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# A new direct updating method for undamped structural systems

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*Abstract*—A new numerical method for simultaneously updating mass and stiffness matrices based on incomplete modal measured data is presented. By using the Kronecker product, all the variables that are to be modified can be found out and then can be updated directly. The optimal approximation mass matrix and stiffness matrix which satisfy the required eigenvalue equation and orthogonality condition are found under the Frobenius norm sense. The physical configuration of the analytical model is preserved and the updated model will exactly reproduce the modal measured data. The numerical example seems to indicate that the method is quite accurate and efficient.

*Keywords*—finite element model, model updating, modal data, optimal approximation.

## I. Introduction

THROUGHOUT this paper, we denote the real $m \times n$ matrix space by $\mathbf{R}^{m\times n}$, the set of all symmetric matrices in $\mathbf{R}^{n\times n}$ by $\mathbf{SR}^{n\times n}$. $A^T$ and $A^+$ stand for the transpose and the Moore-Penrose generalized inverse of a real matrix $A$. For $A, B \in \mathbf{R}^{m\times n}$, an inner product in $\mathbf{R}^{m\times n}$ is defined by $(A, B) = \text{trace}(B^TA)$, then $\mathbf{R}^{m\times n}$ is a Hilbert space. The matrix norm $\|\cdot\|$ induced by the inner product is the Frobenius norm. Given two matrices $A = [a_{ij}] \in \mathbf{R}^{m\times n}$ and $B = [b_{ij}] \in \mathbf{R}^{p\times q}$, the Kronecker product of $A$ and $B$ is defined by $A \otimes B = [a_{ij}B] \in \mathbf{R}^{mp\times nq}$ and the stretching function Vec$(A)$ is defined by $\text{vec}(A) = [a_1^T, a_2^T, \cdots, a_n^T]^T \in \mathbf{R}^{mn}$, where $a_i, i = 1, \cdots, n$, is the $i$-th column vector of $A$. Furthermore, for a matrix $A \in \mathbf{R}^{m\times n}$, let $E_A$ and $F_A$ stand for the two orthogonal projectors $E_A = I_m - AA^+$ and $F_A = I_n - A^+A$.

Using finite element techniques, the undamped free vibration of a structural dynamic system can be described by the second order differential equation as

$$M_a\ddot{x}(t) + K_ax(t) = 0, \quad (1)$$

where $M_a,\ K_a$ are analytical mass and stiffness matrices. Assume that the displacement response of (1) is harmonic,

$$x(t) = x(\omega)e^{i\omega t},$$

then the structural eigenproblem can be written in the form,

$$K_a\phi_j = \mu_jM_a\phi_j,\ j = 1, 2, \cdots, n, \quad (2)$$

where $\mu_j = \omega_j^2$ is the $j$th eigenvalue and $\phi_j$ is the $j$th eigenvector. It is well known that the eigenvalue and eigenvector can be interpreted physically as the square of the natural frequency of vibration and the mode shape respectively. A most important property of the undamped vibration modes is their orthogonality with respect to mass. By premultiplying equation (2) by $\phi_k^T$ we find

$$\phi_k^TK_a\phi_j = \mu_j\phi_k^TM_a\phi_j. \quad (3)$$

By interchanging the subscripts $j$ and $k$, and transposing, we have

$$\phi_k^TK_a\phi_j = \mu_k\phi_k^TM_a\phi_j. \quad (4)$$

If the eigenvalues are distinct $(\mu_j \neq \mu_k)$ then by subtracting equation (4) from equation (3) we obtain

$$\phi_k^TM_a\phi_j = 0,\ j \neq k$$

and

$$\phi_j^TM_a\phi_j = m_j,$$

where $m_j$ is known as the $j$th generalized mass. The result that the product of an eigenvector with a scalar multiple is also an eigenvector leads to the important question of scaling or normalization of eigenvectors. A common and useful approach is to arrange that the eigenvectors are normalized such that

$$\phi_k^TM_a\phi_j = 0,\ j \neq k,\quad \phi_j^TM_a\phi_j = 1,\ j = 1, 2, \cdots, n. \quad (5)$$

Accurate dynamic models are required to establish the dynamic response of complex structures. Unfortunately, due to the inappropriate theoretical assumptions, inaccuracies in estimated material properties, insufficient or incorrect modelling detail, and improper application of solution algorithms, precise mathematical models are rarely available in practice. In other words, natural frequencies and mode shapes of an analytical model described by (2) do not match very well with experimentally measured frequencies and mode shapes obtained from a real-life vibrating structure. Thus, a vibration engineer needs to update the theoretical analytical model of the structure such that the updated model predicts the observed dynamic behavior. The improved model may be considered to

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be a better dynamic representation of the structure. This model can be used with greater confidence for the analysis of the structure under different boundary conditions or with physical structural changes.

Let $X \in \mathbf{R}^{n\times p}$ be the measured modal matrix, $\Lambda \in \mathbf{R}^{p\times p}$ the measured eigenvalues matrix, where $p < n$, and $\Lambda$ is diagonal. The most common approach in finite element model updating is to modify the analytical mass and stiffness matrices to satisfy the basic orthogonality requirement and eigenvalue equation, as shown in Eq. (6).

$$X^T MX = I_p, \quad MX\Lambda = KX, \tag{6}$$

where $M,\ K \in \mathbf{R}^{n\times n}$ are symmetric matrices and represent the corrected mass and stiffness matrices, respectively.

There have been a number of publications of methods which use vibration test data to improve an analytical model of a structure. Friswell and Mottershead [1] provided a comprehensive overview that illustrates many of the different techniques and issues involved in updating a finite element model. For example, Baruch [2], Baruch and Bar-Itzhack [3] proposed a method that the stiffness matrix was corrected based on measured mode shapes from vibration tests by minimizing a norm to use the symmetric positive definite mass matrix as the weighting matrix. Berman [4] described the changes in the mass matrix required to satisfy the orthogonality relationship using a minimum-weighted Euclidean norm and the method of Lagrange multipliers. Berman and Nagy [5] proposed a direct method to identify a set of minimum changes in the analytical matrices which force the eigensolutions to agree with test measurements. Caesar and Pete [6] discussed two methods for the direct updating of mathematical models based on modal test data. Wei [7] proposed a new approach to show the uniqueness of the corrected stiffness matrix in a different way. Wei [8, 9] introduced an approach that can update the mass and stiffness matrices simultaneously using the measured eigenvector matrix as the reference. The constraints imposed are mass orthogonality, the equation of motion and symmetry of the updated matrices. These constraints will force the updated stiffness matrix to satisfy the stiffness orthogonality condition and the effects due to mass and stiffness interaction are clearly determined from the final equations. Recently, Carvalho et al. [10] proposed a direct method which needs the knowledge of only a small number of eigenvalues and eigenvectors of the associated analytical quadratic matrix pencil, which are required to be reassigned to the measured data; while the remaining large number of eigenvalues and eigenvectors which cannot be computed even using the state-of-the-art algorithms and softwares, are guaranteed to remain invariant by means of a proven mathematical result. Generally speaking, only part elements of coefficient matrices have errors. Yuan [11] provided a new local updating method to adjust partial elements of the analytical mass and stiffness matrices $M_a$ and $K_a$ using measured response data. Yang and Chen [12] proposed a direct method for updating the mass and stiffness matrices of a structural model such that it can reproduce the frequencies measured for the structure. Firstly, only the first eigen mode measured for the structure was considered. The updated matrices were derived by utilizing the orthogonality conditions for the eigenvectors, and by replacing the modal vector of concern by the modal matrix, to resolve the problem of lack of available equations. Such a procedure is then generalized to deal with the case where the first few eigen modes are measured for the structure. Taking advantage of the special structure of the constraint sets, Moreno et al. [13] shown that the matrix model updating problem can be formulated as an optimization problem over the intersection of some special subspaces and linear varieties on the space of matrices. Using this formulation, an alternating projection method (APM) is then proposed and analyzed. The projections onto the involved subspaces and linear varieties are characterized. Based on the orthogonality constraints, Yang et al. [14] introduced a direct method for updating the mass and stiffness matrices of the structure first using a single set of modal data. This method hinges on replacement of the modal vector of concern by the modal matrix in computing the correction matrices to solve the problem of insufficient known conditions. Such a method is then extended and applied in a consecutive manner to update the structural model for each of the first few modes that are experimentally made available. All these existing methods can reproduce the given set of measured data while updated matrices may be symmetric, but the analytical mass and stiffness matrices can be dramatically altered. Particularly troublesome is the modification of mass and stiffness coefficients from values of zero to large magnitude nonzero values. Clearly, the introduction of load paths that do not exist in the actual hardware is undesirable.

The purpose of the work presented in this paper is to develop a new method for finite element model updating problems which preserves the connectivity of the original model. That is, only non-zero elements of analytical mass and stiffness matrices are to be modified and zero elements are guaranteed to remain invariant by the proposed method. On the other hand, the analytical matrices are sparse and only contain non-zero elements in a band along the leading diagonal. Therefore, to preserve the structural connectivity means to preserve the bandwidth of the updated matrices. Now, assume that $M_a$ and $K_a$ are real-valued symmetric $(2r+1)$-diagonal matrices, where $r$ represents the half-bandwidth of stiffness and mass matrices. Thus, the problem of updating mass and stiffness matrices simultaneously can be mathematically formulated as follows.

**Problem IP.** Given $X \in \mathbf{R}^{n\times p}$ and a diagonal matrix $\Lambda \in \mathbf{R}^{p\times p}$, find real-valued symmetric $(2r+1)$-diagonal matrices $M$ and $K$ that satisfy the equation of (6).

**Problem II.** Let $\mathcal{S}_{MK}$ be the solution set of IP. Find $(\hat{M},\hat{K}) \in \mathcal{S}_{MK}$ such that

$$\begin{array}{l}\|\hat{M}-M_a\|^2+\|\hat{K}-K_a\|^2 \\ =\min_{(M,K)\in\mathcal{S}_{MK}}(\|M-M_a\|^2+\|K-K_a\|^2).\end{array} \quad (7)$$

The paper is organized as follows. In Section 2, using the Kronecker product and stretching function vec$(\cdot)$ of matrices, we give a necessary and sufficient condition for the solution set $\mathcal{S}_{MK}$ to be nonempty and construct $\mathcal{S}_{MK}$ explicitly when it is nonempty. In Section 3, we show that there exists a unique solution in Problem II and present the expression of the unique solution $(\hat{M},\hat{K})$ of Problem II. Finally, in Section 4, a numerical algorithm to acquire the optimal approximation solution under the Frobenius norm sense is described and a numerical example is provided.

## II. The solution of Problem IP

To begin with, we present two already established lemmas.

*Lemma 1:* [15] If $L\in\mathbf{R}^{m\times q},\ b\in\mathbf{R}^m$, then $Ly=b$ has a solution $y\in\mathbf{R}^q$ if and only if $LL^+b=b$. In this case, the general solution of the equation can be described as $y=L^+b+(I_q-L^+L)z$, where $z\in\mathbf{R}^q$ is an arbitrary vector.

*Lemma 2:* [16] Let $D\in\mathbf{R}^{m\times n}, H\in\mathbf{R}^{n\times l}, J\in\mathbf{R}^{l\times s}$. Then

$$\text{vec}(DHJ)=(J^T\otimes D)\text{vec}(H).$$

Let $S_0$ be the set of all $n\times n$ real-valued symmetric $(2r+1)$-diagonal matrices, then $S_0$ is a linear subspace of $\mathbf{SR}^{n\times n}$, and the dimension of $S_0$ is $N=\frac{1}{2}(2n-r)(r+1)$.
Define $Y_{ij}$ as

$$Y_{ij}=\left\{\begin{array}{l}\frac{\sqrt{2}}{2}(e_ie_j^T+e_je_i^T), \\ \qquad i=1,\cdots,n-1; j=i+1,\cdots,t_i, \\ e_ie_i^T,\ i=j=1,\cdots,n,\end{array}\right. \quad (8)$$

where $t_i=\min\{i+r,n\}$ and $e_i,\ i=1,\cdots,n$, is the $i$th column vector of the identity matrix $I_n$. It is easy to verify that $\{Y_{ij}\}$ forms an orthonormal basis of the subspace $S_0$, that is,

$$(Y_{ij},Y_{kl})=\left\{\begin{array}{l}0,\ i\neq k \text{ or } j\neq l, \\ 1,\ i=k \text{ and } j=l.\end{array}\right. \quad (9)$$

Now, if $M,K\in\mathbf{SR}^{n\times n}$ are $(2r+1)$-diagonal matrices, then $M,K$ can be expressed as

$$M=\sum_{i,j}\alpha_{ij}Y_{ij},\ K=\sum_{i,j}\beta_{ij}Y_{ij}, \quad (10)$$

where the real numbers $\alpha_{ij},\ \beta_{ij},\ i=1,\cdots,n;\ j=i,\cdots,t_i,\ t_i=\min\{i+r,n\}$, are yet to be determined. Substituting (10) into (6), we have

$$\sum_{i,j}\alpha_{ij}X^TY_{ij}X=I_p,\quad \sum_{i,j}\alpha_{ij}Y_{ij}X\Lambda-\sum_{i,j}\beta_{ij}Y_{ij}X=0. \quad (11)$$

Let

$$\begin{array}{rcl}\alpha &=& [\alpha_{11},\cdots,\alpha_{1,r+1},\cdots,\alpha_{n-r,n-r}, \\ && \cdots,\alpha_{n-r,n},\cdots,\alpha_{n-1,n-1},\alpha_{n-1,n},\alpha_{n,n}]^T,\end{array}$$

$$\begin{array}{rcl}\beta &=& [\beta_{11},\cdots,\beta_{1,r+1},\cdots,\beta_{n-r,n-r}, \\ && \cdots,\beta_{n-r,n},\cdots,\beta_{n-1,n-1},\beta_{n-1,n},\beta_{n,n}]^T,\end{array}$$

$$\begin{array}{l}G=[\text{vec}(Y_{11}),\cdots,\text{vec}(Y_{1,r+1}),\cdots, \\ \text{vec}(Y_{n-r,n-r}),\cdots,\text{vec}(Y_{n-r,n}),\cdots, \\ \text{vec}(Y_{n-1,n-1}),\text{vec}(Y_{n-1,n}),\ \text{vec}(Y_{n,n})]\in\mathbf{R}^{n^2\times N}\end{array} \quad (12)$$

and

$$\begin{array}{l}A=(X^T\otimes X^T)G, \\ B=(\Lambda X^T\otimes I_n)G, \\ C=(X^T\otimes I_n)G, \\ h=\text{vec}(I_p).\end{array} \quad (13)$$

Using Lemma 2, we see that the equations of (11) are equivalent to

$$A\alpha = h, \quad (14)$$

$$B\alpha-C\beta = 0. \quad (15)$$

It follows from Lemma 1 that the equation of (14) with unknown vector $\alpha$ has a solution if and only if

$$AA^+h=h. \quad (16)$$

In which case, the general solution of the equation (14) is

$$\alpha=A^+h+F_Az, \quad (17)$$

where $z\in\mathbf{R}^N$ is an arbitrary vector. Substituting (17) into (15), we obtain

$$C\beta=BA^+h+BF_Az. \quad (18)$$

Using Lemma 1 again, we know that the equation of (18) with respect to $\beta$ has a solution if and only if

$$Wz=-E_CBA^+h, \quad (19)$$

where $W=E_CBF_A$. It follows from Lemma 1 that the equation of (19) with unknown vector $z$ has a solution if and only if

$$WW^+E_CBA^+h=E_CBA^+h. \quad (20)$$

In this case, the general solution of the equation (19) is

$$z=-W^+E_CBA^+h+F_Wu, \quad (21)$$

where $u\in\mathbf{R}^N$ is an arbitrary vector. Substituting (21) into (17) and (18) and applying Lemma 1, we obtain

$$\alpha=A^+h-F_AW^+E_CBA^+h+F_AF_Wu, \quad (22)$$

$$\begin{array}{l}\beta=C^+BA^+h \\ \quad -C^+BF_AW^+E_CBA^+h+C^+BF_AF_Wu+F_Cv,\end{array} \quad (23)$$

where $v \in \mathbf{R}^N$ is an arbitrary vector.

As a summary of the above discussion, we have proved the following result.

*Theorem 1:* Suppose that $X \in \mathbf{R}^{n\times p}, \Lambda \in \mathbf{R}^{p\times p}$, and $\Lambda$ is a diagonal matrix. Let $\{Y_{ij}\}, G$ and $A, B, C, h$ be given by (8), (12) and (13), respectively. Write $N = \frac{1}{2}(2n-r)(r+1)$, $W = E_C B F_A$. Then Problem IP is solvable if and only if the conditions (16) and (20) hold, in this case, the solution set $\mathcal{S}_{MK}$ of Problem IP can be expressed as

$$\begin{array}{c}\mathcal{S}_{MK} = \{(M,K) \in \mathbf{SR}^{n\times n} \times \mathbf{SR}^{n\times n} | \\ M = S(\alpha \otimes I_n),\ K = S(\beta \otimes I_n)\},\end{array} \tag{24}$$

where

$$\begin{array}{c}S = [Y_{11}, \cdots, Y_{1,r+1}, \cdots, Y_{n-r,n-r}, \cdots, \\ Y_{n-r,n}, \cdots, Y_{n-1,n-1}, Y_{n-1,n}, Y_{n,n}] \in \mathbf{R}^{n\times nN},\end{array} \tag{25}$$

$\alpha,\ \beta$ are, respectively, given by (22) and (23) with $u,\ v \in \mathbf{R}^N$ being arbitrary vectors.

## III. The solution of Problem II

In order to solve Problem II, we need the following lemma.

*Lemma 3:* Let $F_A = I_N - A^+A, F_W = I_N - W^+W$, where $W = E_C B F_A$. Then

$$F_A F_W = F_W F_A.$$

**Proof.** It follows from $WF_A = W$ that

$$F_W F_A = (I_N - W^+W)F_A = F_A - W^+W.$$

Therefore,

$$\begin{aligned} F_A F_W &= (F_W F_A)^T = (F_A - W^+W)^T \\ &= (F_A)^T - (W^+W)^T = F_A - W^+W, \end{aligned}$$

which implies the conclusion.

It follows from Theorem 1 that the set $\mathcal{S}_{MK}$ is nonempty if the conditions (16) and (20) are satisfied. It is easy to verify that $\mathcal{S}_{MK}$ is a closed convex subset of $\mathbf{SR}^{n\times n} \times \mathbf{SR}^{n\times n}$. From the best approximation theorem [17], we know there exists a unique solution $(\hat{M}, \hat{K})$ in $\mathcal{S}_{MK}$ such that (7) holds.

We now focus our attention on seeking the unique solution $(\hat{M}, \hat{K})$ in $\mathcal{S}_{MK}$. For the real-valued symmetric $(2r+1)$-diagonal matrices $M_a$ and $K_a$, it is easily seen that $M_a,\ K_a$ can be expressed as the linear combinations of the orthonormal basis $\{Y_{ij}\}$, that is,

$$M_a = \sum_{i,j} \gamma_{ij} Y_{ij},\ K_a = \sum_{i,j} \delta_{ij} Y_{ij}, \tag{26}$$

where $\gamma_{ij},\ \delta_{ij},\ i = 1, \cdots, n; j = i, \cdots, t_i,\ t_i = \min\{i+r, n\}$, are uniquely determined by the elements of $M_a$ and $K_a$. Let

$$\begin{array}{c}\gamma = [\gamma_{11}, \cdots, \gamma_{1,r+1}, \cdots, \gamma_{n-r,n-r}, \cdots, \\ \gamma_{n-r,n}, \cdots, \gamma_{n-1,n-1}, \gamma_{n-1,n}, \gamma_{n,n}]^T,\end{array} \tag{27}$$

$$\begin{array}{c}\delta = [\delta_{11}, \cdots, \delta_{1,r+1}, \cdots, \delta_{n-r,n-r}, \cdots, \\ \delta_{n-r,n}, \cdots, \delta_{n-1,n-1}, \delta_{n-1,n}, \delta_{n,n}]^T.\end{array} \tag{28}$$

Then, for any pair of matrices $(M, K) \in \mathcal{S}_{MK}$ in (24), by the relations of (9) and (26) we see that

$$\begin{aligned} f &= \|M - M_a\|^2 + \|K - K_a\|^2 \\ &= \| \sum_{i,j} (\alpha_{ij} - \gamma_{ij}) Y_{ij} \|^2 + \| \sum_{i,j} (\beta_{ij} - \delta_{ij}) Y_{ij} \|^2 \\ &= (\sum_{i,j} (\alpha_{ij} - \gamma_{ij}) Y_{ij}, \sum_{i,j} (\alpha_{ij} - \gamma_{ij}) Y_{ij}) \\ &+ (\sum_{i,j} (\beta_{ij} - \delta_{ij}) Y_{ij}, \sum_{i,j} (\beta_{ij} - \delta_{ij}) Y_{ij}) \\ &= \sum_{i,j} (\alpha_{ij} - \gamma_{ij})(Y_{ij}, \sum_{i,j} (\alpha_{ij} - \gamma_{ij}) Y_{ij}) \\ &+ \sum_{i,j} (\beta_{ij} - \delta_{ij})(Y_{ij}, \sum_{i,j} (\beta_{ij} - \delta_{ij}) Y_{ij}) \\ &= \sum_{i,j} (\alpha_{ij} - \gamma_{ij})^2 + \sum_{i,j} (\beta_{ij} - \delta_{ij})^2 \\ &= \|\alpha - \gamma\|^2 + \|\beta - \delta\|^2. \end{aligned}$$

Substituting (22) and (23) into the relation of $f$, we have

$$\begin{aligned} f &= \|\alpha_0 + F_A F_W u\|^2 + \|\beta_0 + C^+ B F_A F_W u + F_C v\|^2 \\ &= \alpha_0^T \alpha_0 + 2\alpha_0^T F_A F_W u + u^T F_W F_A F_W u + \beta_0^T \beta_0 \\ &+ 2\beta_0^T C^+ B F_A F_W u + 2\beta_0^T F_C v \\ &+ u^T F_W F_A B^T (CC^T)^+ B F_A F_W u + v^T F_C v, \end{aligned}$$

where

$$\begin{array}{l}\alpha_0 = A^+h - F_A W^+ E_C B A^+ h - \gamma, \\ \beta_0 = C^+ B A^+ h - C^+ B F_A W^+ E_C B A^+ h - \delta.\end{array} \tag{29}$$

Therefore,

$$\begin{aligned} \frac{\partial f}{\partial u} &= 2F_W F_A \alpha_0 + 2F_W F_A F_W u + 2F_W F_A B^T (C^+)^T \beta_0 \\ &\quad +2F_W F_A B^T (CC^T)^+ B F_A F_W u, \\ \frac{\partial f}{\partial v} &= 2F_C \beta_0 + 2F_C v. \end{aligned}$$

Since $f$ is a quadratic function with respect to variables $u$ and $v$. It is easy to verify that the function $f = \|M - M_a\|^2 + \|K - K_a\|^2 = \min$ attains the smallest value at

$$\frac{\partial f}{\partial u} = 0,\ \frac{\partial f}{\partial v} = 0. \tag{30}$$

Applying Lemma 3, we get from the equation of (30) that

$$\begin{array}{c}F_A F_W u = -(I_N + F_W F_A B^T (CC^T)^+ B F_A F_W)^{-1} \\ F_W F_A (\alpha_0 + B^T (C^+)^T \beta_0),\end{array} \tag{31}$$

$$F_C v = -F_C \beta_0. \tag{32}$$

Upon substituting (31) and (32) into (22) and (23), we obtain

$$\hat{\alpha} = \alpha_0 + \gamma - F_A F_W (I_N + F_W F_A B^T (CC^T)^+ B F_A F_W)^{-1} F_W F_A (\alpha_0 + B^T (C^+)^T \beta_0), \tag{33}$$

$$\hat{\beta} = \beta_0 + \delta - C^+ B F_A F_W (I_N + F_W F_A B^T (CC^T)^+ B F_A F_W)^{-1} F_W F_A (\alpha_0 + B^T (C^+)^T \beta_0) - F_C \beta_0. \tag{34}$$

By now, we have proved the following result.

*Theorem 2:* Let the real-valued symmetric $(2r+1)$-diagonal matrices $M_a$ and $K_a$ be given. Then Problem II has a unique solution and the unique solution of Problem II can be expressed as

$$\hat{M} = S(\hat{\alpha} \otimes I_n), \tag{35}$$

$$\hat{K} = S(\hat{\beta} \otimes I_n), \tag{36}$$

where $\hat{\alpha},\ \hat{\beta}$ are given by (33) and (34), respectively.

## IV. A NUMERICAL EXAMPLE

Based on Theorem 1 and Theorem 2 we can describe an algorithm for solving Problem IP and Problem II as follows.

**Algorithm 1**.

1) Input $M_a,\ K_a,\ \Lambda,\ X$.
2) Form the orthonormal basis $\{Y_{ij}\}$ by (8).
3) Compute $G$ and $A,\ B,\ C,\ h$ according to (12) and (13), respectively.
4) Compute $E_C = I_{np} - CC^+,\ F_A = I_N - A^+ A$ and $W = E_C B F_A$.
5) If the conditions (16) and (20) are satisfied, go to 6); otherwise, Problem IP has no solution, and stop.
6) Form vectors $\gamma,\ \delta$ by (26), (27) and (28).
7) Compute $\alpha_0,\ \beta_0$ by (29).
8) Compute $S,\ \hat{\alpha},\ \hat{\beta}$ by (25), (33) and (34), respectively.
9) Compute the unique solution $(\hat{M}, \hat{K})$ of Problem II by (35) and (36).

**Example 1**. Consider a five-DOF system modelled analytically with mass and stiffness matrices given by

$$M_a = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix},$$

$$K_a = \begin{bmatrix} 100 & -20 & 0 & 0 & 0 \\ -20 & 120 & -35 & 0 & 0 \\ 0 & -35 & 80 & -12 & 0 \\ 0 & 0 & -12 & 95 & -40 \\ 0 & 0 & 0 & -40 & 124 \end{bmatrix}.$$

That is, $M_a,\ K_a$ are symmetric 3-diagonal matrices. The first two measured modal data are given by

$$\Lambda = \text{diag}\{\lambda_1,\ \lambda_2\} = \text{diag}\{9.9883,\ 16.5605\},$$

$$X = [x_1,\ x_2] = \begin{bmatrix} -0.0643 & 0.1068 \\ -0.1783 & 0.2079 \\ -0.2898 & 0.1642 \\ -0.2091 & -0.2739 \\ -0.1190 & -0.2447 \end{bmatrix}.$$

It is easy to verify that $\|AA^+h - h\| = 7.0436e-016$ and $\|WW^+E_C BA^+h - E_C BA^+h\| = 4.7914e-014$. It follows from Theorem 1 that $\mathcal{S}_{MK}$ is nonempty. Using Algorithm 1, we obtain the unique solution of Problem II as follows.

$$\hat{M} = \begin{bmatrix} 1.6248 & 1.1273 & 0 & 0 & 0 \\ 1.1273 & 3.5604 & 1.3468 & 0 & 0 \\ 0 & 1.3468 & 3.7157 & 1.0068 & 0 \\ 0 & 0 & 1.0068 & 3.6448 & 1.4721 \\ 0 & 0 & 0 & 1.4721 & 3.4253 \end{bmatrix},$$

$$\hat{K} = \begin{bmatrix} 100.4389 & -19.1142 & 0 & 0 & 0 \\ -19.1142 & 122.0577 & -33.0111 & 0 & 0 \\ 0 & -33.0111 & 82.2541 & -12.9018 & 0 \\ 0 & 0 & -12.9018 & 98.0100 & -37.6073 \\ 0 & 0 & 0 & -37.6073 & 126.1021 \end{bmatrix},$$

We can figure out

$$\|\hat{M}X\Lambda - \hat{K}X\| = 2.5418e-014,$$

$$\|X^T \hat{M} X - I_p\| = 5.4177e-016.$$

Therefore, the updated mass and stiffness matrices satisfy the required eigenvalue equation and orthogonality condition and the matrices $\hat{M},\ \hat{K}$ are also symmetric 3-diagonal matrices, which implies that the structural connectivity information of the analytical model is preserved.

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