\dot{u_i}(\xi_i)=0,\dot{u_i}(\eta_i)=0,i=1,2,\ldots,n$. From this and (3), we have

$$\begin{aligned}0 &= a_i(\xi_i)-b_i(\xi_i)e^{u_i(\xi_i)}+\lambda\sum_{j=1,j\neq i}^{n}c_{ij}(\xi_i)e^{u_j(\xi_i)}\\ &\quad -h_i(\xi_i)e^{-u_i(\xi_i)},\ i=1,2,\ldots,n\end{aligned}\tag{5}$$

and

$$\begin{aligned}0 &= a_i(\eta_i)-b_i(\eta_i)e^{u_i(\eta_i)}+\lambda\sum_{j=1,j\neq i}^{n}c_{ij}(\eta_i)e^{u_j(\eta_i)}\\ &\quad -h_i(\eta_i)e^{-u_i(\eta_i)},\ i=1,2,\ldots,n.\end{aligned}\tag{6}$$

By (5) we have

$$\begin{aligned}a_i^M+\sum_{j=1,j\neq i}^{n}c_{ij}^M e^{u_j(\xi_j)} &\geq a_i(\xi_i)+\sum_{j=1,j\neq i}^{n}c_{ij}(\xi_i)e^{u_j(\xi_i)}\\ &= b_i(\xi_i)e^{u_i(\xi_i)}+h_i(\xi_i)e^{-u_i(\xi_i)}\\ &> b_i^l e^{u_i(\xi_i)},\end{aligned}$$

namely

$$b_i^l e^{u_i(\xi_i)}-\sum_{j=1,j\neq i}^{n}c_{ij}^M e^{u_j(\xi_j)}<a_i^M,\ i=1,2,\ldots,n,$$

which can be rewritten by the following matrix form

$$\begin{pmatrix}b_1^l & -c_{12}^M & \cdots & -c_{1n}^M\\ -c_{21}^M & b_2^l & \cdots & -c_{2n}^M\\ \vdots & \vdots & \vdots & \vdots\\ -c_{n1}^M & -c_{n2}^M & \cdots & b_n^l\end{pmatrix}\begin{pmatrix}e^{u_1(\xi_1)}\\ e^{u_2(\xi_2)}\\ \vdots\\ e^{u_n(\xi_n)}\end{pmatrix}<\begin{pmatrix}a_1^M\\ a_2^M\\ \vdots\\ a_n^M\end{pmatrix}.$$

By assumption (H_2) and Lemma 3, we obtain

$$\begin{aligned}\begin{pmatrix}e^{u_1(\xi_1)}\\ e^{u_2(\xi_2)}\\ \vdots\\ e^{u_n(\xi_n)}\end{pmatrix} &< \begin{pmatrix}b_1^l & -c_{12}^M & \cdots & -c_{1n}^M\\ -c_{21}^M & b_2^l & \cdots & -c_{2n}^M\\ \vdots & \vdots & \vdots & \vdots\\ -c_{n1}^M & -c_{n2}^M & \cdots & b_n^l\end{pmatrix}^{-1}\\ &\times\begin{pmatrix}a_1^M\\ a_2^M\\ \vdots\\ a_n^M\end{pmatrix}:=\begin{pmatrix}H_1^+\\ H_2^+\\ \vdots\\ H_n^+\end{pmatrix}\end{aligned}\tag{7}$$

International Journal of Engineering, Mathematical and Physical Sciences
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According to (6) and (7), we obtain

$$\begin{aligned} a_i^M + \sum_{j=1,j\neq i}^{n} c_{ij}^M H_i^+ &> a_i(\eta_i) + \sum_{j=1,j\neq i}^{n} c_{ij}(\eta_i) e^{u_j(\eta_i)} \\ &= b_i(\eta_i) e^{u_i(\eta_i)} + h_i(\eta_i) e^{-u_i(\eta_i)} \\ &> h_i^l e^{-u_i(\eta_i)}, \end{aligned}$$

that is,

$$h_i^l e^{-u_i(\eta_i)} < a_i^M + \sum_{j=1,j\neq i}^{n} c_{ij}^M H_i^+,$$

which implies that

$$e^{u_i(\eta_i)} > \frac{h_i^l}{a_i^M + \sum_{j=1,j\neq i}^{n} c_{ij}^M H_j^+} = \frac{h_i^l}{b_i^l H_i^+} = G_i^-. \tag{8}$$

(7) and (8) give

$$\begin{pmatrix} u_1(\xi_1) \\ u_2(\xi_2) \\ \vdots \\ u_n(\xi_n) \end{pmatrix} < \begin{pmatrix} \ln H_1^+ \\ \ln H_2^+ \\ \vdots \\ \ln H_n^+ \end{pmatrix} \tag{9}$$

and

$$\begin{pmatrix} u_1(\eta_1) \\ u_2(\eta_2) \\ \vdots \\ u_n(\eta_n) \end{pmatrix} > \begin{pmatrix} \ln G_1^- \\ \ln G_2^- \\ \vdots \\ \ln G_n^- \end{pmatrix} \tag{10}$$

respectively. Moreover, according to (5), we have

$$b_i^M e^{u_i(\xi_i)} + h_i^M e^{-u_i(\xi_i)} > a_i^l, \ i = 1, 2, \ldots, n,$$

namely,

$$b_i^M e^{2u_i(\xi_i)} - a_i^l e^{u_i(\xi_i)} + h_i^M > 0, \ i = 1, 2, \ldots, n,$$

which implies that

$$\begin{pmatrix} u_1(\xi_1) \\ u_2(\xi_2) \\ \vdots \\ u_n(\xi_n) \end{pmatrix} > \begin{pmatrix} \ln l_1^+ \\ \ln l_2^+ \\ \vdots \\ \ln l_n^+ \end{pmatrix}$$

or

$$\begin{pmatrix} u_1(\xi_1) \\ u_2(\xi_2) \\ \vdots \\ u_n(\xi_n) \end{pmatrix} < \begin{pmatrix} \ln l_1^- \\ \ln l_2^- \\ \vdots \\ \ln l_n^- \end{pmatrix} \tag{11}$$

Similarly, by (6), we get

$$\begin{pmatrix} u_1(\eta_1) \\ u_2(\eta_2) \\ \vdots \\ u_n(\eta_n) \end{pmatrix} > \begin{pmatrix} \ln l_1^+ \\ \ln l_2^+ \\ \vdots \\ \ln l_n^+ \end{pmatrix}$$

or

$$\begin{pmatrix} u_1(\eta_1) \\ u_2(\eta_2) \\ \vdots \\ u_n(\eta_n) \end{pmatrix} < \begin{pmatrix} \ln l_1^- \\ \ln l_2^- \\ \vdots \\ \ln l_n^- \end{pmatrix} \tag{12}$$

By the assumptions (H_1), (H_2) and Lemma 4, we have

$$\ln G_i^- < \ln l_i^- < \ln l_i^+ < \ln H_i^+, i = 1, 2, \ldots, n. \tag{13}$$

From (9), (10), (11), (12) and (13), we obtain, for all $t \in R$,

$$\ln G_i^- < u_i(t) < \ln l_i^-$$

or

$$\ln l_i^+ < u_i(t) < \ln H_i^+, \ i = 1, 2, \ldots, n. \tag{14}$$

For convenience, we denote

$$G_i = \big(\ln G_i^-, \ln l_i^-\big), \ H_i = \big(\ln l_i^+, \ln H_i^+\big), \ i = 1, 2, \ldots, n.$$

Clearly, $l_i^\pm$, G_i^- and $H_i^+, i = 1, 2, \ldots, n$ are independent of λ. For each $i = 1, 2, \ldots, n$, we choose an interval between two intervals G_i and H_i and denote it as Δ_i, then define the set

$$\big\{u = (u_1, \ldots, u_n)^T \in X : u_i(t) \in \Delta_i, t \in R, i = 1, \ldots, n\big\}.$$

Obviously, the number of the above sets is 2^n. We denote these sets as $\Omega_k, k = 1, 2, \ldots, 2^n$. $\Omega_k, k = 1, 2, \ldots, 2^n$ are bounded open subsets of $X, \Omega_i \cap \Omega_j = \phi, i \neq j$. Thus $\Omega_k(k = 1, 2, \ldots, 2^n)$ satisfies the requirement (a) in Lemma 1.

Now we show that (b) of Lemma 1 holds, i.e., we prove when $u \in \partial\Omega_k \cap \operatorname{Ker} L = \partial\Omega_k \cap R^n, QN(u, 0) \neq (0, 0, \ldots, 0)^T, k = 1, 2, \ldots, 2^n$. If it is not true, then when $u \in \partial\Omega_k \cap \operatorname{Ker} L = \partial\Omega_k \cap R^n, k = 1, 2, \ldots, 2^n$, constant vector $u = (u_1, u_2, \ldots, u_n)^T$ with $u \in \partial\Omega_k, k = 1, 2, \ldots, 2^n$, satisfies

$$\int_0^\omega a_i(t)\,\mathrm{d}t - \int_0^\omega b_i(t) e^{u_i}\,\mathrm{d}t - \int_0^\omega h_i(t) e^{-u_i}\,\mathrm{d}t = 0.$$

In view of the mean value theorem of calculous, there exist n points $t_i (i = 1, 2, \ldots, n)$ such that

$$a_i(t_i) - b_i(t_i) e^{u_i} - h_i(t_i) e^{-u_i} = 0, \ i = 1, 2, \ldots, n. \tag{15}$$

Following the arguments of (5)-(14), we have

$$\ln G_i^- < u_i(t_i) < \ln l_i^- \quad \text{or} \quad \ln l_i^+ < u_i(t_i) < \ln H_i^+. \tag{16}$$

Then u belongs to one of $\Omega_k \cap R^n, k = 1, 2, \ldots, 2^n$. This contradicts the fact that $u \in \partial\Omega_k \cap R^n, k = 1, 2, \ldots, 2^n$. Thus condition (b) in Lemma 1 is satisfied. Finally, we show that (c) in Lemma 1 holds. Note that the system of algebraic equations

$$a_i(t_i) - b_i(t_i) e^{x_i} - h_i(t_i) e^{-x_i} = 0, \ i = 1, 2, \ldots, n$$

has 2^n distinct solutions since (H_1) and (H_2) hold, $(x_1^*, x_2^*, \ldots, x_n^*) = (\ln \hat{x}_1, \ln \hat{x}_2, \ldots, \ln \hat{x}_n)$, where $x_i^\pm = \frac{a_i(t_i) \pm \sqrt{(a_i(t_i))^2 - 4 b_i(t_i) h_i(t_i)}}{2 b_i(t_i)}, \hat{x}_i = x_i^-$ or $\hat{x}_i = x_i^+, i = 1, 2, \ldots, n$. It is easy to verify that

$$\ln G_i^- < \ln x_i^- < \ln l_i^- < \ln l_i^+ < \ln x_i^+ < \ln H_i^+.$$

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Vol:6, No:8, 2012

Therefore, $(x_1^*, x_2^*, \ldots, x_n^*)$ uniquely belongs to the corresponding Ω_k. Since $\mathrm{Ker}\, L = \mathrm{Im}\, Q$, we can take $J = I$. A direct computation gives, for $k = 1, 2, \ldots, 2^n$,

$$\begin{aligned} & \deg\left\{JQN(u,0), \Omega_k \cap \mathrm{Ker}\ L, (0,0)^T\right\} \\ = \ & \mathrm{sign}\left[\prod_{i=1}^{n}\left(-b_i(t_i)x_i^* + \frac{h_i(t_i)}{x_i^*}\right)\right]. \end{aligned}$$

Since $a_i(t_i) - b_i(t_i)x_i^* - \frac{h_i(t_i)}{x_i^*} = 0,\ i = 1, 2, \ldots, n$, then

$$\begin{aligned} & \deg\left\{JQN(u,0), \Omega_k \cap \mathrm{Ker}\, L, (0,0)^T\right\} \\ = \ & \mathrm{sign}\left[\prod_{i=1}^{n}\left(a_i(t_i) - 2b_i(t_i)x_i^*\right)\right] = \pm 1,\ k = 1, 2, \ldots, 2^n. \end{aligned}$$

So far, we have proved that $\Omega_k (k = 1, 2, \ldots, 2^n)$ satisfies all the assumptions in Lemma 1. Hence, system (3) has at least 2^n different ω-periodic solutions. Thus by (2.1) system (1) has at least 2^n different positive ω-periodic solutions. This completes the proof of Theorem 1. ■

III. An example

Now, let us consider the following four species cooperative system with harvesting terms:

$$\begin{aligned} x_i(t) &= \Big(a_i(t) - b_i(t)x_i(t) + \sum_{j=1, j\neq i}^{4} c_{ij}(t)x_j(t) \\ & \quad -h_i(t)\Big), \quad i = 1, 2, 3, 4, \end{aligned} \tag{17}$$

where $a_1(t) = 3 + \sin t$, $b_1(t) = \frac{6+\sin t}{10}$, $h_1(t) = \frac{9+\cos t}{15}$, $a_2(t) = 3 + \cos t$, $b_2(t) = \frac{6+\cos t}{10}$, $h_2(t) = \frac{3+\cos t}{5}$, $a_3(t) = 3 + \sin 2t$, $b_3(t) = \frac{6+\sin 2t}{10}$, $h_3(t) = \frac{8+\cos 2t}{10}$, $a_4(t) = 3 + \cos 2t$, $b_4(t) = \frac{6+\cos 2t}{10}$, $h_4(t) = \frac{8+\sin 2t}{10}$ and $c_{ij}(t) = \frac{1}{10}, i \neq j, i, j = 1, 2, 3, 4$. By the simple calculation, we have

$$\begin{pmatrix} a_1^l \\ a_2^l \\ a_3^l \\ a_4^l \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} b_1^l \\ b_2^l \\ b_3^l \\ b_4^l \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix},$$

$$\begin{pmatrix} h_1^l \\ h_2^l \\ h_3^l \\ h_3^l \end{pmatrix} = \begin{pmatrix} \frac{8}{15} \\ \frac{2}{5} \\ \frac{7}{10} \\ \frac{7}{10} \end{pmatrix}, \begin{pmatrix} a_1^M \\ a_2^M \\ a_3^M \\ a_4^M \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 4 \\ 4 \end{pmatrix},$$

$$\begin{pmatrix} b_1^M \\ b_2^M \\ b_3^M \\ b_4^M \end{pmatrix} = \begin{pmatrix} \frac{7}{10} \\ \frac{7}{10} \\ \frac{7}{10} \\ \frac{7}{10} \end{pmatrix}, \begin{pmatrix} h_1^M \\ h_2^M \\ h_3^M \\ h_3^M \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{4}{5} \\ \frac{9}{10} \\ \frac{9}{10} \end{pmatrix},$$

$$D = \begin{pmatrix} \frac{1}{2} & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} \\ -\frac{1}{10} & \frac{1}{2} & -\frac{1}{10} & -\frac{1}{10} \\ -\frac{1}{10} & -\frac{1}{10} & \frac{1}{2} & -\frac{1}{10} \\ -\frac{1}{10} & \frac{1}{10} & -\frac{1}{10} & \frac{1}{2} \end{pmatrix},$$

$$D^{-1} = \begin{pmatrix} \frac{5}{2} & \frac{5}{6} & \frac{5}{6} & \frac{5}{6} \\ \frac{5}{6} & \frac{5}{2} & \frac{5}{6} & \frac{5}{6} \\ \frac{5}{6} & \frac{5}{6} & \frac{5}{2} & \frac{5}{6} \\ \frac{5}{6} & \frac{5}{6} & \frac{5}{6} & \frac{5}{2} \end{pmatrix}.$$

According to the following calculation,

$$\det\begin{pmatrix} \frac{1}{2} & -\frac{1}{10} \\ -\frac{1}{10} & \frac{1}{2} \end{pmatrix} = 0.24 > 0,$$

$$\det\begin{pmatrix} \frac{1}{2} & -\frac{1}{10} & -\frac{1}{10} \\ -\frac{1}{10} & \frac{1}{2} & -\frac{1}{10} \\ -\frac{1}{10} & -\frac{1}{10} & \frac{1}{2} \end{pmatrix} = 0.108 > 0,$$

$$\det\begin{pmatrix} \frac{1}{2} & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} \\ -\frac{1}{10} & \frac{1}{2} & -\frac{1}{10} & -\frac{1}{10} \\ -\frac{1}{10} & -\frac{1}{10} & \frac{1}{2} & -\frac{1}{10} \\ -\frac{1}{10} & \frac{1}{10} & -\frac{1}{10} & \frac{1}{2} \end{pmatrix} = 0.0432 > 0,$$

we have known that matrix D is positive-definite. In addition, we obtain

$$\begin{pmatrix} a_1^l \\ a_2^l \\ a_3^l \\ a_4^l \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix} > \begin{pmatrix} 2\sqrt{b_1^M h_1^M} \\ 2\sqrt{b_2^M h_2^m} \\ 2\sqrt{b_3^M h_3^M} \\ 2\sqrt{b_4^M h_4^M} \end{pmatrix} = \begin{pmatrix} \frac{2\sqrt{105}}{15} \\ \frac{2\sqrt{14}}{5} \\ \frac{3\sqrt{7}}{5} \\ \frac{3\sqrt{7}}{5} \end{pmatrix}.$$

Therefore, all conditions of Theorem 1 are satisfied. By Theorem 1, system (17) has at least sixteen positive 2π-periodic solutions.

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