

Applications of Stable Distributions in Time series analysis, Computer sciences and Financial markets

Mohammad Ali Baradaran Ghahfarokhi

Parvin Baradaran Ghahfarokhi*

(*science and research branch – Azad University - Tehran- Iran)

Abstract—In this paper, first we introduce the stable distribution, stable process and their characteristics. The α -stable distribution family has received great interest in the last decade due to its success in modeling data, which are too impulsive to be accommodated by the Gaussian distribution. In the second part, we propose major applications of alpha stable distribution in telecommunication, computer science such as network delays and signal processing and financial markets. At the end, we focus on using stable distribution to estimate measure of risk in stock markets and show simulated data with statistical softwares.

Keywords—stable distribution, $S\alpha S$, infinite variance, heavy tail networks, VaR,

I. INTRODUCTION

THE α -stable distribution family has received interest in the last decade due to its success in modeling data, which are too impulsive to be accommodated by the Gaussian distribution. Despite this relatively new interest in the signal processing community, the history of research on this particular distribution family is old starting with the work of Levy (1925). The applications of Alpha-stable distributions have been limited though until much later when Mandelbrot (1963) suggested them as models for financial time series data. Later, Stuck et al.(1974) used them for modeling impulsive noise on telephone lines. It has also been employed as successful model for the atmospheric noise (Nikias, 1995) and found various applications in signal processing (see Nolan and Swami, 1999 for a wide range of recent work). Many physical phenomena are non-Gaussian and if the observed data have frequently occurring extreme values, then the phenomena may be modeled as a random process with an alpha-stable distribution. When positive and negative

outcomes are equally likely, then the process would be symmetric alpha-stable; however, when only positive outcomes are possible, then the process would be positive alpha-stable. Phenomena related to networks, finance and signal processing are examples.

A review of the state of the art on stable processes from a statistical point of view is provided by a collection of papers edited by Cambanis, Samorodnitsky and Taqqu. Several statisticians including Cambanis, Zolotarev, Weron et al have published extensively on the theory and applications of stable processes. They studied the properties of stable processes, their spectral representation as well as prediction and linear filtering problems. Textbooks in the area were written by Samorodnitsky and Taqqu and by Janicki and Weron.

II. STABLE DISTRIBUTIONS

In probability theory, a **Lévy skew alpha-stable distribution** or just **stable distribution**, developed by Paul Lévy, is actually a family of probability distributions which are characterized by four parameters: $\alpha, \beta, \mu, \gamma$, as well as the distributed value, X . The μ and γ are shift and scale parameters which do not determine the shape of the distribution. The stable distribution has the important property of stability: If a number of independent identically distributed (iid) random variables have a stable distribution, then a linear combination of these variables will have the same distribution, except for possibly different shift and scale parameters.

To be more precise:

If X_1 and X_2 are distributed according to a stable distribution $S(x; \alpha, \beta, \mu, \gamma)$, and if $Y = AX_1 + BX_2 + C$ is a linear combination of the two, then there exist values of D and E such that $DY + E$ is distributed according to a stable distribution $S(DY + E; \alpha, \beta, \mu, \gamma)$ or, equivalently, Y is distributed according to a stable distribution

$S(Y; \alpha, \beta, (\mu - E) / D, \gamma / D)$. If $E = 0$ for all A, B , and C then Y is said to have a **strictly stable** distribution. Since the

normal distribution, the Cauchy distribution, and the Lévy distribution all have the above property, it follows that they are special cases of the stable distribution. (Nolan 2005) Stable distributions owe their importance in both theory and practice to the generalization of the Central Limit Theorem (GCLT) to random variables without second (and possibly first) order moments and the concomitant self-similarity of the stable family. It was the demand for self-similarity and the seeming departure from normality of data. All stable distributions are infinitely divisible and with the exception of the normal distribution for which $\alpha=2$, stable distributions are Heavy-tailed distributions.

III. CHARACTERISTICS OF STABLE DISTRIBUTIONS

An α -stable distribution may be thought of as a generalization of the normal distribution where the generalization allows greater concentration close to the mean, more extreme values and possible skewness. The distribution depends on four parameters $\alpha, \beta, \gamma, \delta$. These parameters can be interpreted as follows :

- $\alpha, (0 < \alpha \leq 2)$, is the basic stability parameter. It determines the weight in the tails. The smaller the value of α the greater the frequency and size of extreme events.
- β is a skewness parameter and $-1 \leq \beta \leq 1$. A zero beta implies that the distribution is symmetric. Negative or positive β imply that the distribution is skewed to the left or right respectively
- The parameter γ is positive and measures dispersion. It is similar to the variance of a normal distribution
- The parameter δ is a real number and may be thought of as a location measure. It is similar to the mean of a normal distribution.

The stability property: which states that the random variables X_1, \dots, X_n are independent and symmetrically stable with the same characteristic exponent α if and only if for any constants a_1, \dots, a_n the linear combination $\sum_{i=1}^n a_i X_i$ is also stable distribution.

The generalized central limit theorem (GCLT):

which states that the family of stable distributions contains all limiting distributions of sums of iid random variables. The central limit theorem states that the sum of a number of random variables with finite variances will tend to a normal distribution as the number of variables grows. A generalization due to Gnedenko and Kolmogorov states that the sum of a number of random variables with power-law tail distributions decreasing as $1/|x|^{\alpha+1}$ (and therefore having infinite variance) will tend to a stable Lévy distribution $f(x; \mu, 0, c, 0)$ as the number of variables grows. (Voit 2003 § 5.4.3)

we demonstrate here plots for various parameter values in Figures 1 and 2 for understanding of the behavior of these distributions.

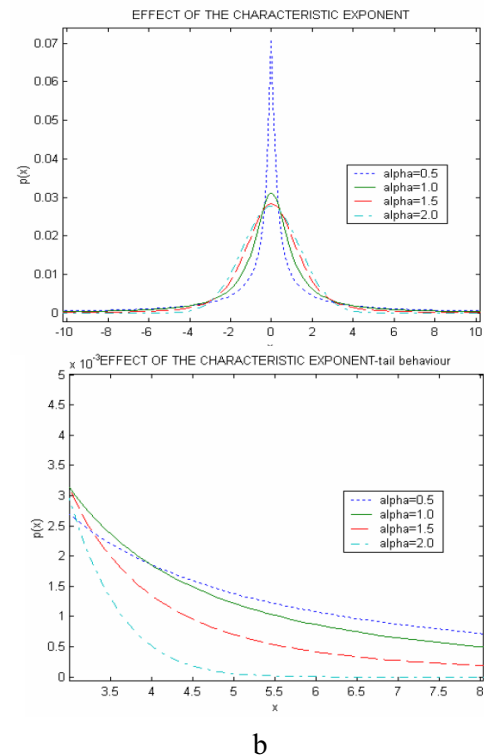


Figure 1: Various α -stable pdfs with varying characteristic exponent (α) values. a) Whole pdfs b) Detail from the tails. Distribution gets more impulsive (heavy-tailed) as α decreases.

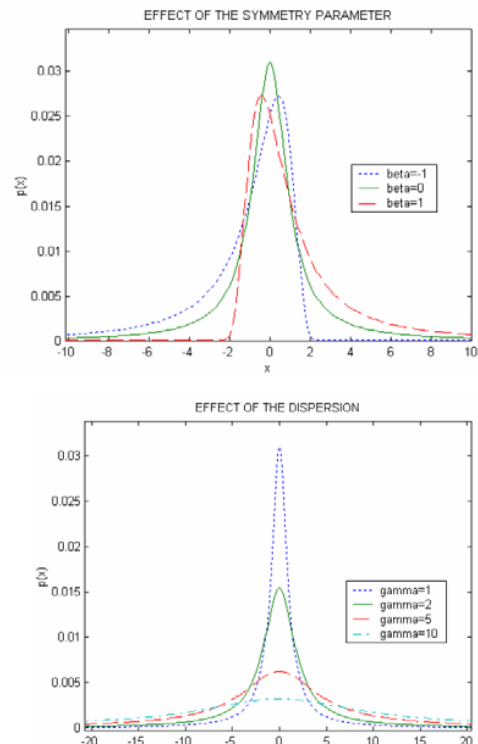


Figure 2: Various α -stable pdfs with varying c) symmetry parameter (β) and d) dispersion (γ).

IV. APPLICATIONS

A. Application in network traffic

Actual network traffic is self-similar or fractal in nature, and therefore could not be modeled by Poisson and Markov processes or variants of Poisson and Markov processes. There are some properties that appear to be important in traffic modeling, such as the traffic burstiness, the heavy tailed distribution and the self-similarity. The alpha-stable process has those properties.

All stable distributions are infinitely divisible and with the exception of the normal distribution for which $\alpha = 2$, stable distributions are Heavy-tailed distributions. This "heavy tail" behavior causes the variance of Lévy distributions to be infinite for all $\alpha < 2$. Alpha-stable model is also able to characterize time-variant delays in Network systems. The modeling of network traffic is important for the design and application of networks, but little is known as to the characteristics of distribution of packets in network traffic.

B. Application in signal and (image processing)

Up to now, the applications of alpha stable distributions in image processing have been very limited. We can see a few works such as: synthesis of textures 2D image models with long-range dependence. (Popescu-Pesquet and Pesquet, 1999). Their model is impulsive but cannot accommodate skewed characteristics. (Tsakalides *et al.*, 2001) considered again symmetric α -stable distributions for modeling the wavelet transform coefficients of subband images. (Achim *et al.*, 2001) employed this model for the removal of speckle noise in SAR images. Work in both signal and image processing have been limited to only symmetric α -stable (SaS) distributions, ignoring skewed distributions in all other than a couple of works on parameter estimation (Dance and Kuruoglu, 1999), (Kuruoglu, 2001) while some real phenomena such as some geophysical signals, teletraffic data and SAR images clearly exhibit skewed characteristics. the α -stable processes provide a very flexible framework for modeling textures in images.

C. Application in finance (stock markets)

The use of the α -stable distribution was first advocated in the 60's by Mandelbrot (Mandelbrot (1962, 1964, 1967, 1997), Mandelbrot and Hudson (2004)) and Fama (1964, 1965, 1976). Mandelbrot examined the variation of prices of cotton (1816-1940), wheat (1883-1936), railroad stock (1857-1936) and interest and exchange rates (similar periods) and found a larger number of extreme values than could be justified by the assumption of a normal distribution. Mandelbrot proposed the stable distribution as a suitable model for price differences, $\xi = S_{i+1} - S_i$, or logarithmic returns, $\xi \log = \log(S_{i+1}) - \log(S_i)$ [4]. In the financial literature, the debate on the stable model

focused on the infinite variance of the distribution, leading to the introduction of subordinated models [5-7];

Fama examined the distribution of daily returns for the 30 stock in the Dow Jones Industrial Average in a period from about the end of 1957 to September 26 1962. There was considerable interest in the α -stable distribution throughout the 60's and the early 70's but interest then declined.

In the physical literature, Mantegna used the model for the empirical analysis of historical stock-exchange indices [8]. Later, Mantegna and Stanley proposed a "truncated" Lévy distribution [9-11], an instance of the so-called KoBoL (Koponen, Boyarchenko and Levendorskii) distributions

Value at Risk (VaR) is the most common measures of risk used in many financial institutions. VaR at a p% level is estimated as the loss that might be exceeded p% of the time. Like many other models in finance it is often based on an assumption that losses follow a normal distribution. It is now well known that extreme losses are greater than, and occur much more often than, a normal distribution would predict.

Value at Risk of the stock or (VaR) : measure of risk can be improved by the use of an α -stable distribution in place of more conventional measures. We show that α -stable based measures are feasible and are better than conventional measures. They are a useful tool for the risk manager and the financial regulator. According to (Frain 2008) We explain why it is a good candidate for the distribution of losses.

The purpose of this part is to show that calculated VaR at various levels assuming that losses follow with α -stable innovations could be estimated with this distribution as a good model. The resulting estimates are compared with estimates obtained from static normal and t-distributions. The portfolios examined are six total returns 1 equity indices (ISEQ, CAC40, DAX30, FTSE100, S&P500, Dow Jones Composite (DJAC)). VaR is estimated at 10%, 5% levels.

Measure of risk estimates with α stable distribution

The quintiles are calculated on the basis of returns following

- an α -stable distribution with parameters estimated by maximum likelihood (MLE)
 - a normal distribution with parameters estimated by maximum likelihood (MLE)
 - a t-distribution with nonzero mean, nonzero scale and degrees of freedom to be estimated by maximum likelihood
- Tables 1 and 2 show the estimates of the VaR at 10%, 5% levels for an investment in each of the six total returns equity indices.

Table 1: 10% VaR for each equity index assuming the specified distribution

Index	Distribution			(1) Sample Quantile	Quantile s.e. (2)
	Stable	Normal	t		
ISEQ	1.04	1.23	1.13	1.03	0.03
CAC40	1.48	1.61	1.52	1.43	0.03
DAX30	1.27	1.49	1.31	1.23	0.02
FTSE100	1.15	1.30	1.22	1.13	0.02
DJC	1.05	1.26	1.11	1.04	0.03
S&P500	1.10	1.20	1.14	1.09	0.03
(1) Harrell and Davis (1982)					
(2) Bootstrap estimate					

Table 2: 5% VaR for each equity index assuming the specified distribution

Index	Distribution			Sample Quantile (1)	Quantile s.e. (2)
	Stable	Normal	t		
ISEQ	1.48	1.60	1.57	1.50	0.05
CAC40	2.02	2.08	2.07	2.04	0.05
DAX30	1.75	1.92	1.81	1.76	0.03
FTSE100	1.59	1.68	1.66	1.55	0.03
DJC	1.46	1.63	1.53	1.47	0.05
S&P500	1.54	1.55	1.56	1.55	0.04
(1) Harrell and Davis (1982)					
(2) Bootstrap estimate					

The estimates for the α -stable distribution are very good at the 10%, 5% levels

V. CONCLUSION

In this paper, we introduced the α -stable distribution and explains why it is a good candidate for the distribution of heavy tail of networks, signals and losses. Although the Stable density behaves approximately like a Gaussian density near the origin, its tails decay at a lower rate than the Gaussian density tails. While the Gaussian density has exponential tails. This implies that random variables following stable distributions with small characteristic exponents are highly impulsive. It is this heavy tail characteristic that makes this densities appropriate for modeling network delays, signals and noise, financial risk or interference which are impulsive in nature.

We also have shown that the estimates of VaR derived from an α -stable distribution are feasible and are a useful addition to the toolbox of a risk manager or a financial regulator.

REFERENCES

- [1] Achim, A., A. Bezerianos, and P. Tsakalides (2002). SAR image denoising: a multiscale robust statistical approach. *IEEE Proc. 14-th Intl. Conf. on Digital Signal Processing (DSP 2002), Santorini, Greece II*, 1235{1238.
- [2] Achim, A., P. Tsakalides, and A. Bezerianos (2003). SAR image denoising via Bayesian wavelet shrinkage based on heavy-tailed modeling. *IEEE Transactions on Geoscience and Remote Sensing* 41, 1773{1784.
- [3] B. V. Gnedenko and A. N. Kolmogorov (1954). *Limit Distributions for Sums of Independent Random Variables*. Addison-Wesley.
- [4] Cambanis, S., R. Keener, and G. Simons (1983). On α -symmetric multivariate distributions. *J. Multivar. Anal.* 13, 213{233.
- [5] Cambanis, S. and M. Maejima (1989). Two classes of self-similar stable processes with stationary increments. *Stoch. Proc. Appl.* 32, 305{329.
- [6] Cambanis, S., M. Maejima, and G. Samorodnitsky (1990). Characterization of linear and harmonizable fractional stable motions. Preprint.
- [7] Cambanis, S. and E. Masry (1991). Wavelet approximation of deterministic and random signals: Convergence properties and rates. Technical Report 352, Center for Stochastic Processes, Univ. of North Carolina.
- [8] Fama, E. (1965). The behavior of stock market prices. *Journal of Business* 38, 34{105.
- [9] Fama, E. (1971). Risk, return and equilibrium. *The Journal of Political Economy* 79, 30{55.
- [10] Fama, E. and R. Roll (1968). Some properties of symmetric stable distributions. *JASA* 63, 817{83.
- [11] Fama, E. and R. Roll (1971). Parameter estimates for symmetric stable distributions. *Journal of the American Statistical Association* 66, 331{338.
- [12] GNU Scientific Library - Reference Manual Edition 1.6, for GSL Version 1.6, 27 December 2004
- [13] Levy, J. and M. Taqqu (1991). A characterization of the asymptotic behavior of stationary stable processes. In S. Cambanis, G. Samorodnitsky, and
- [14] M. Taqqu (Eds.), *Stable Processes and Related Topics*, Volume 25 of *Progress in Probability*, Boston, pp. 181{198. Birkhäuser.
- [15] Levy, J. and M. Taqqu (2005, November). The asymptotic covariance and covariation of log-fractional stable noise. Preprint, BundesBank November 2005 Conference.
- [16] Levy, P. (1924). Théorie des erreurs la loi de Gauss et les lois exceptionnelles. *Bulletin Soc. Math. France* 52, 49{85.
- [17] Levy, P. (1925). *Calcul des Probabilités*. Paris: Gauthier-Villars.
- [18] Levy, P. (1954). *Théorie de l'addition des variables aléatoires*. Paris: Gauthier-Villars. Originally appeared in 1937
- [19] Mandelbrot, B. (1961). Stable Paretian random functions and the multiplicative variation of income. *Econometrica* 29, 517{543.
- [20] Mandelbrot, B. (1962). Sur certain prix spéculatifs: faits empiriques et modèle basé sur les processus stables additifs de Paul Lévy. *Comptes Rendus*. 254, 3968{3970.
- [21] Mandelbrot, B. (1963a). New methods in statistical economics. *Journal of Political Econ.* 71, 421{440.
- [22] Mandelbrot, B. (1963b). The variation of certain speculative prices. *Journal of Business* 26, 394{419.
- [23] Mandelbrot, B. (1965a). Self-similar error clusters in communications systems and the concept of conditional systems and the concept of conditional stationarity. *I.E.E.E. Trans of Communications Technology COM-13*, 71{90.
- [24] Mandelbrot, B. (1965b). Une classe de processus stochastiques homothétiques a soi; application a loi climatologique de H.E. Hurst. *Comptes Rendus Acad. Sci. Paris* 240, 3274{3277.
- [25] Mandelbrot, B. (1967). The variation of some other speculative prices. *Journal of Business* 40, 393{413.
- [26] Mandelbrot, B. (1969). Long-run linearity, locally Gaussian processes, H-spectra and infinite variances. *International Econom. Rev.* 10, 82{113.
- [27] Mandelbrot, B. (1971). A fast fractional Gaussian noise generator. *Water Resour. Res.* 7, 543{553.
- [28] Mandelbrot, B. (1972a). Broken line process derived as an approximation to fractional noise. *Water Resour. Res.* 8, 1354{1356.
- [29] Mandelbrot, B. (1972b). Statistical methodology for non-periodic cycles: from the covariance to r/s analysis. *Ann. Econom. Soc. Measurement* 1, 259{290.
- [30] Mandelbrot, B. (1974). Intermittent turbulence in self-similar cascades; divergence of high moments and dimension of the carrier. *J. Fluid Mechanics* 62, 331{358.
- [31] Mandelbrot, B. (1975a). Fonctions aléatoires pluri-temporelles: approximation poissonienne du cas brownien et généralisations. *Comptes Rendus Acad. Sci.*

Paris 280A, 1075 {1078.

- [32] Mandelbrot, B. and M. Taylor (1967). On the distribution of stock price differences. *Operations Research* 15, 1057{1062.
- [33] Mantegna, R. N. (1994). Fast, accurate algorithm for numerical simulation of Lévy stable stochastic processes. *Physical Review E* 49 (5), 4677 { 4683.
- [34] Nikias, C. L. and M. Shao (1995). *Signal Processing with Alpha-Stable Distributions and Applications*. New York: Wiley.
- [35] Nolan, J. P. (2005a). Identification of stable measures. In progress.
- [36] Nolan, J. P. (2005b). Multivariate stable densities and distribution functions: general and elliptical case. Preprint, BundesBank November 2005 Conference.
- [37] Nolan, J. P. (2006). Multivariate elliptically contoured stable distributions: theory and estimation. Submitted.
- [38] Nolan, J. P. (2007). Metrics for multivariate stable distributions. Preprint.
- [39] Nolan, J. P. (2008a, July). Advances in nonlinear signal processing for heavy tailed noise. IWAP 2008. 78
- [40] Nolan, J. P. (2008b). Multivariate stable cumulative probabilities in polar form and related functions. In progress.
- [41] Nolan, J. P. (2009). *Stable Distributions - Models for Heavy Tailed Data*. Boston: Birkhäuser. Unfinished manuscript, Chapter 1 online at academic2.american.edu/~jpnolan.
- [42] Nolan, J. P. and D. Ojeda (2006). Linear regression with general stable errors. Submitted.
- [43] Nolan, J. P., A. Panorska, and J. H. McCulloch (2001). Estimation of stable spectral measures. *Mathematical and Computer Modelling* 34, 1113 {1122.
- [44] Nolan, J. P. and A. K. Panorska (1997). Data analysis for heavy tailed multivariate samples. *Comm. in Stat. - Stochastic Models* 13, 687 {702.
- [45] Nolan, J. P. and B. Rajput (1995). Calculation of multidimensional stable densities. *Commun. Statist. - Simula.* 24, 551 {556.
- [46] Nolan, J. P. and N. Ravishanker (2009). Simultaneous prediction intervals for ARMA processes with stable time innovations. *J. of Forecasting*.
- [47] Nolan, J. P. and A. Swami (Eds.) (1999). *Proceedings of the ASA-IMS Conference on Heavy Tailed Distributions*, Washington, DC
- [48] _ Peach, G. (1981). "Theory of the pressure broadening and shift of spectral lines
(<http://journalsonline.tandf.co.uk/openurl.asp?genre=article&eissn=1460-6976&volume=30&issue=3&spage=367>)". *Advances in Physics* 30 (3): 367-474.
- [49] _ V.M. Zolotarev (1986). *One-dimensional Stable Distributions*. American Mathematical Society.
- [50] Samorodnitsky, G. (1988). Extrema of skewed stable processes. *Stoch. Proc. Appl.* 30, 17 {39.
- [51] Samorodnitsky, G. (1991). Probability tails of Gaussian extrema. *Stoch. Proc. Appl.* 38, 55 {84.
- [52] Samorodnitsky, G. (1993a). Integrability of stable processes. *Probab. Math. Stat.*. To appear.
- [53] Samorodnitsky, G. (1993b). Possible sample paths of α -stable self-similar processes. *Stat. Probab. Letters*. To appear.
- [54] Samorodnitsky, G. (1995a). Association of infinitely divisible random vectors. *Stochastic Processes and Their Applications* 55, 45 {56.
- [55] Stuck, B. W. (1976). Distinguishing stable probability measures. *Bell System Technical Journal* 55, 1125 {1182.
- [56] Stuck, B. W. and B. Kleiner (1974). A statistical analysis of telephone noise. *Bell Syst. Tech. J.* 53, 1263 {1320
- [57] Taqqu, M. (1978). A representation for self-similar processes. *Stochastic Proc. and their Applic.* 7, 55 {64.
- [58] Tsakalides, P. (1995, December). *Array Signal Processing with Alpha-Stable Distributions*. Ph. D. thesis, University Southern California.
- [59] Tsakalides, P. and C. Nikias (1995). Maximum likelihood localization of sources in noise modeled as a stable processes. *IEEE Trans. on Signal Proc.* 43, 2700 {2713.
- [60] Tsakalides, P. and C. L. Nikias (1999). Robust space-time adaptive processing (stap) in non-gaussian clutter environments. *IEE Proceedings: Radar, Sonar and Navigation* 146 (2), 84 {93.
- [61] Tsakalides, P., R. Raspanti, and C. L. Nikias (1999). Angle/doppler estimation in heavy-tailed clutter backgrounds. *IEEE Transactions on Aerospace and Electronic Systems* 35 (2), 419436.
- [62] Voit Johannes (2003). *The Statistical Mechanics of Financial Markets (Texts and Monographs in Physics)*. Springer-Verlag. ISBN 3-540-00978-
- [63] Zolotarev, V. M. (1986a). Asymptotic behavior of the Gaussian measure in \mathbb{R}^2 . *J. Soviet Math.* 24, 2330 {2334.
- [64] Zolotarev, V. M. (1986b). *One-dimensional Stable Distributions*, Volume 65 of Translations of mathematical monographs. American Mathematical Society. Translation from the original 1983 Russian edition.
- [65] Zolotarev, V. M. (1995). On representation of densities of stable laws by special functions. *Theory Probab. Appl.* 39, 354 {362.

Mr. Mohammad Ali Baradaran Ghahfarokhi .
Statistician and Data Analyst.
Ministry of labor and social affairs. Iran
E- MAIL : MALI.BARADARAN@GMAIL.COM