The Comparison of Finite Difference Methods for Radiation Diffusion Equations

Ren Jian, and Yang Shulin

Abstract—In this paper, the difference between the Alternating Direction Method (ADM) and the Non-Splitting Method (NSM) is investigated, while both methods applied to the simulations for 2-D multimaterial radiation diffusion issues. Although the ADM have the same accuracy orders with the NSM on the uniform meshes, the accuracy of ADM will decrease on the distorted meshes or the boundary of domain. Numerical experiments are carried out to confirm the theoretical predication.

Keywords—Alternating Direction Method, Non-Splitting Method, Radiation Diffusion.

I. INTRODUCTION

THE radiation diffusion is an important process in many complex physical problems, such as astrophysical problems and inertial confinement fusion.

Because of the experience accumulated in solving one-dimensional problems [1], we have the basis for the construction of the algorithms for the complicated multidimensional problems. In the 1960's, the Alternating Direction Method (ADM) were developed by Peaceman, Douglas and Rachford to solving the two-dimensional parabolic equations. The success of ADM is ensured by a simple reduction of the multidimensional problem to a sequence of one-dimensional problems with three-diagonal matrices solved efficiently. So the ADM are very popular in many numerical simulations of industrial application programs. With the rapid progress in computers and the matrix solvers, we need the Non-Splitting Methods (NSM) for the multidimensional problems. In fact, there are so many Non-Splitting Methods solving the two-dimensional diffusion equation during the last four decades.

II. ALTERNATING DIRECTION METHOD AND NON-SPLITTING METHOD

Consider a two-dimensional diffusion equation as follow,

$$\frac{\partial u}{\partial t} = Lu = \sum_{m=1}^{2} L_m u \qquad (1)$$

where

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$$L_1 u = \frac{\partial}{\partial x}(K_1(x,y,t)\frac{\partial u}{\partial x}), L_2 u = \frac{\partial}{\partial y}(K_2(x,y,t)\frac{\partial u}{\partial y})$$

t denotes the time variable, u denotes the temperature, L denotes the spatial differential operator and the diffusion coefficient K is a given positive function of space and time.

So we define the difference operator Λ as the approximation of the differential operator L. Letting Δt be the temporal step and σ be a real number with $0 \le \sigma \le 1$, we can write the general difference formula of (1) as follow (2),

$$\frac{1}{\Delta t}(u^{n+1} - u^n) = \sigma(\Lambda u)^{n+1} + (1-\sigma)(\Lambda u)^{n+1}$$

$$\Lambda \equiv \sum_{m=1}^{2} \Lambda_m, \quad 0 \le \sigma \le 1 \qquad (2)$$

In general, the implicit scheme is applicable to radiation diffusion equation. If we consider the simplest implicit scheme, then we have

$$\frac{1}{\Delta t}(u^{n+1} - u^n) = (\Lambda u)^{n+1}, \quad \Lambda \equiv \sum_{m=1}^{2} \Lambda_m \qquad (3)$$

Based on this non-splitting scheme, we can deduce its alternating direction scheme (4),

$$\frac{1}{\Delta t}(u^* - u^n) = (\Lambda_1 u)^*$$

$$\frac{1}{\Delta t}(u^{n+1} - u^*) = (\Lambda_2 u)^{n+1} \qquad (4)$$

Given u is constant on the boundary of rectangular domain $\Omega = [0,1] \times [0,1]$, the non-splitting difference scheme can be written as follows

$$\frac{1}{\Delta t}(u^{n+1} - u^n) = [(\Lambda_1 + \Lambda_2)u]^{n+1} \qquad (5)$$

$$u^{n+1}(x,0) = U_B, u^{n+1}(x,1) = U_T, u^{n+1}(0,y) = U_L, u^{n+1}(1,y) = U_R$$

In the same boundary condition and domain, the alternating direction difference scheme is (6)

$$\frac{1}{\Delta t}(u^* - u^n) = (\Lambda_1 u)^* \quad , u^*(0,y) = U_L \quad , u^*(1,y) = U_R$$

$$\frac{1}{\Delta t}(u^{n+1} - u^*) = (\Lambda_2 u)^{n+1} \quad , u^{n+1}(x,0) = U_B \quad , u^{n+1}(x,1) = U_T$$

$$(6)$$

If we reduce the intermediate variable u^*,(6) can be changed to (7), as follow

$$\frac{1}{\Delta t}(u^{n+1} - u^n) = [(\Lambda_1 + \Lambda_2)u]^{n+1} - \Delta t(\Lambda_1 \Lambda_2 u)^{n+1}$$

$$(E - \Delta t \Lambda_2)u^{n+1}(0, y) = U_L, \quad (E - \Delta t \Lambda_2) u^{n+1}(1, y) = U_R$$

$$u^{n+1}(x, 0) = U_B , \quad u^{n+1}(x, 1) = U_T$$

$$(7)$$

In theory, they have the same order accuracy if applied on the uniform meshes. But it is not all of the truth. Compared the non-splitting scheme (5) with the alternating direction scheme (7), we can find the temporal step and boundary conditions are important reasons for the different accuracy of the schemes.

III. NINE POINT SCHEME (NPS)

In paper [2], a finite volume scheme solving diffusion equation on convex polygonal meshes is the so-called Nine Point Scheme (NPS) on arbitrary quadrangles. In fact, this non-splitting scheme can be changed to the alternating direction scheme only on the quadrangle meshes. That is to say, this scheme must be used as non-splitting scheme on the unstructured meshes such as triangle meshes.

Here, we do not educe the formulas in detail. As follow, M is the area of meshes. l is the length of mesh edge. The i, j denote x and y directions respectively. The $\vec{q} \cdot \vec{n}$, that we refer to as the flux

$$\vec{q} \cdot \vec{n} = -K\nabla u \cdot \vec{n} = -K\frac{\partial u}{\partial n} \tag{8}$$

So the non-splitting method is

$$\frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} M_{i,j} = -(\vec{q} \cdot \vec{n}l)_{i+\frac{1}{2},j}^{n+1} + (\vec{q} \cdot \vec{n}l)_{i-\frac{1}{2},j}^{n+1} - (\vec{q} \cdot \vec{n}l)_{i,j+\frac{1}{2}}^{n+1} + (\vec{q} \cdot \vec{n}l)_{i,j-\frac{1}{2}}^{n+1} \tag{9}$$

and the alternating direction method is

$$\frac{u_{i,j}^* - u_{i,j}^n}{\Delta t} M_{i,j} = -(\vec{q} \cdot \vec{n}l)_{i+\frac{1}{2},j}^* + (\vec{q} \cdot \vec{n}l)_{i-\frac{1}{2},j}^* \tag{10a}$$

$$\frac{u_{i,j}^{n+1} - u_{i,j}^*}{\Delta t} M_{i,j} = -(\vec{q} \cdot \vec{n}l)_{i,j+\frac{1}{2}}^{n+1} + (\vec{q} \cdot \vec{n}l)_{i,j-\frac{1}{2}}^{n+1} \tag{10b}$$

Obviously, the coefficient matrix of the non-splitting method (9) is a sparse matrix. The alternating direction method (10) will generate three-diagonal matrices.

In fact, the Non-Splitting Method (9) and the Alternating Direction Method (10) have the same formula of flux $\vec{q} \cdot \vec{n}$ of NinePoint Scheme in [2].

IV. NUMERICAL RESULTS

Consider the 2-D linear diffusion equation on rectangular domain $\Omega = [0,1] \times [0,1]$,

$$u_t = \Delta u, \quad in \quad \Omega \times (0,T] \tag{11a}$$

$$u(x, y, t) = 1, \quad on \quad \partial\Omega \times (0,T] \tag{11b}$$

$$u(x, y, 0) = \varphi(x, y), \quad on \quad \Omega \tag{11c}$$

The analytic solution for this problem is

$$u(x, y, t) = e^{-2\pi^2 t} \sin(\pi x)\sin(\pi y) + 1 \tag{12}$$

The figures of analytic solution are concentric circles.

Given the point $P \in \Omega$ and the temporal step $n \in (0, T]$, we define the relative error as follows

$$E_P^n = \frac{|u(P, t^n) - u_K^n|}{u(P, t^n)} \tag{13}$$

where u_P^n is the numerical solution and $u(P, t^n)$ is the true solution. So the relative error in the L_2-norm is

$$\|E^n\|_2 = \left(\sum_{P \in T} |E_P^n|^2 m(P)\right)^{\frac{1}{2}}, 0 \le n \le N+1 \tag{14}$$

and in the L_{max}-norm is

$$\|E^n\|_\infty = \max_{P \in T} | E_P^n |, 0 \le n \le N+1 \tag{15}$$

The number of mesh in the x and y directions are I and J respectively. In the following numerical examples, the total number of mesh is 32×32 ($I = J = 32$). The length of spatial step is about $1/32$ and $\Delta x \Delta y = 1/(I \cdot J)$. Δt denotes the temporal step. Then we fix the ratio of the temporal step and the spatial step $r = \Delta t /(\Delta x \Delta y)$. So we have $\Delta t = r \Delta x \Delta y$, if given r at first.

The terminal time is 0.5 in every example.

We calculate the same problem on three different types of distorted meshes, respectively RandomMesh, KershawMesh and ShestakovMesh. Mesh figures are Fig. 1.
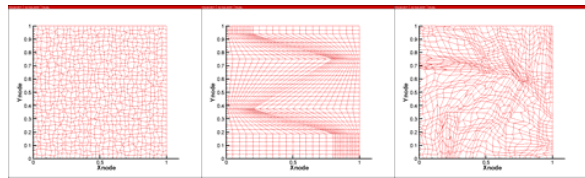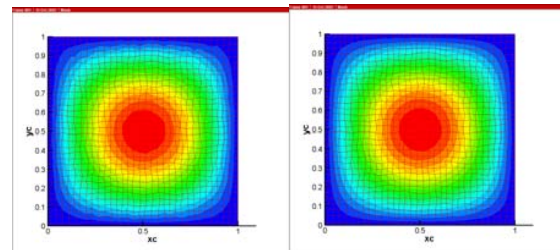

Fig. 1 Random(L), Kershaw(M), Shestakov(R) Mesh figures

TABLE I
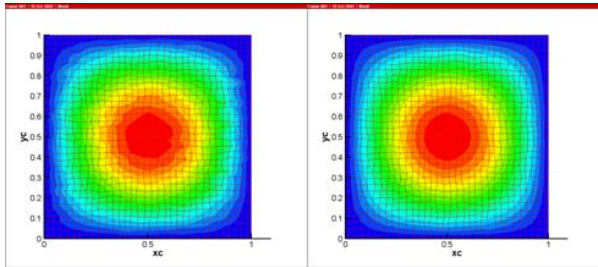NUMERICAL ERROR ANALYSIS AND CPU TIME ON RANDOMMESH

Ratio	Scheme	$\|E^n\|_2$	$\|E^n\|_\infty$	CPU Time (s)
r=1.0	ADM	0.109317E-04	0.217642E-04	0.7343
	NSM	0.194241E-05	0.387848E-05	2.9062
r=10.0	ADM	0.336819E-04	0.670350E-04	9.375E-02
	NSM	0.249547E-04	0.493799E-04	0.750

A. RandomMesh

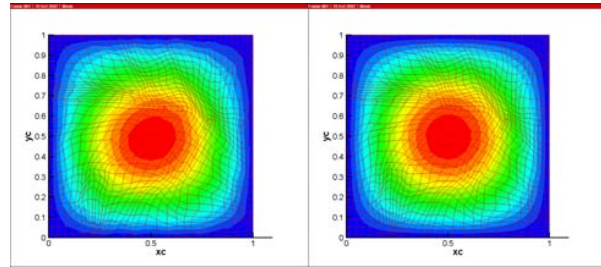ADM (Ratio=1.0)   NSM (Ratio=1.0)

ADM (Ratio=10.0)   NSM (Ratio=10.0)
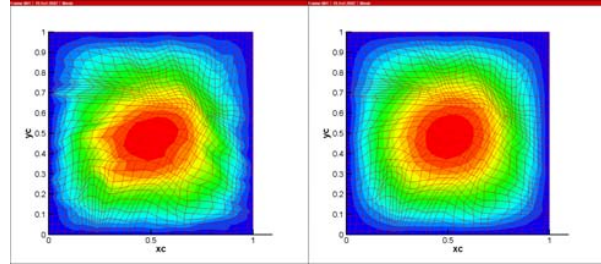
Fig.2 Numerical solution on RandomMesh

B. KershawMesh

TABLE II
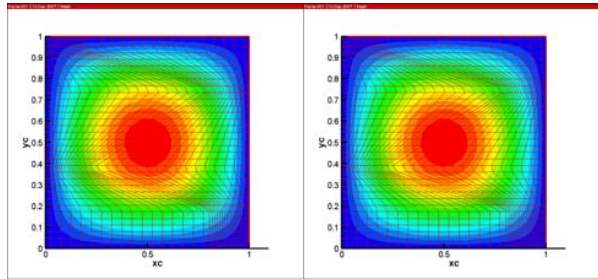NUMERICAL ERROR ANALYSIS AND CPU TIME ON KERSHAWMESH

Ratio	Scheme	$\|E^n\|_2$	$\|E^n\|_\infty$	CPU Time (s)
r=1.0	ADM	0.158973E-03	0.283334E-04	0.7656
	NSM	0.100721E-03	0.180074E-05	6.3750
r=10.0	ADM	0.173888E-02	0.329751E-04	9.375E-02
	NSM	0.136125E-02	0.249642E-02	1.6406

ADM (Ratio=1.0)   NSM (Ratio=1.0)

ADM (Ratio=10.0)   NSM (Ratio=10.0)

Fig. 3 Numerical solution on KershawMesh

C. ShestakovMesh

TABLE III
NUMERICAL ERROR ANALYSIS AND CPU TIME ON SHESTAKOVMESH

Ratio	Scheme	$\|E^n\|_2$	$\|E^n\|_\infty$	CPU Time (s)
r=1.0	ADM	0.812127E-03	0.164725E-04	0. 875
	NSM	0.104844E-05	0.220449E-05	6.359
r=10.0	ADM	0.348986E-04	0.710542E-04	7.812E-02
	NSM	0.132744E-04	0.264603E-04	1.328

ADM (Ratio=1.0)   NSM (Ratio=1.0)

ADM (Ratio=10.0)   NSM (Ratio=10.0)

Fig. 4 Numerical solution on ShestakovMesh

V. CONCLUSION

The theoretical analysis and numerical results show that:

1) The accuracy of NSM is less than ADM on distorted meshes.
2) The CPU time of NSM is more than ADM. When the ratio of the temporal step and the spatial step increase, the numerical errors of both methods increase. Moreover, the CPU time of NSM rise up obviously.
3) The error distribution of NSM connect with the mesh shape. But the error distribution of ADM connect with no only the mesh shape, but also the alternating direction during the process of calculation.
4) NSM have no limit to the type of meshes. But ADM can not be applied to certain types such as triangle meshes.

REFERENCES

[1] P.G. Ciarlet and J.L. Lions, Handbook of Numerical Analysis, Volume I, Finite Difference Methods (Part 1), Splitting and Alternating Direction Methods, G.I. Marchuk (North-Holland, 1990).
[2] Deyuan Li, Hongshou Shui, Minjun Tang, On the finite difference scheme of two-dimensional parabolic equation in a non-rectangular mesh, J. Numer. Methods Comput. Appl. 4 (1980) 217.
[3] Guangwei Yuan, Zhiqiang Sheng, Analysis of accuracy of a finite volume scheme for diffusion equations on distorted meshes. J. Comput. Phys. (2007) 224.