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Iterative Methods for An Inverse Problem

Minghui Wang and Shanrui Hu

Abstract**—**An inverse problem of doubly center matrices is discussed. By translating the constrained problem into unconstrained problem, two iterative methods are proposed. A numerical example illustrate our algorithms.

Keywords**—**doubly center matrix, electric network theory, iterative methods, least-square problem.

I. INTRODUCTION

MATRIX inverse problem is an important field. Doubly center matrices have widely applications in the electric network theory, in which are called the indefinite admittance matrices, see [1]-[4]. In recent years, some results have been obtained for solving the inverse problem $AX = B$ and the existence of the solution has been discussed in [5]. However, the matrix X and B are often derived from experiment and measure, we cannot make sure that the problem has the exact solution. So, we have to discuss its least-square solution.

Definition 1:[5] A matrix $A = (a_{ij}) \in R^{n\times n}$ is called a doubly center matrix if the sum of all the elements in each row and the sum of all the elements in each column are equal to zero, i.e.,

$$\sum_{i=1}^{n} a_{ij} = 0 (j = 1,2,\cdots,n), \qquad \sum_{j=1}^{n} a_{ij} = 0 (i = 1,2,\cdots,n).$$

The set of all such matrices is denoted by $DCR^{n\times n}$. If A is also a real symmetric matrix meanwhile, it is called a symmetric doubly center matrix. The set of all such matrices is denoted by $DCSR^{n\times n}$.

Suppose that an electric network system has n terminals. The input currents and the corresponding voltages are $i_1, i_2, \cdots i_n$ and $u_1, u_2, \cdots u_n$, respectively. Here, we denote $i = (i_1, i_2, \cdots i_n)^T$ as current vector and $u = (u_1, u_2, \cdots u_n)^T$ as voltage vector. There exists a linear relationship between them and we express it as follows

$$i = Au .$$

According to the Kirchoff electric current law and the relative law of potentials, A satisfy the following relations: $Ae_n = 0, e_n^T A = 0$ (i.e. $A \in DCR^{n\times n}$). A is called the indefinite admittance matrix in electric network theory. If phase shifter branches don't exist, A is a symmetric matrix. Each element of A has the dimension of admittance and we can get the parameter by connecting one terminal to the voltage source and shorting other terminals with the reference node. For example, if keeping the voltage source of terminal k, and short the others, we get

$$a_{jk} = \frac{i_j}{u_k}\Big|_{u_j=0}, \qquad j \neq k, \qquad j = 1,2,\cdots,n .$$

Affected by the measurement error or random disturbance, we can hardly get a doubly center matrix of A. Here, we consider some iterative methods to solve the problems.

In this paper, we will discuss the following problem

$$\|AX - B\| = \min \qquad \text{(problem 1)}$$

where $X, B \in R^{n\times m}$ are given, $A \in DCR^{n\times n}$ (doubly center matrix) is to find.

The notations used in this paper can be summarized as follows. The set of all n dimensional column vectors is written as R^n and the identity matrix in $R^{n\times n}$ is written as I_n; The set of all real matrices is denoted by $R^{n\times n}$. $tr(A)$ and A^+ represent the trace and the Moore-Penrose pseudo-inverse of A, respectively. $A \otimes B$ represent the Kronecker product of A and B and $vec(A)$ means to straighten the matrix A according to its columns to form a vector. In addition, the Frobenius norm of A is denoted by $\|A\|$. We define the inner product $(A,B) = \sqrt{tr(B^T A)}$, and thus $\|A\| = \sqrt{tr(A^T A)}$.

II. ITERATIVE METHODS FOR PROBLEM 2

The general form of the solutions has been given and the necessary and sufficient conditions for the solvability of the problem 1 have been discussed in [5]. However, the general form is complex and we have to compute P^+ using singular value decomposition. When it comes to large matrices, it's difficult to get the solutions. Here, we consider iterative methods.

Zhou and Wu have discussed the structure of $DCR^{n\times n}$ and $DCSR^{n\times n}$ in [5]. Let us review it as follows.

Lemma 1:[5] Let $e_n = (1,1,\cdots 1)^T \in R^n$, then $A \in DCR^{n\times n}$ if and only if

$$Ae_n = 0, \qquad e_n^T A = 0 .$$

Proof. Use Definition 1.1, we can get the result.

Lemma 2:[6] Let $A \in R^{n\times n}, B \in R^{n\times m}, C \in R^{m\times l}, D \in R^{n\times l}$, then linear matrix equations $AX = B, XC = D$ have common solutions if and only if each equation has solution and $AD = BC$. When the common solutions exist, and a special solution is

$$X_c = A^+B + DC^+ - A^+ADC^+ ,$$

the common solutions are

$$X = X_c + (I_n - A^+A)Y(I_m - CC^+), \qquad \forall Y \in R^{n\times m} .$$

Lemma 3:[5] Let $e_n = (1,1,\cdots 1)^T \in R^n$, then

M. Wang and S. Hu are with the Department of Mathematics, Qingdao University of Science and Technology, Shandong 266061, China, e-mail: (mhwang@yeah.net, mhwang@qust.edu.cn).
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$$e_n^+ = \frac{1}{n} e_n^T, \qquad (I_n - \frac{1}{n} e_n e_n^T)^+ = I_n - \frac{1}{n} e_n e_n^T,$$

$$(I_n - \frac{1}{n} e_n e_n^T)^2 = I_n - \frac{1}{n} e_n e_n^T.$$

Proof. Use of the Moore-Penrose pseudo-inverse, we can easily get the results.

Theorem 1:[5] Suppose $A \in DCR^{n\times n}$, $e_n = (1,1,\cdots 1) \in R^n$. Then

$$A = (I_n - \frac{1}{n} e_n e_n^T) Y (I_n - \frac{1}{n} e_n e_n^T). \qquad (1)$$

Especially, when $Y \in R^{n\times n}$ is a symmetric matrix, $A \in DCSR^{n\times n}$.

Proof. By Lemma 1-3, we can get (1). Especially, when Y is symmetric,

$$A^T = (I_n - \frac{1}{n} e_n e_n^T) Y^T (I_n - \frac{1}{n} e_n e_n^T)$$
$$= (I_n - \frac{1}{n} e_n e_n^T) Y (I_n - \frac{1}{n} e_n e_n^T) = A$$

holds, i.e., $A \in DCSR^{n\times n}$.

Now we consider problem 1. Let

$$S = S^T = I_n - \frac{1}{n} e_n e_n^T, L = SX = (I_n - \frac{1}{n} e_n e_n^T) X \qquad (2)$$

then, we can rewrite the problem as follows.

Theorem 2: The solution of problem 1 is

$$A = SYS.$$

Here, $Y \in R^{n\times n}$ is the solution of the least squares problem

$$\|SYL - B\| = \min \qquad \text{(problem 2)},$$

where $S \in R^{n\times n}, L \in R^{n\times m}, B \in R^{n\times m}$ are given, $Y \in R^{n\times n}$ is to find.

It's easy to prove the theorem when we replace A with (1) and (2). By this way, we can translate the constrained problem into the unconstrained problem. We only need to discuss problem 2.

A. CG-like Method

Y. Peng has proposed an iterative method to find the solution of $AXB = C (X \in R^{n\times m})$ in his PhD thesis [7]. We will use his idea to solve our problem.

Lemma 4: Problem 2 is equivalent to the linear matrix equation

$$S^T SYLL^T = S^T BL^T, \qquad Y \in R^{n\times n} \qquad (3)$$

Proof. By $vec(SYL) = (L^T \otimes S) vec(Y)$, we get that problem 2 is equivalent to the least squares problem

$$\|(L^T \otimes S) vec(Y) - vec(B)\|_2 = \min \qquad (4)$$

Considering the normal equation of (4), we get

$$(L^T \otimes S)^T (L^T \otimes S) vec(Y) = (L^T \otimes S)^T vec(B)$$
$$\Leftrightarrow (L \otimes S^T)(L^T \otimes S) vec(Y) = (L \otimes S^T) vec(B)$$
$$\Leftrightarrow (LL^T \otimes S^T S) vec(Y) = (L \otimes S^T) vec(B)$$
$$\Leftrightarrow S^T SYLL^T = S^T BL^T.$$

Now, we apply the CG-like method to (3), then we get the following algorithm.

Algorithm 1:

(1). Initialization

$R_1 = SBL^T - SSY_1 LL^T$

$P_1 = S(SR_1 L)L^T$

$Q_1 = P_1$.

(2). Iteration. For $i = 1, 2, \cdots$

$$Y_{i+1} = Y_i + \frac{\|R_i\|^2}{\|Q_i\|^2} Q_i$$

$R_{i+1} = SBL^T - SSY_{i+1} LL^T$

$P_{i+1} = SSR_{i+1} LL^T$

$$Q_{i+1} = P_{i+1} - \frac{trace(P_{i+1}^T Q_i)}{\|Q_i\|^2} Q_i.$$

(3). Check convergence.

Remark 1. Algorithm 1 will terminate in finite iterations and we can get the minimum norm solution of problem 2, see [7] for details.

Remark 2. The minimum norm solution of problem 1 is $A = SYS$, where Y can be obtained by Algorithm 1.

B. LSQR Method

In 1982, Paige and Sauders proposed the LSQR method [8] to solve the following problem

$$\|Mx - f\|_2 = \min \qquad (5)$$

where $M \in R^{m\times n}, f \in R^m$ are given, $x \in R^n$ is to find.

In order to use the LSQR method, we first change problem 2 into a similar form with (5). It is easy to obtain the following result.

Theorem 3: Problem 2 is equivalent to the following problem.

$$\|My - b\|_2 = \min \qquad \text{(problem 3)}$$

where $M = L^T \otimes S, y = vec(Y), b = vec(B)$.

Let us denote $mat(x) = mat(vec(X)) = X$, that is, $mat(x)$ is the inverse of $vec(X)$. Then we get

$$mat(Mv) = mat(L^T \otimes S \cdot v) = mat(L^T \otimes S \cdot vec(V)) = SVL$$
$$mat(M^T u) = mat((L^T \otimes S)^T u) = mat(L \otimes S^T \cdot vec(U)) = S^T UL^T = SUL^T$$

Now we write the LSQR algorithm to problem 3 as follows.

Algorithm 2:

(1). Initialization.

$Y_0 = 0$, $\beta_1 = \|B\|$, $U_1 = B / \beta_1$

$\bar{V}_1 = SU_1 L^T$, $\alpha_1 = \|\bar{V}_1\|$, $V_1 = \bar{V}_1 / \alpha_1$

$H_1 = V_1$, $\bar{\varsigma} = \beta_1$, $\bar{\rho} = \alpha_1$

(2). Iteration. For $i = 1, 2, \cdots$

$\bar{U}_{i+1} = SV_i L - \alpha_i U_i$

$\beta_{i+1} = \|\bar{U}_{i+1}\|$, $U_{i+1} = \bar{U}_{i+1} / \beta_{i+1}$

$\bar{V}_{i+1} = SU_{i+1} L^T - \beta_{i+1} V_i$

$\alpha_{i+1} = \|\bar{V}_{i+1}\|$, $V_{i+1} = \bar{V}_{i+1} / \alpha_{i+1}$

$$\rho_i = \sqrt{\bar{\rho}_i^2 + \beta_{i+1}^2}$$

$c_i = \bar{\rho}_i / \rho_i$, $s_i = \beta_{i+1} / \rho_i$, $\theta_{i+1} = s_i \alpha_{i+1}$

$\bar{\rho}_{i+1} = -c_i \alpha_{i+1}$, $\varsigma_i = c_i \bar{\varsigma}_i$, $\bar{\varsigma}_{i+1} = s_i \bar{\varsigma}_i$

$Y_i = Y_{i-1} + (\varsigma_i / \rho_i) H_i$

$H_{i+1} = V_{i+1} - (\theta_{i+1} / \rho_i) H_i$.

(3). Check convergence.

Remark 3. If $AX = B$ is consistent, we can get the minimum norm solution of problem 3,see [8] for details.

Remark 4. The minimum norm solution of problem 1 is $A = SYS$, where Y can be obtained by Algorithm 2.

III. Numerical Example

In this section, we will use an example to illustrate our algorithm. All the tests are performed by MATLAB 7.0 and the initial iterative matrices are chosen as zero matrices in suitable size.

Example. When designing the network system, three groups of energizing voltages and response currents are given (see TABLE I). Find the indefinite admittance matrix.

TABLE I
Three Groups Of Energizing Voltages And Response Currents

n terminals	1	2	3	4	5
u_1	1	2	1	2	2
I_1	0.8000	-0.6000	-0.6000	2.4000	-2.0000
u_2	2	2	1	3	5
I_2	3.0000	-2.4000	0.6000	8.6000	-9.8000
u_3	9	5	3	2	8
I_3	-5.0000	2.4000	5.4000	2.4000	-5.2000

According to TABLE I, we have

$$X = \begin{bmatrix} 1 & 2 & 9 \\ 2 & 2 & 5 \\ 1 & 1 & 3 \\ 2 & 3 & 2 \\ 2 & 5 & 8 \end{bmatrix}, B = \begin{bmatrix} 0.8000 & 3.0000 & -5.0000 \\ -0.6000 & -2.4000 & 2.4000 \\ -0.6000 & 0.6000 & 5.4000 \\ 2.4000 & 8.6000 & 2.4000 \\ -2.0000 & -9.8000 & -5.2000 \end{bmatrix}.$$

We use Algorithm 1 and Algorithm 2 to compute the indefinite admittance matrix. After iterations, we get the minimum norm solution

$$A = \begin{bmatrix} -0.9714 & -0.9143 & 0.1714 & 0.9857 & 0.7286 \\ 0.6000 & 0.6000 & -0.0000 & -0.6000 & -0.6000 \\ 0.6571 & -0.4286 & -0.0571 & -0.5286 & 0.3571 \\ -1.0572 & -0.5714 & -1.3429 & 0.8286 & 2.1429 \\ 0.7714 & 1.3143 & 1.2286 & -0.6857 & -2.6286 \end{bmatrix}$$

Furthermore, if we consider the normal equation of problem 1, we get $AXX^T = BX^T$ (similar to the proof of Lemma 4). And we denote

$$\xi_k = \log 10(\|BX^T - AXX^T\|)$$

as the residual of Algorithm 1 and Algorithm 2 after k steps. Then, we compare the two algorithms in Fig 1.

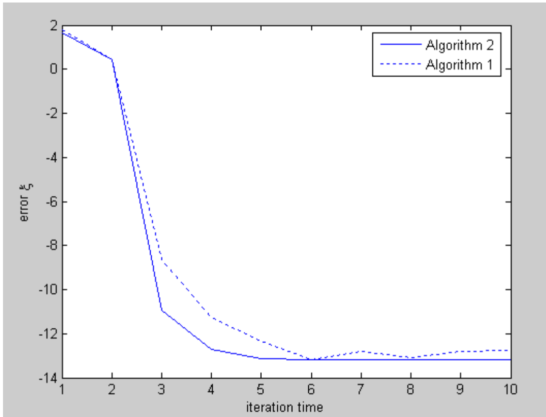

Fig. 1 The Residual Curves Generated By CG Method And LSQR Method

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Minghui Wang is Ph.D, Professor of the Department of Mathematics in Qingdao University of Science and Technology. His research interests covers numerical algebra and matrix theory.