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# Some properties of superfuzzy subset of a fuzzy subset

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***Abstract*****—**In this paper, we define permutable and mutually permutable fuzzy subgroups of a group. Then we study their relation with permutable and mutually permutable subgroups of a group. Also we study some properties of fuzzy quasinormal subgroup. We define superfuzzy subset of a fuzzy subset and we study some properties of superfuzzy subset of a fuzzy subset.

## I. Introduction

Applying the concept of fuzzy sets of Zadeh [7] to group theory, Rosenfeld [6] introduced the notion of a fuzzy group as early as 1971. Let G be a group and let $\mu$ and $\nu$ be fuzzy subgroups of G. We say that $\mu$ is permuted by $\nu$ if for any $a, b \in G$, there exists $x \in G$ such that $\mu(x^{-1}ab) \geq \mu(a), \nu(x) \geq \nu(b)$ and we say $\mu$ and $\nu$ are permutable if $\mu$ is permuted by $\nu$ and $\nu$ is permuted by $\mu$. Also we say that $\mu$ is permuted by $\nu$ mutually if for any subgroup L of $\nu_b$ that $b \in Im\nu$, we have been for any $a \in G, l \in L$, there exist $l_1, l_2$ of L such that $\mu(l_1^{-1}al) \geq \mu(a)$ and $\mu(lal_2^{-1}) \geq \mu(a)$ and we say $\mu$ and $\nu$ are mutually permutable if $\mu$ is permuted by $\nu$ mutually and $\nu$ is permuted by $\mu$ mutually. Let $\mu$ and $\nu$ be fuzzy subgroups of G. We determine that $\mu$ and $\nu$ are permutable(mutually permutable) if and if for any $t \in Im\mu, s \in Im\nu$, $\mu_t, \nu_s$ are permutable(mutually permutable). We know $\mu o \nu$ is a fuzzy subgroup of G if and only if $\mu o \nu = \nu o \mu$. We obtain sufficient condition such that $\mu o \nu$ is a fuzzy subgroup. But it is not necessary condition. Ajmal and Thomas [1] introduced the notion of a fuzzy quasinormal subgroup. Fuzzy quasinormal subgroup arising out of fuzzy normal subgroup. Also we prove that $\mu$ is a fuzzy quasinormal subgroup of group G if and only if for every subgroup L of G, we have been that for any $a \in G, l \in L$ there exist $l_1, l_2$ of L such that $\mu(l_1^{-1}al) \geq \mu(a)$ and $\mu(lal_2^{-1}) \geq \mu(a)$. Finally we define superfuzzy subset of a fuzzy subset and we study some properties of superfuzzy subset of a fuzzy subset.

## II. Preliminaries

We use [0,1], the real unit interval as a chain the usual ordering in R which $\wedge$ stands for infimum( or intersection) and $\vee$ stands for supremum ( or union) for the degree of membership. A fuzzy subset of a set X is mapping $\mu :\rightarrow [0, 1]$. The union and intersection of two fuzzy subset are defined using sup and inf point wise. We denote the set of all fuzzy subset of X by $I^X$. Further, we denote fuzzy subsets by the Greek letters $\mu, \nu, \eta$, etc. Let $\mu, \nu \in I^X$. If $\mu(x) \leq \nu(x) \forall x \in X$, then we say that $\mu$ is contained in $\nu$ ( or $\nu$ contains $\mu$) and we write $\mu \subseteq \nu$. Let $\mu \in I^X$ for $a \in I$, define $\mu_a$ as follow:
$\mu_a = \{x \mid x \in X, \mu(x) \geq a\}$. $\mu_a$ is called a-cut( or a-level) set of $\mu$.
It is easy to verify that for any $\mu, \nu \in I^X$:
1) $\mu \subseteq \nu, a \in I \Rightarrow \mu_a \subseteq \nu_a$.
2) $a \leq b, a, b \in I \Rightarrow \mu_b \subseteq \mu_a$.
3) $\mu = \nu \Leftrightarrow \mu_a = \nu_a \forall a \in I$.
Let G is an arbitrary group with a multiplicative binary operation and identity. We define the binary operation $o$ on $I^G$ as follow:
$\forall \mu, \nu \in I^G$, $\forall x \in G$
$(\mu o \nu)(x) = \vee\{\mu(y) \wedge \nu(z) \mid y, z \in G, yz = x\}$.
We call $\mu o \nu$ the product of $\mu$ and $\nu$. Fuzzy subset $\mu$ of G is called a fuzzy subgroup of G if
$(G_1)$ $\mu(xy) \geq \mu(x) \wedge \mu(y) \forall x, y \in G$;
$(G_2)$ $\mu(x^{-1} \geq \mu(x) \forall x \in G$.

**Proposition II.1.** *([4;Lemma 1.2.5]). Let $\mu \in I^G$. Then $\mu$ is a fuzzy subgroup of G if and only if $\mu_a$ is a subgroup of G, $\forall a \in \mu(G) \bigcup \{b \in I \mid b \leq \mu(e)\}$.*

**Theorem II.2.** *([4;Theorem 1.2.9]). let $\mu \in I^G$. Then $\mu o \nu$ is a fuzzy subgroup if and only if $\mu o \nu = \nu o \mu$.*

**Definition II.3.** ([1]).Let $\mu$ is a fuzzy subgroup of group G, $\mu$ is said to be fuzzy normal subgroup of G if $\mu(xy) = \mu(yx) \forall x, y \in G$.

**Definition II.4.** ([2]). Let G be a group and let H and K be subgroups of G.
(a) We say that H and K are permutable if $HK = KH =< H, K >$.
(b) We say that H and K are mutually permutable if H permutes with every subgroup of K and K permutes with every subgroup of H.

**Definition II.5.** ([2]). Let G be a group and let H be a subgroup of G, H is said to be quasinormal in G, if H permutes whit every subgroup of G.

## III. Permutable and mutually permutable on fuzzy subgroups of a group

**Definition III.1.** Let G be a group and let $\mu$ and $\nu$ be fuzzy subgroups of G.
(a) We say that $\mu$ is permuted by $\nu$ if for any $a, b \in G$, there exists $x \in G$ such that $\mu(x^{-1}ab) \geq \mu(a), \nu(x) \geq \nu(b)$.
(b) We say that $\mu$ is permuted by $\nu$ mutually if for any subgroup L of $\nu_b$ that $b \in Im\nu$, we have been for any $a \in G, l \in L$, there exist $l_1, l_2$ of L such that $\mu(l_1^{-1}al) \geq \mu(a)$ and $\mu(lal_2^{-1}) \geq \mu(a)$.

**Definition III.2.** Let G be a group and let $\mu$ and $\nu$ be fuzzy subgroups of G.

(a) We say $\mu$ and $\nu$ are permutable if $\mu$ is permuted by $\nu$ and $\nu$ is permuted by $\mu$.
(b) We say $\mu$ and $\nu$ are mutually permutable if $\mu$ is permuted by $\nu$ mutually and $\nu$ is permuted by $\mu$ mutually.

**Corollary III.3.** *Let $\mu$ and $\nu$ be fuzzy subgroups of G. If $\mu$ and $\nu$ are mutually permutable then $\mu$ and $nu$ are permutable.*

*Proof:* Straightforward. ■

**Corollary III.4.** *Let $\mu$ is a fuzzy normal subgroup of G. Then $\mu$ permutes with every fuzzy subgroup of G mutually.*

*Proof:* Straightforward. ■

**Theorem III.5.** *Let $\mu$ and $\nu$ be fuzzy subgroups of G, then $\mu$ and $\nu$ are permutable if and if for any $t \in Im\mu, s \in Im\nu$, $\mu_t, \nu_s$ are permutable.*

*Proof:* Let $\mu$ and $\nu$ be permutable. Let $t \in Im\mu, s \in Im\nu$. If $a \in \mu_t$ and $b \in \nu_s$ then $\mu(a)$
$geqt, \nu(b) \geq s$. We know that $\mu$ is permuted by $\nu$. Then there that exists $x \in G$ such that $\mu(x^{-1}ab) \geq t$ and $\nu(x) \geq s$, this means that $x^{-1}ab \in \mu_t$ and $x\nu_s$. So that $ab = x(x^{-1}ab)$. If $a \in \nu_s, b \in \mu_t$, then $\mu(b) \geq t, \nu(a) \geq s$. So that there exists $y \in G$ such that $\nu(y^{-1}ab) \geq \nu(a) \geq s$ and $\mu(y) \geq \mu(b) \geq t$, this means that $y^{-1}ab \in \nu_s$ and $y \in \mu_t$. So that $ab = y(y^{-1}ab)$, consequently $\mu_t\nu_s = \nu_s\mu_t$. Now let $\mu_t\nu_s = \nu_s\mu_t, \forall t \in Im\mu, s \in Im\nu$ and let a and b be two arbitrary elements of G. Let $r = \mu(a), s = \nu(b)$, then elements exist for example $a' \in \mu_t, b' \in \nu_s$ such that $ab = a'b'$, then $b'^{-1}ab = a'$, this implies $\mu(b'^{-1}ab) = \mu(a') \geq t = \mu(a)$. Hence $b' \in \nu_s$, then $\nu(b') \geq s = \nu(b)$. Therefore $\mu$ is permuted by $\nu$. Similarly $\nu$ is permuted by $\mu$. ■

**Proposition III.6.** *Let $\mu$ and $\nu$ be fuzzy subgroups of G and $t \in Im\mu, s \in Im\nu$ if $\mu$ and $\nu$ be permutable then*
*(1) If $t \leq s$ then there exists $a \in G$ such that $\nu(a) \geq t$.*
*(2) If $s \leq t$ then there exists $b \in G$ such that $\mu(b) \geq s$.*

*Proof:* We know that $\mu_t, \nu_s \neq \emptyset$ then there exist a and b in G such that $\mu(a) \geq t$ and $\nu(b) \geq s$. Hence $\mu$ and $\nu$ are permutable then $\mu_t\nu_s = \nu_s\mu_t$, then there are $a' \in \mu_t$ and $b' \in \nu_s$ such that $ab = a'b'$. Therefore $\mu(aa') \geq \min\{\mu(a), \mu(a')\} \geq t$. Similarly $\nu(bb') \geq s$. If $t \leq s$ then $\nu(bb') \geq s \geq t$ and if $s \leq t$ then $\mu(aa') \geq t \geq s$. ■

**Proposition III.7.** *Let $\mu$ and $\nu$ be fuzzy subgroups of G. If $\mu$ and $\nu$ be permutable then $\mu o \nu$ is a fuzzy subgroup of G.*

*Proof:* Let $\mu$ and $\nu$ be permutable and $x \in G$. If $y \in G$ be an arbitrary element then there exists $t \in G$ such that $\mu(t^{-1}yy^{-1}x) \geq \mu(y)$ and $\nu(t) \geq \nu(y^{-1}x)$, so that $\mu(y) \wedge \nu(y^{-1}x) \leq \mu(t^{-1}x) \wedge \nu(t)$. Therefore $\mu(y) \wedge \nu(y^{-1}x) \leq \sup_{z \in G}\{\nu(z) \wedge \mu(z^{-1}\}$, this means that $(\mu o \nu)(x) \leq (\nu o \mu)(x)$. Similarly $(\nu o \mu)(x) \leq (\mu o \nu)(x)$ because $\nu$ is permuted by $\mu$. ■

**Example III.8.** Let G be symmetric group $S_3$. Define $\mu$ and $\nu$ as follow:

$$\mu(x) = \begin{cases} 1 & x = e \\ \frac{1}{2} & x = b \\ \frac{1}{3} & else \end{cases}, \qquad \nu(x) = \begin{cases} 1 & x = e \\ \frac{1}{2} & x = ab \\ \frac{1}{3} & else \end{cases}$$

Clearly, $\mu o \nu = \mu$, but $\mu$ is not permuted by $\nu$.

**Theorem III.9.** *Let $\mu$ and $\nu$ be fuzzy subgroups of G, then $\mu$ and $\nu$ are mutually permutable if and if for any $t \in Im\mu, s \in Im\nu$, $\mu_t, \nu_s$ are mutually permutable.*

*Proof:* Let $\mu$ and $\nu$ be mutually permutable. Let $a \in Im\mu$ and $b \in Im\nu$. Also let $L \leq \nu_b, x \in \mu_a$ and $l \in L$, then $\mu(x) \geq a$. We known that exists $l_1 \in L$ such that $\mu(l_1^{-1}xl) \geq \mu(x)$, this means that $l_1^{-1}xl \in \mu_a$, so that $xl = l_1(l_1^{-1}xl)$. Therefore $\mu_aL \subseteq L\mu_a$ and also there exists $l_2 \in L$ such that $\mu(lxl_2^{-1}) \geq \mu(x) \geq a$. That is, $lxl_2^{-1} \in \mu_a$. So that $lx = (lxl_2^{-1})l_2$, therefore $L\mu_a \subseteq \mu_aL$. So $\mu_aL$ is a subgroup of G. Similarly, we know that $\nu$ is permuted by $\mu$ mutually then for any subgroup H of $\mu_a$, $H\nu_b = \nu_bH$. So $\mu_a$ and $\nu_b$ are mutually permutable. Now let for any $a \in Im\mu$ and $b \in Im\nu$, $\mu_a$ and $\nu_b$ be mutually permutable. Let $b \in Im\nu$ and $L \leq \nu_b$ and also $x \in G$ and $l \in L$. Let $r = \mu(x)$, so that $\mu_r$ and $\nu_b$ are mutually permutable, therefore exist $l_1 \in L$ and $y \in \mu_r$ such that $lx = yl_1$, then $lxl_1^{-1} = y$, this implies $lxl_1^{-1} \in \mu_r$ and $\mu(lxl_1^{-1}) \geq r = \mu(x)$. Also there exist $l_2 \in L$ and $y' \in \mu_r$ such that $xl = l_2y'$, then $l_2^{-1}xl = y'$, this implies $l_2^{-1}xl \in \mu_r$ and $\mu(l_2^{-1}xl) \geq \mu(x)$. Therefore $\mu$ is permuted by $\nu$ mutually. Similarly $\nu$ is permuted by $\mu$ mutually. ■

## IV. Some properties of fuzzy quasinormal subgroup of a group

**Definition IV.1.** ([5]). A fuzzy subgroup $\mu$ of G is called quasinormal if its level subgroups are quasinormal subgroups of G.

**Theorem IV.2.** *If $\mu$ is a fuzzy subgroup of group G, then the following properties are equivalent:*
*($q_1$) For every subgroup L of G, we have been that for any $a \in G, l \in L$ there exist $l_1, l_2$ of L such that $\mu(l_1^{-1}al) \geq \mu(a)$ and $\mu(lal_2^{-1}) \geq \mu(a)$. ($q_2$) For any $a \in Im\mu$, $\mu_a$ is a quasinormal subgroup of G.*

*Proof:* Assume firstly the validity of ($q_1$). Let $a \in Im\mu$ and $L \leq G$. If $x \in \mu_a, l \in L$ then there exists $l_1 \in L$ such that $\mu(l_1^{-1}xl) \geq \mu(x) \geq a$, this means that $l_1^{-1}xl \in \mu_a$. So that $xl = l_1(l_1^{-1}xl)$. Also let $y \in \mu_a, l' \in L$, therefore there exists $l_2 \in L$ such that $\mu(l'yl_2^{-1}) \geq \mu(y)$. So $\mu(l'yl_2^{-1}) \geq a$, this means that $l'yl_2^{-1} \in \mu_a$, Therefore $l'y = (l'yl_2^{-1})l_2$, consequently $L\mu_a = \mu_aL$. Hence ($q_1$) implies ($q_2$). Assume next the validity of ($q_2$). Let $L \subseteq G$ and $x \in G, l \in L$. If $r = \mu(x)$ then there exist $y \in \mu_r$ and $l_1 \in L$ such that $xl = l_1y$, so $\mu(l_1^{-1}xl) \geq r = \mu(x)$. Similarly there exist $y' \in \mu_r, l_2 \in L$ such that $ix = y'l_2$. Then $\mu(lxl_2^{-1}) \geq \mu(x)$. Hence ($q_2$) implies ($q_1$). ■

**Corollary IV.3.** *Let $\mu$ be a fuzzy subgroup of G. Then $\mu$ is a fuzzy quasinormal subgroup if and only if for every subgroup L of G, we have been that for any $a \in G, l \in L$ there exist $l_1, l_2$ of L such that $\mu(l_1^{-1}al) \geq \mu(a)$ and $\mu(lal_2^{-1}) \geq \mu(a)$.*

*Proof:* Straightforward. ■

**Theorem IV.4.** *([5;Theorem 4.3.13]). Let $\mu$ be a fuzzy subgroup of G with finite image. Then $\mu$ is fuzzy quasinormal if and only if $\mu o \nu = \nu o \mu$, for all fuzzy subgroups $\nu$ of group G.*

**Corollary IV.5.** *Let $\mu$ be a fuzzy subgroup of G with finite image. Then $\mu o \nu = \nu o \mu$, for all fuzzy subgroups $\nu$ of group G if and only if for every subgroup L of G, we have been that for any $a \in G, l \in L$ there exist $l_1, l_2$ of L such that $\mu(l_1^{-1} al) \geq \mu(a)$ and $\mu(lal_2^{-1}) \geq \mu(a)$.*

*Proof:* Straightforward. ■

**Corollary IV.6.** *Let $\mu$ be a fuzzy normal subgroup of group G. Then $\mu$ is fuzzy quasinormal subgroup of G.*

*Proof:* Straightforward. ■

**Corollary IV.7.** *Let $\mu$ be a fuzzy quasi subgroup of group G. Then $\mu$ is permuted by every fuzzy subgroup of G.*

*Proof:* Straightforward. ■

## V. Superfuzzy subset of a fuzzy subset

**Definition V.1.** Let $\mu, \nu \in I^X$. We say $\nu$ is a superfuzzy subset of fuzzy subset $\mu$, if $\mu \subseteq \nu$ and thus, there be a unique element $a \in G$ such that for any $x \in G$, $\nu(a) \leq \mu(x)$ and a is denoted by $\nu_\mu$. Also superfuzzy subset $\nu$ of fuzzy subset $\mu$ is denoted by $\mu \preceq \nu$.

**Lemma V.2.** *Let $\mu$ and $\nu$ be fuzzy subgroups of group G. If $\mu \preceq \nu$ then for any $t \in Im\mu$, there exists $s \in Im\mu$ such that $\mu_t \leq \nu_s$.*

*Proof:* Let $t \in Im\mu$. There exists $a \in G$ such that for any $x \in G$, $\nu(a) \leq \mu(x)$. Let $s = \nu(a)$ and $x \in \mu_t$. We know that $\nu(x) \geq \mu(x) \geq t$ then there exists $x_0 \in G$ such that $\mu(x_0) = t$. Then $s = \nu(a) \leq \mu(x_0) = t$, therefore $x \in \nu_s$ and the proof is completed. ■

**Theorem V.3.** *Let G is a finite group and $\mu, \nu, \eta$ and $\theta$ be fuzzy subgroups of G and $\mu \wedge \nu \preceq \eta \preceq \nu, \mu \wedge \nu \preceq \theta \preceq \mu$. If mu and $\nu$ be mutually permutable and for any $a \in Im\mu$ and $b \in Im\nu$, $G = \mu_a \nu_b$ and $(\mu \wedge \nu)(\theta_{\mu\wedge\nu}) \leq \min\{a,b\}, (\mu \wedge \nu)(\eta_{\mu\wedge\nu}) \leq \min\{a,b\}$. Then $\theta$ and $\eta$ are mutually permutable.*

*Proof:* Let $t \in Im\theta$ and $s \in Im\eta$. By lemma 5.2, there exist $a \in Im\mu$ and $b \in Im\nu$ such that $\theta_t \leq \mu_a$ and $\eta_s \leq \nu_b$. Let $x \in \mu_a \cap \mu_b$ then $(\mu \wedge \nu)(x) \geq \min\{a,b\}$. Let $z_1 = \theta_{\mu\wedge\nu}$ and $z_2 = \eta_{\mu\wedge\nu}$, then $\theta(x) \geq (\mu \wedge \nu)(x) \geq \min\{a,b\} \geq (\mu \wedge \nu)(z_1)$ and $\eta(x) \geq (\mu \wedge \nu)(z_2)$. Let $t = \theta(x_0)$ and $s = \eta(y_0)$, so that $(\mu \wedge \nu)(z_1) \geq \theta(x_0) \geq t$ and $(\mu \wedge \nu)(z_2) \leq \eta(y_0) \geq s$, then $\theta(x) \geq t$ and $\eta(x) \geq s$. This means that $\mu_a \cap \nu_b \leq \eta_s$ and $\mu_a \cap \nu_b \leq \theta_t$, therefore $\mu_a \cap \nu_b \leq \theta_t \leq \mu_a$ and $\mu_a \cap \nu_b \leq \eta_s \leq \nu_b$. By theorem [3;3.5] $\theta_t$ and $\eta_s$ are mutually permutable and the proof is completed. ■

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