International Journal of Mechanical, Industrial and Aerospace Sciences
ISSN: 2517-9950
Vol:5, No:6, 2011

Study of Coupled Lateral-Torsional Free Vibrations of Laminated Composite Beam: Analytical Approach

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***Abstract*—**In this paper, an analytical approach is used to study the coupled lateral-torsional vibrations of laminated composite beam. It is known that in such structures due to the fibers orientation in various layers, any lateral displacement will produce a twisting moment. This phenomenon is modeled by the bending-twisting material coupling rigidity and its main feature is the coupling of lateral and torsional vibrations. In addition to the material coupling, the effects of shear deformation and rotary inertia are taken into account in the definition of the potential and kinetic energies. Then, the governing differential equations are derived using the Hamilton's principle and the mathematical model matches the Timoshenko beam model when neglecting the effect of bending-twisting rigidity. The equations of motion which form a system of three coupled PDEs are solved analytically to study the free vibrations of the beam in lateral and rotational modes due to the bending, as well as the torsional mode caused by twisting. The analytic solution is carried out in three steps: 1) assuming synchronous motion for the kinematic variables which are the lateral, rotational and torsional displacements, 2) solving the ensuing eigenvalue problem which contains three coupled second order ODEs and 3) imposing different boundary conditions related to combinations of simply, clamped and free end conditions. The resulting natural frequencies and mode shapes are compared with similar results in the literature and good agreement is achieved.

***Keywords*—**Free vibration, Laminated composite beam, Material coupling, State space

I. Introduction

COMPOSITE beams are extensively used in many structures in aerospace, mechanical, civil and mining engineering. Mechanical properties such as high strength/stiffness to weight ratio and excellent fatigue strength of composite materials have increased applications of them in the construction of several structures. Therefore, the vibrational behavior of composite beams has been studied by many researchers in recent years. Free vibration analysis of the simple laminated composite beams started by Abarcar [1], Mansfield [2] and Teh [3] in 1970's. They neglected the effects of shear deformation and rotary inertia in their studies. When cross-sectional dimensions are large or higher frequencies are studied in the vibrational analysis of the beams, the effects of shear deformation and rotary inertia (similar to the Timoshenko beam theory) should be taken into account. Furthermore, low shear moduli of fibrous composites that results in low shear stiffness of the beam, intensifies this requirement. Also in composite beams, because of the ply orientation and stacking sequence of the fibers imbedded in continuous resin media, the effect of bending-torsion material coupling should be considered [1, 4-6]. This effect adds some additional terms and an additional equation to the equations of motion of the metallic Timoshenko beam and so they become more complicated to solve. Using some numerical approaches, the vibrational behavior of composite beams according to Timoshenko beam theory was studied [5, 7-8]; Bank and Kao [9] studied the free and forced vibrations of the thin-walled fiber-reinforced composite material beams using the Timoshenko beam theory. Banerjee and Williams [8] and Banerjee [10] developed the dynamic stiffness matrix method for the problems of free vibration of composite Timoshenko beam and axially loaded composite Timoshenko beam, respectively. Moreover the later is analyzed by Kaya and Ozdemir Ozgumus [11] using the differential transform method (DTM). Also Banerjee [12] represented frequency equation and mode shape formulae for clamped-free boundary conditions composite beam using symbolic computations. As known by authors, the problem of coupled lateral-torsional vibrations of laminated composite beam has been analyzed using various approaches for cantilever beam which is usually used as a model for composite aircraft wing or helicopter blade. In many other applications, a combination of simply supported, clamped and free boundary conditions will take into account. In this paper, the three coupled PDEs of motion for a laminated composite beam subjected to several combinations of boundary conditions are studied using an analytical approach.

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II. Formulation

The differential equations of motion for free vibrations of a laminated composite beam can be easily derived using Hamilton's principle. According to the Hamilton's principle the integration of the Lagrangian of a dynamical system on any arbitrary interval of time is stationary, i.e.

$$\delta\int_{t_1}^{t_2}(U_k - U_p)\,dt = 0 \tag{1}$$

Where U_k and U_p are the kinetic and the potential energies, respectively. Considering w as transverse deflection, θ as bending rotation and ψ as twist angle of the beam, the total kinetic energy U_k of the beam is given by

$$U_k = \tfrac{1}{2}\int_0^L \left[\rho A(w_t)^2 + I_\alpha(\psi_t)^2 + \rho I(\theta_t)^2\right]dx \tag{2}$$

and the total potential energy of the beam is

$$U_p = \tfrac{1}{2}\int_0^L \left[\kappa AG(w_x-\theta)^2 + GJ(\psi_x)^2 + 2K\theta_x\psi_x + EI(\theta_x)^2\right]dx \tag{3}$$

Where, ρ is the density of the material, A is the cross-sectional area, I_α is the polar mass moment of inertia per unit length, I is the second moment of area of the beam cross-section, EI is the bending rigidity, GJ is torsional rigidity, K is bending-torsion coupling rigidity, L is the length of the beam and κAG is shear rigidity of the material (includes shear correction factor) and differentiation with respect to space and time are shown by indices x and t respectively. Substituting Eqs. (2) and (3) into Eq. (1), using integration by parts and simplifying the results, the equations of motion for the laminated composite beam are derived in the following form

$$EI\theta_{xx} + \kappa AG(w_x-\theta) + K\psi_{xx} - \rho I\theta_{tt} = 0 \tag{4}$$

$$\kappa AG(w_{xx}-\theta_x) - \rho A w_{tt} = 0 \tag{5}$$

$$GJ\psi_{xx} + K\theta_{xx} - I_\alpha\psi_{tt} = 0 \tag{6}$$

Beside the above coupled PDEs, geometric and natural BCs must be taken into account. Notice that natural BCs include the values of shear force S, bending moment M and twisting torque T at the boundaries. These quantities are represented by the following expressions

$$S = \kappa AG(w_x-\theta)\,,\quad M = EI\theta_x + K\psi_x\,,\quad T = K\theta_x + GJ\psi_x \tag{7}$$

Now, assuming synchronous motion in which the general shape of the beam does not change with time. Mathematically, this implies that the unknown functions w, θ and ψ are separable in space and time,

$$w(x,t) = W(x)\Phi(t),\ \theta(x,t) = \Theta(x)\Phi(t),\quad \psi(x,t) = \Psi(x)\Phi(t) \tag{8}$$

Substituting these functions into Eqs. (4-6) leads to

$$\Phi_{tt} + \omega^2\Phi = 0 \tag{9}$$

which shows a harmonic motion and the following coupled ODEs

$$EI\Theta_{xx} + \kappa AG(W_x-\Theta) + K\Psi_{xx} + \rho I\omega^2\Theta = 0 \tag{10}$$

$$\kappa AG(W_{xx}-\Theta_x) + \rho A\omega^2 W = 0 \tag{11}$$

$$GJ\Psi_{xx} + K\Theta_{xx} + I_\alpha\omega^2\Psi = 0 \tag{12}$$

These ODEs have the following non-dimensional form

$$s\tilde{\theta}_{\tilde{x}\tilde{x}} + \tilde{w}_{\tilde{x}} + (brs-1)\tilde{\theta} + k_b s\tilde{\psi}_{\tilde{x}\tilde{x}} = 0 \tag{13}$$

$$\tilde{w}_{\tilde{x}\tilde{x}} - \tilde{\theta}_{\tilde{x}} + bs\tilde{w} = 0 \tag{14}$$

$$\tilde{\psi}_{\tilde{x}\tilde{x}} + k_t\tilde{\theta}_{\tilde{x}\tilde{x}} + a\tilde{\psi} = 0 \tag{15}$$

Where the non-dimensional parameters are defined as

$$\begin{gathered}\tilde{x} = \frac{x}{L}\,,\quad \tilde{w} = \frac{W}{L}\,,\quad \tilde{\theta} = \Theta\,,\quad \tilde{\psi} = \Psi \\ k_t = \frac{K}{GJ}\,,\quad k_b = \frac{K}{EI}\,,\quad a = \frac{I_\alpha\omega^2L^2}{GJ}\,,\quad b = \frac{\rho A\omega^2L^4}{EI} \\ r = \frac{I}{AL^2}\,,\quad s = \frac{EI}{\kappa AGL^2}\end{gathered} \tag{16}$$

Consequently, the non-dimensional shear force, bending moment and twisting torque are

$$\tilde{S} = \tilde{w}_{\tilde{x}} - \tilde{\theta}\,,\qquad \tilde{M} = \tilde{\theta}_{\tilde{x}} + k_b\tilde{\psi}_{\tilde{x}}\,,\qquad \tilde{T} = \tilde{\theta}_{\tilde{x}} + \tilde{\psi}_{\tilde{x}} \tag{17}$$

III. ANALYTIC SOLUTION

Free vibration analysis of the beam deals with the solution of an eigenvalue problem which consists of Eqs. (10-12 or 13-15) with corresponding boundary conditions of the beam. To solve this problem the state space approach is used by introducing the state variables vector $\tilde{Q}$,

$$\tilde{Q} = \begin{bmatrix}\tilde{w} & \tilde{\theta} & \tilde{\psi} & \tilde{w}_{\tilde{x}} & \tilde{\theta}_{\tilde{x}} & \tilde{\psi}_{\tilde{x}}\end{bmatrix}^T \tag{18}$$

Via this vector, the equations of motion can be simplified in the state space as

$$\tilde{Q}_{\tilde{x}} = \tilde{A}\tilde{Q}\,,\qquad \tilde{A} = \begin{bmatrix}\tilde{0} & \tilde{I} \\ \tilde{\Lambda} & \tilde{\Gamma}\end{bmatrix} \tag{19}$$

where, $\tilde{Q}_{\tilde{x}}$ is stand for the derivative of $\tilde{Q}$ with respect to $\tilde{x}$ (the non-dimensional coordinate), $\tilde{I}$ is the identity matrix and the non-zeros components of 3×3 matrices $\tilde{\Lambda}$ and $\tilde{\Gamma}$ are

$$\begin{gathered}\tilde{\Lambda}_{11} = -bs\ ,\tilde{\Lambda}_{22} = \frac{brs-1}{s(k_bk_t-1)}\ ,\tilde{\Lambda}_{23} = \frac{a\,k_b}{1-k_bk_t} \\ \tilde{\Lambda}_{32} = \frac{k_t(brs-1)}{s(1-k_bk_t)}\ ,\tilde{\Lambda}_{33} = \frac{a}{k_bk_t-1} \\ \tilde{\Gamma}_{12} = 1\ ,\tilde{\Gamma}_{21} = \frac{1}{s(k_bk_t-1)}\ ,\tilde{\Gamma}_{31} = \frac{k_t}{s(k_bk_t-1)}\end{gathered} \tag{20}$$

The equations of motion which are transformed to the state space as a system of first order ODEs can be solved analytically. It easy to show that the analytical solution is equal to

$$\tilde{Q} = \sum_{i=1}^{6} c_i\tilde{v}_i e^{\lambda_i\tilde{x}} \tag{21}$$

Where, λ_is and $\tilde{v}_i$s which are functions of ω indicate the eigenvalues and the eigenvectors of the system respectively. In addition, unknown coefficients c_is can be determined by applying boundary conditions. It should be noticed that at each end of the beam three boundary conditions encounter the problem. These BCs are any triple proper-combinations of the geometric or the natural BCs. By a simple calculation the total number of possible BCs is thirty six. In this paper, among these possible BCs only a few cases are studied and the complete procedure to find the natural frequencies and mode shapes is briefly explained for a cantilever beam (the same route is applicable for other cases). In the case of cantilever beam the boundary conditions are given by

$$\begin{array}{lll} \tilde{w}(0)=0\,, & \tilde{\theta}(0)=0\,, & \tilde{\psi}(0)=0 \\ \tilde{S}(1)=0\,, & \tilde{M}(1)=0\,, & \tilde{T}(1)=0 \end{array} \tag{22}$$

and the following system of algebraic equations is carried out by substitution of these end conditions to the definition of the state variables (Eq. 18), shear force, bending moment, twisting torque (Eq. 17) and the general solution (Eq. 21)

$$\begin{bmatrix} \tilde{v}_{11} & \tilde{v}_{12} & \tilde{v}_{13} & \tilde{v}_{14} & \tilde{v}_{15} & \tilde{v}_{16} \\ \tilde{v}_{21} & \tilde{v}_{22} & \tilde{v}_{23} & \tilde{v}_{24} & \tilde{v}_{25} & \tilde{v}_{26} \\ \tilde{v}_{31} & \tilde{v}_{32} & \tilde{v}_{33} & \tilde{v}_{34} & \tilde{v}_{35} & \tilde{v}_{36} \\ f_1 & f_2 & f_3 & f_4 & f_5 & f_6 \\ g_1 & g_2 & g_3 & g_4 & g_5 & g_6 \\ d_1 & d_2 & d_3 & d_4 & d_5 & d_6 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{Bmatrix} = \tilde{0} \tag{23}$$

$$f_i = (\tilde{v}_{5i} + k_b \tilde{v}_{6i}) e^{\lambda_i}\,, \quad i = 1,2,...,6$$
$$g_i = (\tilde{v}_{4i} - \tilde{v}_{2i}) e^{\lambda_i}\,, \quad i = 1,2,...,6$$
$$d_i = (k_t \tilde{v}_{5i} + \tilde{v}_{6i}) e^{\lambda_i}\,, \quad i = 1,2,...,6$$

All information that are required in the free vibration analysis can be found from the above system of algebraic equations. This system has non-zero solutions if and only if the determinant of its coefficient matrix is equal to zero. This condition gives the characteristic equation and its roots determine the natural frequencies. Consequently, mode shapes can be obtained by substituting the corresponding natural frequencies in the coefficient matrix and solving the problem for unknown c_i s.

IV. RESULTS

The coupled lateral-torsional free vibration of the laminated composite cantilever beam was analyzed by Banerjee [13]. To validate and confirm the accuracy of solution procedure, the numerical results are calculated for the glass-epoxy composite beam with those data used in Ref. [13] (These data are represented in table 1). For the cantilever beam, comparison of the results with Ref. [13] shows a good agreement. Indeed, the two approaches which are introduced here and Ref. [13] have the same nature. The first three normalized mode shapes of the cantilever beam are shown in Fig. 1. Similar figures are presented in Ref [13] with slight differences. Also, the same method is applied for free vibration analyses of different types of boundary conditions which are free-free (Fig. 2), clamped-clamped (Fig. 3) and pseudo-simply supported (Fig. 4). In addition, the first four natural frequencies for these cases and some other types of BCs are presented in Table 2. For semi-definite system only non-zero mode shapes are shown (e.g. in the case of Free-Free ends).

TABLE I
PHYSICAL PROPERTIES OF GLASS-EPOXY COMPOSITE BEAM WITH ALL FIBER ANGLES SET TO +15° AND CROSS-SECTIONAL DIMENSIONS: THICKNESS (H=3.18 MM) & WIDTH (B=12.7 MM)

$EI(Nm^2)$	$GJ(Nm^2)$	$K(Nm^2)$	$\rho A(kg/m^3)$	$I_\alpha(kgm)$	$\kappa AG(N)$	$L(mm)$
0.2865	0.1891	0.1143	0.0544	0.777×10^{-6}	6343.3	190.5

TABLE II
THE FIRST FOUR NATURAL FREQUENCIES FOR VARIOUS BOUNDARY CONDITIONS

BCs	$\omega_1(rad/s^2)$	$\omega_2(rad/s^2)$	$\omega_3(rad/s^2)$	$\omega_4(rad/s^2)$
C-F	193.19	1192.42	3259.65	4073.21
C-C	1203.4998	3223.9494	6100.0963	8125.0104
F-F	1220.3711	3300.8862	6262.8822	8183.6200
S-S	540.7359	2128.7098	4669.6608	8030.4573
at ξ=0, 1: w=0, θ=0, T=0	1203.4877	3224.0146	6100.0627	8129.6382
at ξ=0, 1: w=0, M=0, ψ=0	602.5454	2116.4747	4715.4566	7304.1033
at ξ=0, 1: S=0, θ=0, T=0	541.0425	2128.5001	4671.3989	7903.5821
at ξ=0, 1: S=0, M=0, ψ=0	1339.1478	3262.8299	6406.3463	7497.8193
at ξ=0, 1: S=0, θ=0, ψ=0	540.7359	2128.7098	4669.6608	8030.4573

IV. CONCLUSIONS

The coupled lateral-torsional vibrations of laminated composite beam are studied using an analytical approach. This approach is based on the state space description of the vibrational system and its solution. The effect of boundary conditions on the vibrational phenomena is investigated. The results show that how the torsional motion is affected by the lateral vibration. In the formulation of the problem, bending-twisting material coupling, the effects of shear deformation and rotary inertia are taken into account. The natural frequencies and mode shapes for cantilever beam are compared with similar results in the literature and good agreement is achieved. In this paper, the results for some other BCs which are listed in table (2) are presented for the first time by an analytical approach. Focusing on the data in table 2

International Journal of Mechanical, Industrial and Aerospace Sciences
ISSN: 2517-9950
Vol:5, No:6, 2011

indicate that the increase/decrease of the natural frequencies are compatible with the nature of BCs. Although the numerical results for the composite beam with rectangular cross-section are represented, the approach can be similarly applied to other cross-sections such as box or airfoil.

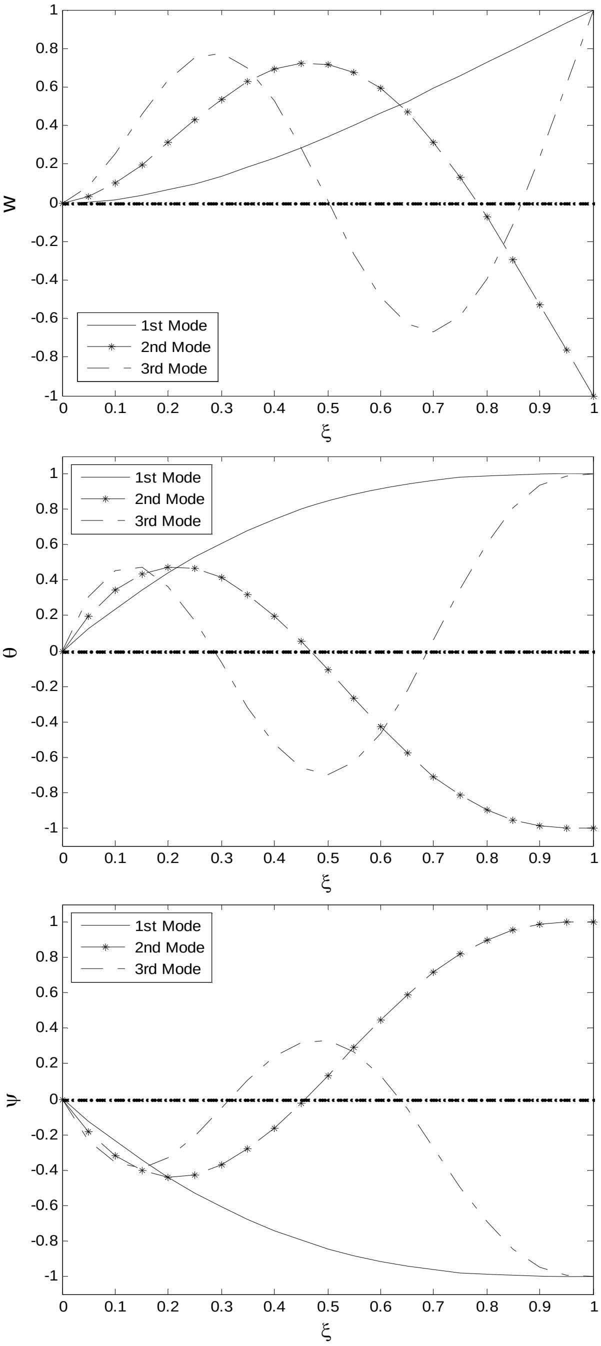


Fig. 1 Mode shapes of cantilever beam

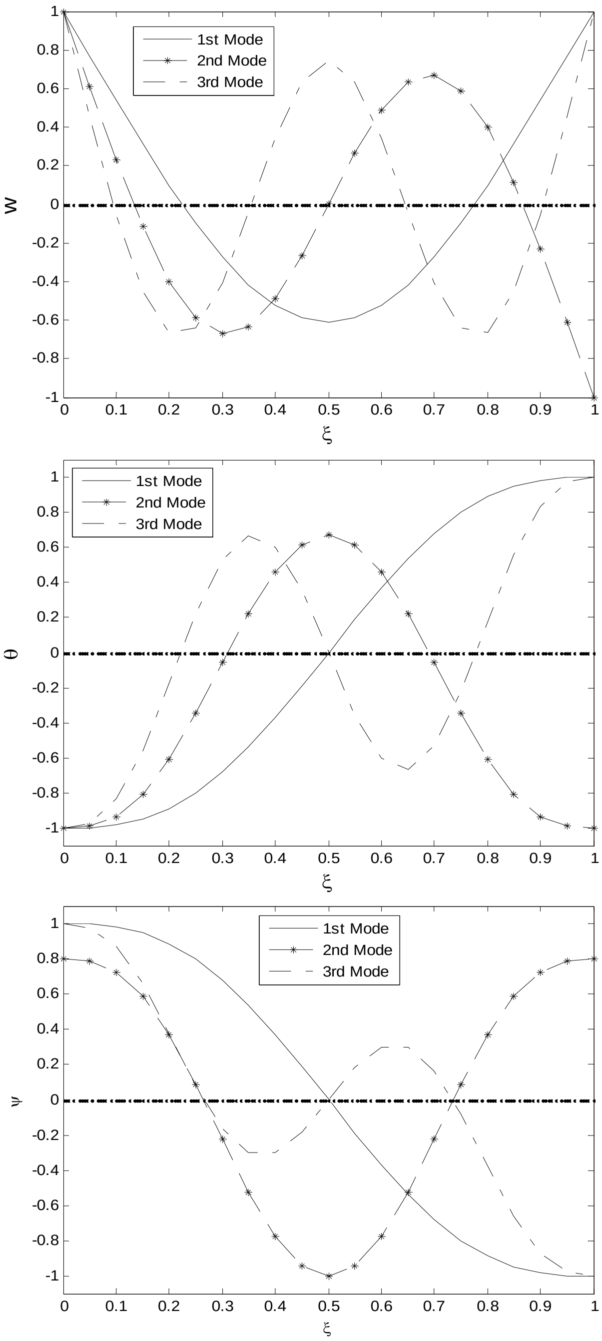


Fig. 2 Mode shapes of Free-Free beam

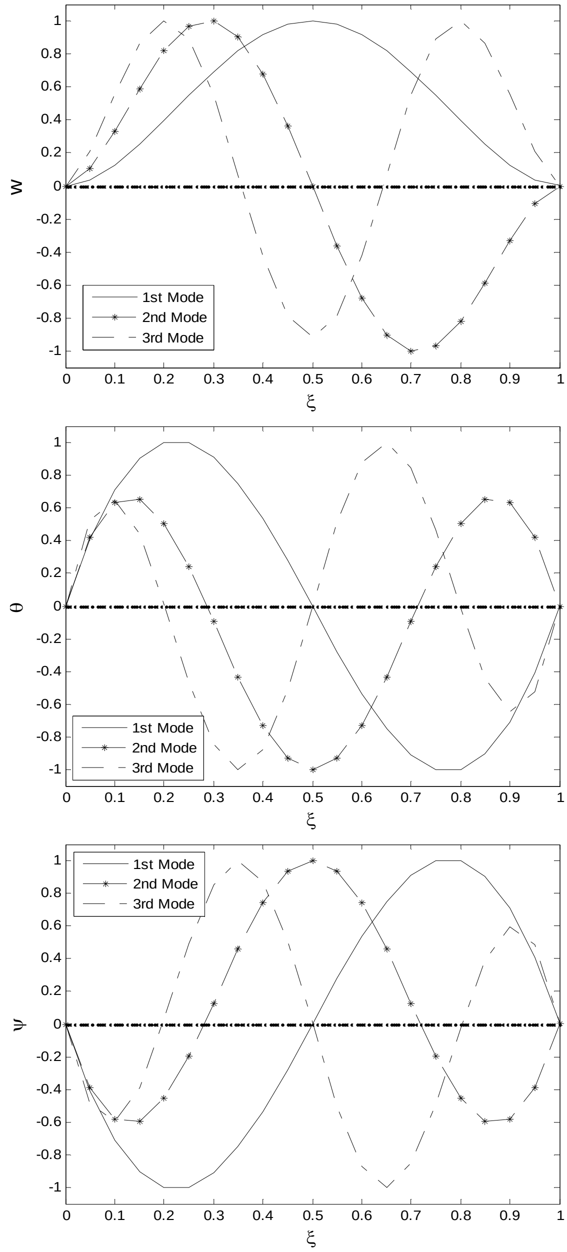
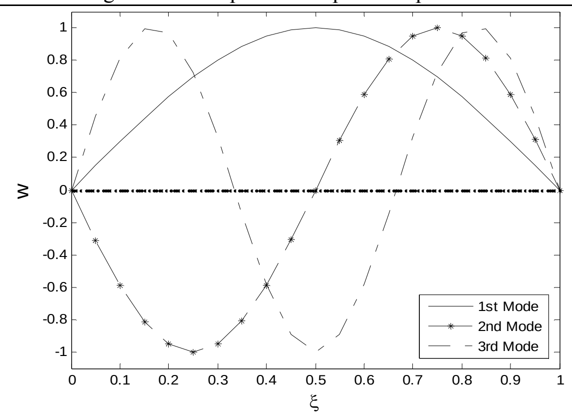


Fig. 3 Mode shapes of clamped-clamped beam



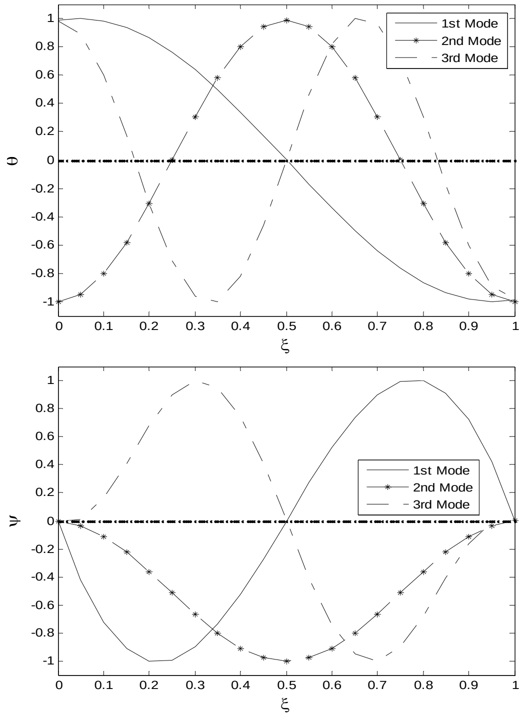


Fig. 4 Mode shapes of beam with BCs: at ξ=0, 1: w=0, M=0, ψ=0

REFERENCE

[1] R.B. Abarcar, P.F. Cunniff, "The vibration of cantilever beams of fiber reinforced material," *Journal of Composite Materials*, vol. 6, pp. 504-517, 1972.

[2] E.H. Mansfield, A.J. Sobey, "The fibre composite helicopter blade, part 1: stiffness properties, part 2: prospects for aeroelastic tailoring," *Aero Quart*., vol. 30, pp. 413–49, 1979.

[3] K.K. Teh, C.C. Huang, "The vibrations of generally orthotropic beams, a finite element approach," *J. Sound. Vib*., vol. 62, pp. 195–206, 1979.

[4] K. Chandrashekhara, K. Krishanamurthy, S. Roy, "Free vibration of composite beams including rotatory inertia and shear deformation," *Composite Structures*, vol. 14, pp. 269-279, 1990.

[5] H. Abramovich, "Shear deformation and rotatory inertia effects of vibrating composite beams," *Composite Structures*, vol. 20, pp. 165-173, 1992.

[6] V. Yildirim, "Rotary inertia, axial and shear deformation effects on the in-plane natural frequencies of symmetric cross-ply laminated circular arches," *J. Sound. Vib.*, vol. 224, no. 4, pp. 575-589, 1999.

[7] L.S. Teoh, C. C. Huang, "The vibration of beams of fibre reinforced material," *J. Sound Vib*. vol. 51, pp. 467–73, 1977.

[8] J.R. Banerjee, F. W. Williams, "Exact dynamics stiffness matrix for composite Timoshenko beams with applications," *J. Sound Vib.*, vol. 194, no. 4, pp. 573-585, 1996.

[9] L.C. Bank, C.H. Kao, "Dynamic response of composite beam, in: D. Hui, J.R. Vinson, (Eds.), Recent Advances in the Macro- and Micro-Mechanics of Composite Material Structures," AD 13, The Winter Annual Meeting of the 1975 ASME, Chicago, IL, November–December 1977.

[10] J.R. Banerjee, "Free vibration of axially loaded composite Timoshenko beams using the dynamic stiffness matrix method," *Computers and Structures,* vol. 69, pp. 197-208, 1998.

International Journal of Mechanical, Industrial and Aerospace Sciences
ISSN: 2517-9950
Vol:5, No:6, 2011

[11] M.O. Kaya, O. Ozdemir Ozgumus, "Flexural–torsional-coupled vibration analysis of axially loaded closed-section composite Timoshenko beam by using DTM," *J. Sound Vib*., vol. 306, pp. 495–506, 2007.
[12] J.R. Banerjee, "Frequency equation and mode shape formulae for composite Timoshenko beams," *Composite Structures,* vol. 51, pp. 381-388, 2001.