International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:4, No:8, 2010

Induced Acyclic Graphoidal Covers in a Graph

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Abstract—An induced acyclic graphoidal cover of a graph G is a collection ψ of open paths in G such that every path in ψ has atleast two vertices, every vertex of G is an internal vertex of at most one path in ψ, every edge of G is in exactly one path in ψ and every member of ψ is an induced path. The minimum cardinality of an induced acyclic graphoidal cover of G is called the induced acyclic graphoidal covering number of G and is denoted by $\eta_{ia}(G)$ or η_{ia}. Here we find induced acyclic graphoidal cover for some classes of graphs.

Keywords—Graphoidal cover, Induced acyclic graphoidal cover, Induced acyclic graphoidal covering number.

I. Introduction

A graph is a pair $G = (V, E)$, where V is the set of vertices and E is the set of edges. Here, we consider only nontrivial, simple, finite and connected graphs. The order and size of G are denoted by p and q respectively. The concept of graphoidal cover was introduced by B.D. Acharya and E. Sampathkumar [1] and the concept of induced acyclic graphoidal cover was introduced by S. Arumugram [4]. The reader may refer [3], [5] and [6] for the terms not defined here.

Definition I.1. *[1] A graphoidal cover of a graph G is a collection ψ of (not necessarily open) paths in G satisfying the following conditions:*
(i) Every path in ψ has atleast two vertices.
(ii) Every vertex of G is an internal vertex of at most one path in ψ.
(iii) Every edge of G is in exactly one path in ψ.

The minimum cardinality of a graphoidal cover of G is called the graphoidal covering number of G and is denoted by $\eta(G)$.

Definition I.2. *[4] An induced graphoidal cover of a graph G is a collection ψ of (not necessarily open) paths in G satisfying the following conditions:*
(i) Every path in ψ has atleast two vertices.
(ii) Every vertex of G is an internal vertex of at most one path in ψ.
(iii) Every edge of G is in exactly one path in ψ.
(iv) Every member of ψ is an induced cycle or an induced path.

The minimum cardinality of an induced graphoidal cover of G is called the induced graphoidal covering number of G and is denoted by $\eta_i(G)$ or η_i.

Let ψ be a graphoidal cover of G and Z be a cycle of G. Then for any edge e of Z, the family $\psi_1 = (\psi \setminus \{Z\}) \cup \{Z - e\} \cup \{e\}$ is again a graphoidal cover of G. Thus one can successively break up every cycle member of ψ into paths eventually yielding a graphoidal cover ψ_1 of G that has only path members. Motivated by this observation, S. Arumugam and Suresh Suseela [2] introduced the concept of acyclic graphoidal cover and acyclic graphoidal covering number of a graph.

Definition I.3. *[2] A graphoidal cover ψ of a graph G is called an acyclic graphoidal cover if every member of ψ is a path. The minimum cardinality of an acyclic graphoidal cover of G is called the acyclic graphoidal covering number of G and is denoted by $\eta_a(G)$ or η_a.*

Definition I.4. *[4] A graphoidal cover ψ of a graph G is called an induced acyclic graphoidal cover if every member of ψ is an induced path. The minimum cardinality of an induced acyclic graphoidal cover of G is called the induced acyclic graphoidal covering number of G and is denoted by $\eta_{ia}(G)$ or η_{ia}.*

Definition I.5. *Let ψ be a collection of internally edge disjoint paths in G. A vertex of G is said to be an internal vertex of ψ if it is an internal vertex of some path in ψ, otherwise it is called an external vertex of ψ.*

II. Main Results

The following result for graphoidal covering number also holds for induced acyclic graphoidal covering number.

Theorem II.1. *[3] For any induced acyclic graphoidal cover ψ of a (p,q)- graph G, let t_ψ denote the number of external vertices of ψ and let $t = min\ t_\psi$, where the minimum is taken over all induced acyclic graphoidal covers ψ of G then $\eta_{ia}(G) = q - p + t$.*

Corollary II.2. *For any graph G, $\eta_{ia}(G) \geq q - p$. Moreover, the following are equivalent*
(i) $\eta_{ia}(G) = q - p$.
(ii) There exists an induced acyclic graphoidal cover of G without external vertices.
(iii) There exists a set Q of internally disjoint and edge disjoint induced acyclic graphoidal path without exterior vertices (From such a set Q of paths, the required induced acyclic graphoidal cover can be obtained by adding the edges which are not covered by the paths in Q).

Corollary II.3. *If there exists an induced acyclic graphoidal cover ψ of a graph G such that every vertex of G with degree atleast two is internal to ψ, then ψ is a minimum induced acyclic graphoidal cover of G and $\eta_{ia}(G) = q - p + n$, where n is the number of pendant vertices of G.*

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Corollary II.4. *Since every graphoidal cover of a tree T is also an induced acyclic graphoidal cover of T, we have* $\eta_{ia}(T) = n - 1$*, where n is the number of pendant vertices of T.*

Theorem II.5. *Let G be a complete graph* K_p*. Then* $\eta_{ia}(K_p) = q$*.*

Proof: The result follows from the fact that every member in an induced acyclic graphoidal cover ψ of K_p is an edge. ■

Theorem II.6. *Let G be a complete bipartite graph* $K_{m,n}$*, then*

(i) $\eta_{ia}(K_{1,n}) = n - 1, \quad n \geq 2$.

(ii) $\eta_{ia}(K_{2,n}) = q - p + 2, \quad n \geq 2$.

$$(iii) \quad \eta_{ia}(K_{3,n}) = \begin{cases} q - p + 2 & \text{if } n = 3, 4, 5; \\ q - p & \text{if } n \geq 6. \end{cases}$$

(iv) $\eta_{ia}(K_{m,n}) = q - p$ *if* $m, n \geq 4$.

Proof: Let $X = \{v_1, v_2, v_3, \ldots, v_m\}$ and $Y = \{w_1, w_2, w_3, \ldots, w_n\}$ be a bipartition of $K_{m,n}$.
(i). Since $K_{1,n}$ is a tree with n pendant vertices. Hence $\eta_{ia}(K_{1,n}) = n - 1$.
(ii). When $n \geq 2$. Let $X = \{v_1, v_2\}$ and $Y = \{w_1, w_2, w_3, \ldots, w_n\}$ be a bipartition of $K_{2,n}$.

Let $P_i = (v_1, w_i, v_2), i = 1, 2, \ldots, n$. Then $\psi = \{P_i | i = 1, 2, \ldots, n\}$ is an induced acyclic graphoidal cover of $K_{2,n}$ and $|\psi| = q - p + 2$. Hence $\eta_{ia}(K_{2,n}) \leq q - p + 2$. Further, for any induced acyclic graphoidal cover ψ of $K_{2,n}$ atleast two vertices are external vertices so that $t \geq 2$. Hence $\eta_{ia}(K_{2,n}) = q - p + 2$.
(iii). When $n = 3, 4, 5$. Let $X = \{v_1, v_2, v_3\}$ and $Y = \{w_1, w_2, w_3, \ldots, w_n\}$ be a bipartition of $K_{3,n}$.

Let $P_i = (v_1, w_i, v_2), i = 1, 2, \ldots, 5$ and $Q = (w_1, v_3, w_2)$. Then $\psi = \{P_i, Q\} \cup S, \ i = 1, 2, \ldots, 5$, where S is the set of edges not covered by the paths $P_i, i = 1, 2, 3, 4, 5$ and Q is an induced acyclic graphoidal cover of $K_{3,n}$ and $|\psi| = q - p + 2$. Hence $\eta_{ia}(K_{3,n}) \leq q - p + 2$. Further, for any induced acyclic graphoidal cover ψ of $K_{3,n}$ atleast two vertices are external vertices so that $t \geq 2$. Hence $\eta_{ia}(K_{3,n}) = q - p + 2$.

When $n \geq 6$. Let $X = \{v_1, v_2, v_3\}$ and $Y = \{w_1, w_2, w_3, \ldots, w_n\}$ be a bipartition of $K_{3,n}$.

Let $P_1 = (w_1, v_1, w_2), P_2 = (w_3, v_2, w_4)$, $P_3 = (w_5, v_3, w_6), P_4 = (v_2, w_1, v_3), P_5 = (v_2, w_2, v_3)$, $P_6 = (v_1, w_3, v_3), P_7 = (v_1, w_4, v_3), P_{i+3} = (v_1, w_i, v_2), i = 5, 6, \ldots, n$. Then $\psi = \{P_1, P_2, \ldots, P_{i+3}\} \cup S$, where S is the set of edges not covered by $P_1, P_2, P_3, \ldots, P_{n+3}$ is an induced acyclic graphoidal cover of $K_{3,n}$ and every vertex is an internal vertex of some path in ψ. Hence, $\eta_{ia}(K_{3,n}) = q - p$.

(iv). When $m, n \geq 4$. Let $X = \{v_1, v_2, \ldots, v_m\}$ and $Y = \{w_1, w_2, w_3, \ldots, w_n\}$ be a bipartition of $K_{m,n}$.

Let $P_1 = (w_1, v_1, w_2), P_2 = (w_2, v_2, w_3)$, $P_3 = (w_3, v_3, w_4), P_4 = (w_1, v_4, w_4), P_5 = (v_2, w_1, v_3)$, $P_6 = (v_3, w_2, v_4), P_7 = (v_1, w_3, v_4), P_8 = (v_1, w_4, v_2)$, $Q_{i-4} = (v_1, w_i, v_2), \ i = 5, 6, \ldots, n$, $R_{j-4} = (w_1, v_j, w_2), j = 5, 6, \ldots, m$.. Then $\psi = \{P_l, Q_{i-4}, R_{j-4}\} \cup S, l = 1, 2, \ldots, 8, i = 5, 6, \ldots, m; j = 5, 6, \ldots, m$ and S is the set of edges not covered by P_l, Q_{i-4}, R_{j-4} is an induced acyclic graphoidal cover of $K_{m,n}$ and every vertex is an internal vertex of some path in ψ. Hence, $\eta_{ia}(K_{m,n}) = q - p$. ■

Theorem II.7. *For the wheel* $W_p = K_1 + C_{p-1}$*, we have*

$$\eta_{ia}(W_p) = \begin{cases} 6 & \text{if } p = 4; \\ p & \text{if } p \geq 5. \end{cases}$$

Proof: Let $V(W_p) = \{v_0, v_1, \ldots, v_{p-1}\}$ and $E(W_p) = \{v_0 v_i : 1 \leq i \leq p - 1\} \cup \{v_i v_{i+1} : 1 \leq i \leq p - 2\} \cup \{v_1 v_{p-1}\}$
If $p = 4$ then $W_4 = K_4$ and so $\eta_{ia}(W_p) = 6$.
If $p \geq 5$. Let $P_1 = (v_1, v_2, v_3), P_2 = (v_1, v_0, v_3), P_3 = (v_3, v_4, \ldots, v_{p-1}, v_1)$. Then $\psi = \{P_1, P_2, P_3\} \cup S$, where S is the set of edges of W_p not covered by P_1, P_2 and P_3 is an induced acyclic graphoidal cover of W_p and $|\psi| = p$. Hence, $\eta_{ia}(G) \leq p$. On the other hand, for any induced acyclic graphoidal cover ψ of W_p atleast two vertices are external vertices so that $t \geq 2$. Hence $\eta_{ia}(W_p) \geq q - p + 2 = p$. Thus $\eta_{ia}(W_p) = p$. ■

Theorem II.8. *If G is a unicyclic graph with n pendant vertices and the unique cycle* C_k*, and j denote the number of vertices of degree greater than or equal to 3 in* C_k *then when*
(i) $k = 3$

$$\eta_{ia}(G) = \begin{cases} 3 & \text{if } j = 0; \\ n + 2 & \text{if } j = 1; \\ n + 1 & \text{if } j = 2; \\ n & otherwise. \end{cases}$$

(ii) $k \geq 4$

$$\eta_{ia}(G) = \begin{cases} 2 & \text{if } j = 0; \\ n + 1 & \text{if } j = 1; \text{ or } j = 2 \text{ and the two vertices} \\ & \text{of deg} \geq 3 \text{ are adjacent in } C_k; \\ n & otherwise. \end{cases}$$

Proof: Let $C_k = \{v_1, v_2, v_3, \ldots, v_k, v_1\}$ be the unique cycle in G.

(i). Case(a). When $j = 0$ then $G = C_3$ so that $\eta_{ia}(G) = 3$.

Case(b). When $j = 1$. Let v_1 be the unique vertex of deg ≥ 3 in C_3. Let $T = G - \{v_1 v_2, v_2 v_3\}$ be the tree with $n + 1$ pendant vertices so that $\eta_{ia}(T) = n$. Let ψ_1 be a minimum induced acyclic graphoidal cover of T. Then $\psi = \psi_1 \cup \{v_1 v_2, v_2 v_3\}$ is an induced acyclic graphoidal cover of G and so $|\psi| = n + 2$. Hence, $\eta_{ia}(G) \leq n + 2$. On the other hand, for any induced acyclic graphoidal cover ψ of G, the n pendant vertices of G and atleast two vertices in C_k are external vertices so that $t \geq n + 2$. Hence, $\eta_{ia}(G) = q - p + t \geq q - p + n + 2 = n + 2$.

Case(c). When $j = 2$. Let v_1 and v_2 be the vertices of deg $\geq$ 3 in C_3. Let $T = G - \{v_2 v_3, v_3 v_1\}$ be the tree with n pendant vertices so that $\eta_{ia}(T) = n - 1$. Let ψ_1 be a minimum induced acyclic graphoidal cover of T. Then $\psi = \psi_1 \cup \{v_2 v_3, v_3 v_1\}$ is an induced acyclic graphoidal cover of G and so $|\psi| = n + 1$. Hence, $\eta_{ia}(G) \leq n + 1$. On the other hand, for any induced

International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:4, No:8, 2010

acyclic graphoidal cover ψ of G, the n pendant vertices of G and atleast one vertex in C_k are external vertices so that $t \geq n+1$. Hence, $\eta_{ia}(G) = q-p+t \geq q-p+n+1 = n+1$.

Case(d). When all the vertices in C_3 are of deg ≥ 3. Let $T = G-(v_1v_2)$ be the tree with n pendant vertices. Let T_1 be the induced subgraph of T formed by v_2 along with vertices connected to v_2 such that v_3 occurs as an pendant vertex. Then T_1 has n_1+1 pendant vertices so that $\eta_{ia}(T_1) = n_1$. Let ψ_1 be a minimum induced acyclic graphoidal cover of T_1. Also, $T_2 = T - T_1$ is also a tree with n_2 pendant vertices so that $n_1 + n_2 = n$ and $\eta_{ia}(T_2) = n_2 - 1$. Let ψ_2 be a minimum induced acyclic graphoidal cover of T_2. Then $\psi = \psi_1 \cup \psi_2 \cup (v_1v_2)$, is an induced acyclic graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ. Hence, $\eta_{ia}(G) = n$.

(ii). Case(a). When $j = 0$, then $G = C_k$ so that $\eta_{ia}(G) = 2$.

Case(b). When $j = 1$. Let v_1 be the unique vertex of deg$\geq$3 in C_k. Let $P = (v_1, v_k, v_{k-1}, \ldots, v_4, v_3)$ be an induced path of length atleast 2. Then $T = G-P$ is a tree with $n+1$ pendant vertices so that $\eta_{ia}(T) = n$, with ψ_1 as a minimum induced acyclic graphoidal cover. Then $\psi = \psi_1 \cup P$ is an induced acyclic graphoidal cover of G and so $|\psi| = n+1$. Hence, $\eta_{ia}(G) \leq n+1$. Further, for any induced acyclic graphoidal cover ψ of G, the n pendant vertices of G and atleast one vertex in C_k are external vertices so that $t \geq n+1$. Hence, $\eta_{ia}(G) = q-p+t \geq q-p+n+1 = n+1$.

When $j = 2$ and the two vertices of deg$\geq$3 are adjacent vertices in C_k, the proof is similar to that for $j = 1$.

Case(c). When $j = 2$. Suppose v_1, v_3 are the two non adjacent vertices of deg ≥ 3. Let $P = (v_1, v_k, v_{k-1}, \ldots, v_4, v_3)$ be an induced path of length atleast 2. Then $T = G-P$ is a tree with n pendant vertices so that $\eta_{ia}(T) = n-1$, with ψ_1 as a minimum induced acyclic graphoidal cover. Then $\psi = \psi_1 \cup P$ is an induced acyclic graphoidal cover of G such that every vertex of degree greater than 1 is an internal vertex of some path in ψ. Hence, $\eta_{ia}(G) = n$.

When $j \geq 3$. Take two non adjacent vertices v_i and v_j of deg$\geq$3 in C_k. let T be the induced subgraph of G containing all vertices on one side of the arc v_i and v_j of G such that these two vertices appear as pendant vertices and T has n_1+2 pendant vertices so that $\eta_{ia}(T_1) = n_1+1$. Let ψ_1 be the minimum induced acyclic graphoidal cover of T. Then $T' = G-T$ is a tree with n_2 pendant vertices so that $n = n_1 + n_2$ and $\eta_{ia}(T') = n_2 - 1$. Let ψ_2 be the minimum induced acyclic graphoidal cover of T'. Then $\psi = \psi_1 \cup \psi_2$, is an induced acyclic graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ. Hence, $\eta_{ia}(G) = n$. ∎

Theorem II.9. *Let G be a bicyclic graph with n pendant vertices containing a $U(l;m)$ and j be the number of vertices of degree greater than or equal to 3 in $U(l;m)$. Then when (i) $l, m = 3$*

$$\eta_{ia}(G) = \begin{cases} 5 & \text{if } G = U(l;m); \\ n+6-j & \text{if } 1 \leq j \leq 5. \end{cases}$$

(ii) $l = 3, m \geq 4$

$$\eta_{ia}(G) = \begin{cases} 4 & \text{if } G = U(l;m); \\ n+4 & \text{if } j = 1; \text{ or } j = 2 \text{ and the other vertex of deg}\geq 3 \text{ is adjacent to } u_0 \text{ in } C_m; \\ n+3 & \text{if } j = 2 \text{ and } u_0 \text{ is adjacent to the other vertex of deg}\geq 3 \text{ in } C_l; \text{ or } j \geq 2 \text{ and all vertices of deg}\geq 3 \text{ are in } C_m. \\ n+2 & \text{if } j = 3 \text{ in } C_l; \text{ or } j = 4 \text{ and } C_m \text{ has only one vertex other than } u_0 \text{ of deg}\geq 3 \text{ which is adjacent to } u_0; \\ n+1 & \text{otherwise.} \end{cases}$$

(iii) $l, m \geq 4$

$$\eta_{ia}(G) = \begin{cases} 3 & \text{if } G = U(l;m); \\ n+3 & \text{if } j = 1; \text{ or } j = 2 \text{ and the other vertices of deg}\geq 3 \text{ is adjacent to } u_0; \text{ or } j = 3 \text{ and the other vertices of deg}\geq 3 \text{ are adjacent to } u_0 \text{ in } C_m; \\ n+2 & \text{if } j \geq 2 \text{ and all vertices of deg}\geq 3 \text{ are in } C_l \text{ or } C_m; \text{ or } j = 2 \text{ and the other vertex of deg}\geq 3 \text{ is nonadjacent to } u_0; \\ n+1 & \text{otherwise.} \end{cases}$$

Proof: Let the $l-$cycle be $C_l = \{u_0, u_1, \ldots, u_{l-1}, u_0\}$ and the $m-$cycle be $C_m = \{u_0, u_l, u_{l+1}, \ldots, u_{l+m-2}, u_0\}$ in G.

(i). If $G = U(l;m)$ then $\eta_{ia}(G) = 5$.

Otherwise, Take $G' = G - \{e\}$, where e is an edge with end vertices of degree 2 in G.

If $j = 1$ then G' is a unicyclic graph with $n+2$ pendent vertices so that $\eta_{ia}(G') = n+4$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G'. Then $\psi = \psi_1 \cup e$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 1 = n+5$. Hence, $\eta_{ia}(G) \leq n+5$. Again, for any induced acyclic graphoidal cover ψ of G, the n pendant vertices of G and atleast four vertices in $U(l;m)$ are external vertices so that $t \geq n+4$. Hence, $\eta_{ia}(G) = q-p+t \geq 1+n+4 = n+5$.

If $j = 2$, similar as above.

If $j = 3$, similar as above if no vertex of C_l except u_0 is of deg$\geq$3.

If $j = 3$ each of C_l and C_m has a vertex other than u_0 of deg$\geq$3. Let e be an edge in $U(l;m)$ not adjacent to u_0, then $G' = G-e$ is a unicyclic graph with $n+1$ pendent vertices so that $\eta_{ia}(G') = n+2$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G'. Then $\psi = \psi_1 \cup e$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 1 = n+3$. Hence, $\eta_{ia}(G) \leq n+3$. Again, for any induced acyclic graphoidal cover ψ of G, the n pendant vertices of G and atleast two vertices in $U(l;m)$ are external vertices so that $t \geq n+2$. Hence, $\eta_{ia}(G) = q-p+t \geq 1+n+2 = n+3$.

If $j = 4$, similar as above.

If $j = 5$. Let e be an edge in $U(l;m)$ not adjacent to u_0. Then $G_1 = G-e$ is a unicyclic graph with n pendant vertices so that $\eta_{ia}(G_1) = n$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1. Then $\psi = \psi_1 \cup e$ is an induced acyclic graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ. Hence, $\eta_{ia}(G) = q - p + t = n + 1$.

(ii). Case(a). $G = U(l; m)$. Then $\eta_{ia}(G) = 4$.

Case(b). When $j = 1$. Then $G_1 = G - C_m$ is a unicyclic graph with n pendant vertices so that $\eta_{ia}(G_1) = n+2$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1. Then $\psi = \psi_1 \cup C_m$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 2 = n + 4$. Hence, $\eta_{ia}(G) \leq n + 4$. Again, for any induced acyclic graphoidal cover ψ of G, the n pendant vertices of G and atleast three vertices in $U(l; m)$ are external vertices so that $t \geq n + 3$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 3 = n + 4$.

When $j = 2$ and u_0 is adjacent to the other vertex of deg≥3 in C_m. Then $G_1 = G - C_l$ is a unicyclic graph with n pendant vertices so that $\eta_{ia}(G_1) = n+1$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1. Then $\psi = \psi_1 \cup C_l$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 3 = n + 4$. Hence, $\eta_{ia}(G) \leq n + 4$. Again, for any induced acyclic graphoidal cover ψ of G, the n pendant vertices of G and atleast three vertices in $U(l; m)$ are external vertices so that $t \geq n + 3$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 3 = n + 4$.

Case(c). When $j = 2$ and the other vertex of deg≥3 is in C_l. Then $G_1 = G - C_m$ is a unicyclic graph with n pendant vertices so that $\eta_{ia}(G_1) = n+1$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1. Then $\psi = \psi_1 \cup C_m$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 2 = n + 3$. Hence, $\eta_{ia}(G) \leq n + 3$. Again, for any induced acyclic graphoidal cover ψ of G, the n pendant vertices of G and atleast two vertices in $U(l; m)$ are external vertices so that $t \geq n + 2$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 2 = n + 3$.

When $j \geq 2$ and all vertices of deg≥3 are in C_m. Then $G_1 = G - C_l$ is a unicyclic graph with n pendant vertices so that $\eta_{ia}(G_1) = n$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1. Then $\psi = \psi_1 \cup C_l$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 3 = n + 3$. Hence, $\eta_{ia}(G) \leq n + 3$. Again, for any induced acyclic graphoidal cover ψ of G, the n pendant vertices of G and atleast two vertices in $U(l; m)$ are external vertices so that $t \geq n + 2$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 2 = n + 3$.

Case(d). When $j = 3$ and all vertices of deg≥3 are in C_l. Then $G_1 = G - C_m$ is a unicyclic graph with n pendant vertices so that $\eta_{ia}(G_1) = n$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1. Then $\psi = \psi_1 \cup C_m$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 2 = n + 2$. Hence, $\eta_{ia}(G) \leq n + 2$. Again, for any induced acyclic graphoidal cover ψ of G, the n pendant vertices of G and atleast one vertex in $U(l; m)$ are external vertices so that $t \geq n + 1$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 1 = n + 2$.

When $j = 4$ and exactly one vertex, say u_l, of deg≥3 is adjacent to u_0 in C_m. Let e be an edge not adjacent to u_0 in C_l. Then $G_1 = G - e$ is a unicyclic graph with n pendant vertices so that $\eta_{ia}(G_1) = n+1$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1. Then $\psi = \psi_1 \cup e$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 1 = n + 2$. Hence, $\eta_{ia}(G) \leq n + 2$. Again, for any induced acyclic graphoidal cover ψ of G, the n pendant vertices of G and atleast one vertex in $U(l; m)$ are external vertices so that $t \geq n + 1$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 1 = n + 2$.

Case(e). When $j \geq 4$. Let e be an edge in C_l not adjacent to u_0. Then $G_1 = G - e$ is a unicyclic graph with n pendant vertices so that $\eta_{ia}(G_1) = n$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1. Then $\psi = \psi_1 \cup e$ is an induced acyclic graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ. Hence, $\eta_{ia}(G) = q - p + t = n + 1$.

(iii). Case(a). $G = U(l; m)$. Then $\eta_{ia}(G) = 3$.

Case(b). When $j = 1$. Here, $G_1 = G - C_m$ is a unicyclic graph with n pendant vertices so that $\eta_{ia}(G_1) = n+1$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1. Then $\psi = \psi_1 \cup C_m$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 2 = n + 3$. Hence, $\eta_{ia}(G) \leq n + 3$. Again, for any induced acyclic graphoidal cover ψ of G, the n pendant vertices of G and atleast two vertices in $U(l; m)$ are external vertices so that $t \geq n + 2$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 2 = n + 3$.

Similarly, we can prove for $j = 2$ and the other vertex of deg≥3 is adjacent to u_0.

When $j = 3$. Suppose u_l in C_l and u_{l+m-2} in C_m are of deg≥3 and both are adjacent to u_0. Let P be an induced path $u_l - u_{l+m-2}$ of length atleast two in C_m such that $G_1 = G - P$ is a unicyclic graph with $n + 1$ pendant vertices and so $\eta_{ia}(G_1) = n + 2$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1. Then $\psi = \psi_1 \cup P$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 1 = n + 3$. Hence, $\eta_{ia}(G) \leq n + 3$. Again, for any induced acyclic graphoidal cover ψ of G, the n pendant vertices of G and atleast two vertices in $U(l; m)$ are external vertices so that $t \geq n + 2$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 2 = n + 3$.

Case(c). When $j \geq 2$ and all vertices of deg≥3 except u_0 are in C_m or $j = 2$ with the other vertex v of deg≥3 is nonadjacent to u_0 in G. Then $G_1 = G - C_l$ is a unicyclic graph with n pendant vertices so that $\eta_{ia}(G_1) = n$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1. Then $\psi = \psi_1 \cup C_l$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 2 = n + 2$. Hence, $\eta_{ia}(G) \leq n + 2$. Again, for any induced acyclic graphoidal cover ψ of G, the n pendant vertices of G and atleast one vertex in $U(l; m)$ are external vertices so that $t \geq n + 1$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 1 = n + 2$.

Case(d). When $j = 3$ and $v \in C_l$, $w \in C_m$ of deg≥3 are non adjacent to u_0. Let $P = (u_0, \ldots, v)$ be a path of length atleast 2 in C_l such that $G_1 = G - P$ is a unicyclic graph with n pendant vertices and so $\eta_{ia}(G_1) = n$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1. Then $\psi = \psi_1 \cup P$ is an induced acyclic graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ. Hence, $\eta_{ia}(G) = n + 1$.

When $j \geq 4$, Take two non adjacent vertices v_i and v_j of deg≥3 in C_l (or C_m). Let T be the induced subgraph of G containing all vertices on one side of the arc $v_i - v_j$ of C_l (or C_m) such that these two vertices appear as pendant vertices and T has $n_1 + 2$ pendant vertices so that $\eta_{ia}(T) = n_1 + 1$. Let ψ_1 be the minimum induced acyclic graphoidal cover of T. Then $G_1 = G - T$ is a unicyclic graph with n_2 pendant vertices so that $n = n_1 + n_2$ and $\eta_{ia}(G_1) = n_2$. Let ψ_2 be the minimum induced acyclic graphoidal cover of G_1. Then $\psi = \psi_1 \cup \psi_2$, is an induced acyclic graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ. Hence, $\eta_{ia}(G) = n + 1$. ■

Theorem II.10. *Let G be a bicyclic graph with n pendant vertices containing a $D(l, m; i)$ and j be the number of vertices of degree greater than or equal to 3 in cycles in $D(l, m; i)$. Then when*
(i) $l, m = 3$

$$\eta_{ia}(G) = \begin{cases} 5 & \text{if } G = D(l, m; i); \\ n + 7 - j & \text{if } 2 \leq j \leq 6. \end{cases}$$

(ii) $l = 3, m \geq 4$

$$\eta_{ia}(G) = \begin{cases} 4 & \text{if } G = D(l, m; i); \\ n + 4 & \text{if } j = 2\text{; or } j = 3 \text{ and the third vertex of deg}{\geq}3 \text{ is adjacent to } u_{l+i-1} \text{ in } C_m; \\ n + 3 & \text{if } j = 3 \text{ and the third vertex of deg}{\geq}3 \text{ is in } C_l\text{; or } j \geq 3 \text{ and all vertices of deg}{\geq}3 \text{ are in } C_m\text{; or } j = 4 \text{ and } v \text{ in } C_l \text{ and } w \text{ adjacent to } u_{l+i-1} \text{ in } C_m \text{ are of deg}{\geq}3; \\ n + 2 & \text{if } j = 4 \text{ and } C_m \text{ has no vertex of deg}{\geq}3 \text{ other than } u_{l+i-1}\text{; or } j = 5 \text{ and } C_m \text{ has exactly one vertex of deg}{\geq}3 \text{ which is adjacent to } u_{l+i-1}; \\ n + 1 & otherwise. \end{cases}$$

(iii) $l, m \geq 4$

$$\eta_{ia}(G) = \begin{cases} 3 & \text{if } G = D(l, m; i); \\ n + 3 & \text{if } j = 2\text{; or } j = 3 \text{ and the third vertex of deg}{\geq}3 \text{ is adjacent to either } u_{l-1} \text{ or } u_{l+i-1}\text{; or } j = 4 \text{ and from the vertices of deg}{\geq}3 \text{ other than } u_{l-1} \text{ and } u_{l+i-1} \text{ one is adjacent to } u_{l-1} \text{ in } C_l \text{ and other is adjacent to } u_{l+i-1} \text{ in } C_m; \\ n + 2 & \text{if } j \geq 3 \text{ and all vertices of deg}{\geq}3 \text{ are in } C_l \text{ or } C_m \text{ only; or } j = 3 \text{ and the other vertex of deg}{\geq}3 \text{ is adjacent to neither } u_{l-1} \text{ nor } u_{l+i-1}; \\ n + 1 & otherwise. \end{cases}$$

Proof: Let $C_l = u_0u_1 \ldots u_{l-1}u_0$, $P_i = u_{l-1}u_l \ldots u_{l+i-1}$ and $C_m = u_{l+i-1}u_{l+i} \ldots u_{l+m+i-2}u_{l+i-1}$ in G.

(i). If $G = D(l, m; i)$ then $\eta_{ia}(G) = 5$.

Otherwise, Take $G' = G - \{e\}$, where e is an edge with end vertices of degree 2 in G.

If $j = 2$ then G' is a unicyclic graph with $n + 2$ pendent vertices so that $\eta_{ia}(G') = n+4$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G'. Then $\psi = \psi_1 \cup e$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 1 = n + 5$. Hence, $\eta_{ia}(G) \leq n + 5$. Again, for any induced acyclic graphoidal cover ψ of G, the n pendant vertices of G and atleast four vertices in $D(l, m; i)$ are external vertices so that $t \geq n + 4$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 4 = n + 5$.

If $j = 3$, similar as above.

If $j = 4$, similar as above if no vertex of C_l except u_{l-1} is of deg≥3.

If $j = 4$ and each of C_l and C_m has a vertex other than u_{l-1} and u_{l+i-1} are of deg≥3. Let e be an edge in $D(l, m; i)$ not adjacent to either u_{l-1} or u_{l+i-1}, then $G' = G - e$ is a unicyclic graph with $n+1$ pendent vertices so that $\eta_{ia}(G') = n+2$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G'. Then $\psi = \psi_1 \cup e$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 1 = n + 3$. Hence, $\eta_{ia}(G) \leq n + 3$. Again, for any induced acyclic graphoidal cover ψ of G, the n pendant vertices of G and atleast two vertices in $D(l, m; i)$ are external vertices so that $t \geq n+2$. Hence, $\eta_{ia}(G) = q-p+t \geq 1 + n + 2 = n + 3$.

If $j = 5$, similar as above.

If $j = 6$. Let e be an edge in C_l not adjacent to u_{l-1}. Then $G_1 = G - e$ is a unicyclic graph with n pendant vertices so that $\eta_{ia}(G_1) = n$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1. Then $\psi = \psi_1 \cup e$ is an induced acyclic graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ. Hence, $\eta_{ia}(G) = q - p + t = n + 1$.

(ii). Case(a). If $G = D(l, m; i)$ then $\eta_{ia}(G) = 4$.

Case(b). When $j = 2$. Then $G_1 = G - C_m$ is a unicyclic graph with n pendant vertices so that $\eta_{ia}(G_1) = n + 2$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1. Then $\psi = \psi_1 \cup C_m$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 2 = n + 4$. Hence, $\eta_{ia}(G) \leq n + 4$. Again, for any induced acyclic graphoidal cover ψ of G, the n pendant vertices of G and atleast three vertices in $D(l, m; i)$ are external vertices so that $t \geq n + 3$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 3 = n + 4$.

Similarly, we can prove for $j = 3$ and the third vertex of deg≥3 is adjacent to u_{l+i-1} in C_m.

Case(c). When $j = 3$ and the third vertex of deg≥3 is in C_l. Then $G_1 = G - C_m$ is a unicyclic graph with n pendant vertices so that $\eta_{ia}(G_1) = n+1$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1. Then $\psi = \psi_1 \cup C_m$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 2 = n+3$. Hence, $\eta_{ia}(G) \leq n+3$. Again, for any induced acyclic graphoidal cover ψ of G, the n pendant vertices of G and atleast two vertices in $D(l, m; i)$ are external vertices so that $t \geq n + 2$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 2 = n + 3$.

Similarly, we can prove for $j \geq 3$ and all vertices of deg≥3 are in C_m by taking $G_1 = G - C_l$.

When $j = 4$ and v in C_l and w adjacent to u_{l+i-1} in C_m are of deg≥3. Let e be an edge in $D(l, m; i)$ not adjacent to u_{l-1} so that $G_1 = G - e$ is a unicyclic graph with $n + 1$ pendant vertices. Then $\eta_{ia}(G_1) = n+2$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1. Then $\psi = \psi_1 \cup e$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 1 = n+3$. Hence, $\eta_{ia}(G) \leq n+3$. Again, for any induced acyclic graphoidal cover ψ of G, the n pendant vertices of G and atleast two vertices in $D(l, m; i)$ are external vertices so that $t \geq n + 2$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 2 = n + 3$.

Case(d). When $j = 4$ and C_m has no vertex of deg≥3 other than u_{l+i-1}. Then $G_1 = G - C_m$ is a unicyclic graph with n pendant vertices so that $\eta_{ia}(G_1) = n$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1. Then $\psi = \psi_1 \cup C_m$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 2 = n+2$. Hence, $\eta_{ia}(G) \leq n+2$. Again, for any induced acyclic graphoidal cover ψ of G, the n pendant vertices of G and atleast one vertex in $D(l, m; i)$ are external vertices so that $t \geq n+1$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 1 = n + 2$.

When $j = 5$ and C_m has exactly one vertex of deg≥3 which is adjacent to u_{l+i-1}. Let e be an edge in C_l not adjacent to u_{l-1}. Then $G_1 = G - e$ is a unicyclic graph with n pendant vertices so that $\eta_{ia}(G_1) = n+1$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1. Then $\psi = \psi_1 \cup e$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 1 = n + 2$. Hence, $\eta_{ia}(G) \leq n + 2$. Again, for any induced acyclic graphoidal cover ψ of G, the n pendant vertices of G and atleast one vertex in $D(l, m; i)$ are external vertices so that $t \geq n+1$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 1 = n + 2$.

Case(e). When $j \geq 5$. Let e be an edge in $D(l, m; i)$ not incident to any vertex of P_i in C_l. Then $G_1 = G - e$ is a unicyclic graph with n pendant vertices so that $\eta_{ia}(G_1) = n$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1. Then $\psi = \psi_1 \cup e$ is an induced acyclic graphoidal cover of $D(l, m; i)$ and every vertex of degree greater than 1 is an internal vertex of some path in ψ. Hence, $\eta_{ia}(G) = n + 1$.

(iii). Case(a). If $G = D(l, m; i)$ then $\eta_{ia}(G) = 3$.

Case(b). When $j = 2$. Then $G_1 = G - C_m$ is a unicyclic graph with n pendant vertices so that $\eta_{ia}(G_1) = n + 1$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1. Then $\psi = \psi_1 \cup C_m$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 2 = n + 3$. Hence, $\eta_{ia}(G) \leq n + 3$. Again, for any induced acyclic graphoidal cover ψ of G, the n pendant vertices of G and atleast two vertices in $D(l, m; i)$ are external vertices so that $t \geq n+2$. Hence, $\eta_{ia}(G) = q-p+t \geq 1 + n + 2 = n + 3$.

Similarly, we can prove for $j = 3$ and the third vertex of deg≥3 is adjacent to either u_{l-1} or u_{l+i-1}.

When $j = 4$ and from the vertices of deg≥3 other than u_{l-1} and u_{l+i-1} one is adjacent to u_{l-1} in C_l and other is adjacent to u_{l+i-1} in C_m. Suppose v with deg≥3 in C_m is adjacent to u_{l+i-1}. Let P be an induced path $v - u_{l+m+i-2}$ of length atleast two in C_m. Then $G_1 = G - P$ is a unicyclic graph with $n + 1$ pendant vertices so that $\eta_{ia}(G_1) = n + 2$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1. Then $\psi = \psi_1 \cup P$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 1 = n + 3$. Hence, $\eta_{ia}(G) \leq n + 3$. Again, for any induced acyclic graphoidal cover ψ of G, the n pendant vertices of G and atleast two vertices in $D(l, m; i)$ are external vertices so that $t \geq n+2$. Hence, $\eta_{ia}(G) = q-p+t \geq 1 + n + 2 = n + 3$.

Case(c). When $j \geq 3$ and all vertices of deg≥3 are only in C_l or C_m only; or $j = 3$ and the other vertex of deg≥3 is adjacent to neither u_{l-1} nor u_{l+i-1} in G. Suppose v with deg≥3 lies in C_m. Then $G_1 = G - C_l$ is a unicyclic graph with n pendant vertices so that $\eta_{ia}(G_1) = n$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1. Then $\psi = \psi_1 \cup C_l$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 2 = n+2$. Hence, $\eta_{ia}(G) \leq n+2$. Again, for any induced acyclic graphoidal cover ψ of G, the n pendant vertices of G and atleast one vertex in $D(l, m; i)$ are external vertices so that $t \geq n+1$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 1 = n + 2$.

Case(d). Take two non adjacent vertices v_i and v_j of deg≥3 in C_l (or C_m). Let T be the induced subgraph of G containing all vertices on one side of the arc $v_i - v_j$ of C_l (or C_m) such that these two vertices appear as pendant vertices and T has $n_1 + 2$ pendant vertices. Then $\eta_{ia}(T_1) = n_1 + 1$. Let ψ_1 be the minimum induced acyclic graphoidal cover of T. Then $G_1 = G - T$ is a unicyclic graph with n_2 pendant vertices so that $n = n_1 + n_2$ and $\eta_{ia}(G_1) = n_2$. Let ψ_2 be the minimum induced acyclic graphoidal cover of G_1. Then $\psi = \psi_1 \cup \psi_2$, is an induced acyclic graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ. Hence, $\eta_{ia}(G) = n + 1$. ■

Remark II.11. *In case P_i in $D(L, m; i)$ has any intermediate vertex(ices) of degree greater than or equal to 3 there will be no change in the minimum induced acyclic graphoidal covering number.*

Theorem II.12. *Let G be a bicyclic graph with n pendant vertices containing a $C_m(i; l)$ and j be the number of vertices of degree greater than or equal to 3 in $C_m(i; l)$. Then*

(i) $\eta_{ia}(G) = 3 \quad if \quad G = C_m(i; l).$

and when

(ii) $l = 1$

$$\eta_{ia}(G) = \begin{cases} n+3 & \text{if } j = 2; \\ n+2 & \text{if } j \geq 3 \text{ and all the vertices of deg}\geq 3 \\ & \text{are in one side of } P_l; \\ n+1 & otherwise. \end{cases}$$

(iii) $l \geq 2$

$$\eta_{ia}(G) = \begin{cases} n+2 & deg u_0 = 3 \text{ and either } j = 2 \text{ or } j = 3 \text{ with} \\ & \text{the third vertex of deg}\geq 3 \text{ is adjacent to } u_i; \\ n+1 & otherwise. \end{cases}$$

Proof: $G = C_m(i; l)$, so it contains atleast $C_m = \{u_0, u_1, \ldots, u_i, u_{i+1}, \ldots, u_{m-1}, u_0\}$ with $m \geq 4$ and the chord $P_l = \{u_0, u_m, u_{m+1}, \ldots, u_{m+l-2}, u_i\}$, $l \geq 1$ and $2 \leq i \leq m - 2$.

(i). If $G = C_m(i; l)$ then $\eta_{ia}(G) = 3$.

(ii) Case(a). When $j = 2$. let u_s, $0 < s < i$, be any vertex in $C_m(i; l)$. Then $P_1 = (u_0, u_s)$, $P_2 = (u_s, u_{s+1}, \ldots, u_i)$ be induced paths in $C_m(i; l)$. Let $G_1 = G - \{P_1, P_2\}$ is a unicyclic graph with n pendant vertices so that $\eta_{ia}(G_1) = n+1$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1. Then $\psi = \psi_1 \cup P_1 \cup P_2$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 2 = n + 1 + 2$. Hence, $\eta_{ia}(G) \leq n + 3$. Again, for any induced acyclic graphoidal cover ψ of G, the n pendant vertices of G and atleast two vertices in $C_m(i; l)$ are external vertices so that $t \geq n + 2$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 2 = n + 3$.

Case(b). When $j \geq 3$ and all the vertices of deg≥3 are in one side of P_l, say $(u_i, u_m, \ldots, u_0)$. Take a vertex u_s, $0 <$

$s < i$ in $C_m(i;l)$. Then the paths $P_1 = u_0 - u_s$ and $P_2 = u_s - u_i$ are induced paths in G. Let $G_1 = G - \{P_1, P_2\}$ is a unicyclic graph with n pendant vertices so that $\eta_{ia}(G_1) = n$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1. Then $\psi = \psi_1 \cup P_1 \cup P_2$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 2 = n + 2$. Hence, $\eta_{ia}(G) \leq n + 2$. Again, for any induced acyclic graphoidal cover ψ of G, the n pendant vertices of G and atleast one vertex in $C_m(i;l)$ are external vertices so that $t \geq n+1$. Hence, $\eta_{ia}(G) = q-p+t \geq 1 + n + 1 = n + 2$.

Case(c). When $j \geq 4$, suppose u_s $(0 < s < i)$ and u_t $(i < t < m-1)$ are two vertices of $deg \geq 3$ in $C_m(i;l)$. Take the tree T of $C_m(i;l)$ on one side of P_l containing u_s such that u_0 and u_i are pendant vertices. Again, bifurcate T into two trees T_1 and T_2 containing $\{u_0, u_1, \dots, u_s\}$ and $\{u_s, u_{s+1}, \dots, u_i\}$ along with the subgraphs of T incident to these vertices such that deg$u_s = 1$ in T_1. Suppose T_1 and T_2 have $n_1 + 2$ and $n_2 + 1$ pendant vertices respectively. Then $\eta_{ia}(T_1) = n_1 + 1$ and $\eta_{ia}(T_2) = n_2$. Let ψ_1 and ψ_2 be respectively the minimum induced acyclic graphoidal cover of T_1 and T_2. Also, $G' = G - T$ is a unicyclic graph with n_3 pendant vertices so that $n_1 + n_2 + n_3 = n$ and $\eta_{ia}(G') = n_3$. Let ψ_3 be a minimum induced acyclic graphoidal cover of G'. Then $\psi = \psi_1 \cup \psi_2 \cup \psi_3$ is an induced acyclic graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ. Hence, $\eta_{ia}(G) = n_1 + n_2 + n_3 + 1 = n + 1$.

(iii). Case(a). When $j = 2$ and $deg\ u_0 = 3$.
Let $P = \{u_0 u_m u_{m+1} \dots u_{l+m-2} u_i\}, 2 \leq i \leq m-2$, be the chord in $C_m(i;l)$ such that $\eta_{ia}(P) = 1$. Then $G_1 = G - P$ is a unicyclic graph with n pendant vertices so that $\eta_{ia}(G_1) = n+1$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1. Then $\psi = \psi_1 \cup P$ is an induced acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 1 = n + 2$. Hence, $\eta_{ia}(G) \leq n + 2$. Again, for any induced acyclic graphoidal cover ψ of G, the n pendant vertices of G and atleast one vertex in $C_m(i;l)$ are external vertices so that $t \geq n+1$. Hence, $\eta_{ia}(G) = q-p+t \geq 1 + n + 1 = n + 2$.

Similarly, we can prove for $j = 3$ and the third vertex of deg$\geq$3 is adjacent to u_0.

Case(b). Let $P = \{u_0 u_m u_{m+1} \dots u_{l+m-2} u_i\}, 2 \leq i \leq m-2$, be the chord in $C_m(i;l)$ such that $\eta_{ia}(P) = 1$. Then $G_1 = G - P$ is a unicyclic graph with n pendant vertices so that $\eta_{ia}(G_1) = n$. Let ψ_1 be a minimum induced acyclic graphoidal cover of G_1. Then $\psi = \psi_1 \cup P$ is an induced acyclic graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ. Hence, $\eta_{ia}(G) = n+1$.

Otherwise, let T be the induced subgraph of G with vertex set $\{u_0 u_m u_{m+1} \dots u_{l+m-2} u_i\}$, $2 \leq i \leq m-2$, along with vertices incident to this vertex set such that deg $u_0, u_{l+m-2} = 1$. Then T has n_1+2 pendant vertices so that $\eta_{ia}(T) = n_1+1$. Let ψ_1 be the minimum induced acyclic graphoidal cover of T. Then $G_1 = G - T$ is a unicyclic graph with n_2 pendant vertices so that $n = n_1 + n_2$ and $\eta_{ia}(G_1) = n_2$. Let ψ_2 be the minimum induced acyclic graphoidal cover of G_1. Then $\psi = \psi_1 \cup \psi_2$, is an induced acyclic graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ. Hence, $\eta_{ia}(G) = n + 1$. ■

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