Reliability-Based Topology Optimization Based on Evolutionary Structural Optimization

Sang-Rak Kim, Jea-Yong Park, Won-Goo Lee, Jin-Shik Yu, and Seog-Young Han

Abstract—This paper presents a Reliability-Based Topology Optimization (RBTO) based on Evolutionary Structural Optimization (ESO). An actual design involves uncertain conditions such as material property, operational load and dimensional variation. Deterministic Topology Optimization (DTO) is obtained without considering of the uncertainties related to the uncertainty parameters. However, RBTO involves evaluation of probabilistic constraints, which can be done in two different ways, the reliability index approach (RIA) and the performance measure approach (PMA). Limit state function is approximated using Monte Carlo Simulation and Central Composite Design for reliability analysis. ESO, one of the topology optimization techniques, is adopted for topology optimization. Numerical examples are presented to compare the DTO with RBTO.

Keywords—Evolutionary Structural Optimization, Performance Measure Approach, Reliability-Based Topology Optimization, Reliability Index Approach.

I. INTRODUCTION

DUE to the inherent uncertainties such as external loading, material properties, and manufacturing quality, the prototypes or manufactured products may not satisfy the necessary performance requirements. To alleviate the possible degradation of performance in the production process, these uncertainties must be considered during the process of topology design optimization.

In probabilistic optimization, cost minimization and bringing probabilistic constraints on target should be done simultaneously. The main difference between the deterministic optimization (DO) and reliability-based design optimization (RBDO) are their constraints, as RBDO has the same objective as DO. However, in RBDO, probabilistic constraints are formulated so as to construct approximated linear function to closely represent the nonlinear limit state function for the

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reliability index calculation and optimization by using the appropriate transformations [1]-[8].

In order to determine whether or not the probabilistic constraint is satisfied, simulation technique and various approximation methods are developed. Simulation technique such as Monte Carlo simulation (MCS)[9] directly estimate the failure probability, and what is known as the moment methods such as reliability index approach (RIA) and performance measure approach (PMA) calculate the reliability index as a measure of the probabilistic structural safety.

However, the simulation techniques sometimes require a prohibitively large amount of structural analyses in spite of their robustness. In addition, they do not produce any information regarding the sensitivity for more efficient search of the optimum structural design.

For this reason, moment methods, RIA and PMA, are frequently used to estimate the probabilistic constraints with acceptable computations. Probabilistic constraints of RIA are formulated in terms of the reliability index. Reliability analysis in PMA can be formulated as the inverse of reliability analysis in RIA [4], [5].

In this study, the evolutionary structural optimization (ESO) method is considered as a topology optimization method. ESO was introduced by Y. M. Xie and G. P. Steven. ESO is based on the simple idea that the optimal structure (maximum stiffness, minimum weight) can be produced by gradually removing the ineffectively used material from the design domain. The design domain is constructed by the FE method, and furthermore, external loads and support conditions are applied to the element model. Considering the engineering aspects, ESO seems to have some attractive features: the ESO method is very simple to program via the FEA packages and requires a relatively small amount of FEA time. Additionally, the ESO topologies have been compared with analytical ones, e.g. Michell trusses, and so far the results are quite promising. The wide range of ESO applications have become proof of its versatility and its potential as a design tool [10], [11].

The two methods, RIA and PMA, are used in order to compute the probabilistic constraints, as they are both representative analysis methods of probabilistic constraints in reliability-based topology optimization (RBTO). The limit state function is approximate using MCS and central composite design (CCD) at each iteration in order to evaluate the probabilistic constraints. For the implementation, MATLAB is used as an optimizer and reliability analysis. For the finite element analysis, ANSYS is used. Numerical examples are

presented to compare the deterministic topology optimization (DTO) with RBTO.

II. EVOLUTIONARY STRUCTURAL OPTIMIZATION

Despite the significant effort directed towards structural optimization over the past four decades, most techniques developed so far are restricted to sizing optimization or shape optimization with fixed topology. The search for a general method capable of performing simultaneous shape and topology optimization has been a great challenge. An important recent development in this area was made by Bndsøe and Kikuchi who proposed the homogenization method, where the structure was represented by a model with microvoids and the objective was to seek the optimal porosity of the porous medium using an optimality criterion.

Recently, a simple method for shape and layout optimization, called Evolutionary Structural Optimization (ESO), has been proposed by Xie and Steven, which is based on the concept of gradually removing redundant elements to achieve an optimal design.

ESO can be easily implemented into any general purpose finite element analysis program. In contrast to most other methods, the ESO involves no mathematical programming techniques in the optimization process.

Sensitivity number, which indicates the change in the overall stiffness or a specified displacement due to removal of an element, is formulated using results form a finite element analysis. Then a number of elements with the lowest sensitivity numbers will be eliminated from the structure. The optimal design of the structure will be obtained by repeating the cycle of finite element analysis, calculation of sensitivity numbers and element elimination until the overall stiffness or specified displacements reach their given limits.

In the finite element method, the static behavior of a structure is represented by

$$[K]\{u\} = \{P\} \tag{1}$$

where $[K]$ is the global stiffness matrix, $\{u\}$ is the global nodal displacement vector and $\{P\}$ is the nodal load vector.

The strain energy of the structure, which is defined as:

$$C = \frac{1}{2}\{P\}^T\{u\} \tag{2}$$

is commonly used as the inverse measure of the overall stiffness of the structure. It is obvious that maximizing the overall stiffness us equivalent to minimizing the strain energy.

Consider the removal of the i th element from a structure comprising n finite elements. The stiffness matrix will change by $[\Delta K] = [K^*] - [K] = -[k^i]$, where $[K^*]$ is the stiffness matrix of the resulting structure after removal of the i th element and $[k^i]$ is the stiffness matrix of the i th element. It is assumed that the removal of the element has no effect on the load vector $\{P\}$. By ignoring a higher order term, we can find the change of the displacement vector from Eq. (1) as

$$\{\Delta u\} = -[K]^{-1}[\Delta K]\{u\} \tag{3}$$

From Eqs. (2) and (3) we get

$$\Delta C = \frac{1}{2}\{P\}\{\Delta u\} = -\frac{1}{2}\{P\}^T[K]^{-1}\{\Delta u\} = \frac{1}{2}\{u^i\}[K^i]\{u^i\} \tag{4}$$

where $\{u^i\}\{\}$ is the displacement vector of the i th element. We thus define

$$\alpha_i = \frac{1}{2}\{u^i\}[K^i]\{u^i\} \tag{5}$$

as the sensitivity number for problems with an overall stiffness constraint, which indicates the change in the strain energy due to the removal of the i th element. It should be noted that α_i is the element strain energy. Both C and α_i are always positive values. In general, when an element is removed, the stiffness of a structure reduces and correspondingly the strain energy increases. To achieve this objective through element removal, it is obviously most effective to remove the element which has the lowest value of α_i so that the increase in C is minimum.

The ESO procedure for optimization with overall stiffness or displacement constraints is as follows:

Step1: Discrete the structure using a fine mesh of finite elements

Step2: Analyze the structure for the given loads

Step3: Calculate the sensitivity number for each element

Step4: Remove elements which have the lowest sensitivity numbers

Step5: Repeat Step 3 to Step 4 until one of the constraints reaches its limit.

In Step 3 only one of the formulae form Eq. (5) is used depending on the type and the number of constraints involved. The number of elements to be removed at each iteration, in Step 4, can be prescribed by its ratio to the total number of elements of the initial or the current FEA model [10], [11]. This ratio is called the removal ratio. For the purposes of this paper a removal ratio of 2% has been adopted.

III. RELIABILITY-BASED DESIGN OPTIMIZATION

In the system parameter design, the RBDO model [12]-[15] can be generally formulated as

$$\begin{aligned}&Min. &&f(\mathbf{d})\\&S.t. &&P[G_i(\mathbf{d},\mathbf{x}) \leq 0] \leq P_i &&(i = 1, 2, ..., p)\end{aligned} \tag{6}$$

where $\mathbf{d}$ is the design variable vector, $\mathbf{x}$ is the random variables vector and the probabilistic constraints are described by the limit state function $G_i(\mathbf{d},\mathbf{x})$ is the i th limit state equation.

The statistical description of the failure of the limit state

function $G_i(\mathbf{d},\mathbf{x})$ is characterized by the cumulative distribution function $F_{G_j}(\bullet)$ as

$$F_{G_j}(0) = P[G_i(\mathbf{d},\mathbf{x}) \le 0] = \int_{G_j \le 0} \cdots \int f_x(\mathbf{d},\mathbf{x}) dx_1 \ldots dx_n \qquad (7)$$

In Eq. (7) $f_x(\mathbf{d},\mathbf{x})$ is the joint probability density function of all random variables. The evaluation of Eq. (7) requires reliability analysis where the multiple integration is involved as shown in Eq. (7). The evaluation of the integral in Eq. (7) is not easy, because it represents a very small quantity and all the necessary information for the joint density function are not available. For these reasons, the simulation technique or approximate reliability analysis methods. Simulation technique, MCS [4],[5], directly performs the reliability analysis. However, it sometimes requires a prohibitively large amount of structural analyses in spite of their robustness. In addition, they do not produce any information regarding the sensitivity for more efficient search of the optimum structural design. Thus approximate probability integration methods, such as the first-order reliability method (FORM), are widely used for RBDO applications.

In FORM, the reliability analysis requires a transformation $\mathbf{T}$ from the original random parameter $\mathbf{X}$ to the independent and standard normal random parameter $\mathbf{U}$. The limit state function $G(\mathbf{d},\mathbf{x})$ in X-space can then be mapped onto $G(\mathbf{d},\mathbf{T}(\mathbf{x})) \equiv G(\mathbf{U})$ in U-space.

As described in Section I, the probabilistic constraint in Eq. (6) can be further expressed in two different ways through inverse transformations [7] as

$$\beta_i = -\Phi^{-1}(F_{G_j}(0)) \ge \beta_{i,\text{target}} \qquad (8)$$

$$G_i = F_{G_j}^{-1}(\Phi(\beta_{i,\text{target}})) \qquad (9)$$

where β_i and $\beta_{i,\text{target}}$ are the reliability index at current design and the target reliability index of the i th probabilistic constraint, respectively. G_i is the probabilistic performance measure for i th the probabilistic constraint. Φ is the cumulative distribution function in standard normal distribute space. Eq. (8) can be used to describe the probabilistic constraint in Eq. (6) using the reliability index, i.e., the RIA. Similarly, Eq. (9) can be used in Eq. (6) as the probabilistic constraint.

The general form of RBTO for static problems is described [1] as follows

$$
\begin{aligned}
&Min. &&Volume\\
&S.t. &&P[G \le 0] \le P_t
\end{aligned}
\qquad (10)
$$

where $G = \delta_{\text{target}} - \delta_{\text{max}} \ge 0$ is the limit state function, $P[G \le 0]$ means the probability of failure at current design, P_t is the target probability of failure.

A. Reliability Index Approach (RIA)

When probabilistic constraints are estimated in terms of the reliability index, the probability structural design optimization of Eq. (10) may be expressed as

$$
\begin{aligned}
&Min. &&Volume\\
&S.t. &&\beta \le \beta_{\text{target}}
\end{aligned}
\qquad (11)
$$

In order to evaluate the probabilistic constraint for RIA, nested optimization loop is necessary. The definition of reliability index is the minimum distance form origin to approximate limit state function. Therefore the first order reliability index $\beta_{s,FORM}$ is formulated as an optimization problem with a equality constraint in U-space as follows

$$
\begin{aligned}
&Min. &&\beta = \|\mathbf{u}\| = \sqrt{\mathbf{u}^T \mathbf{u}}\\
&S.t. &&G(\mathbf{u}) = 0
\end{aligned}
\qquad (12)
$$

where the optimum point on the failure surface is called the most probable point (MPP) $u^*_{G(U)=0}$. Either an MPP search algorithm that is specifically developed for first-order reliability analysis or general optimization algorithms, SLP or SQP etc., can be used to solve this equation. In this paper, the advanced first order reliability method [16], [17] is employed to perform reliability analyses in RIA.

B. Performance Measure Approach (PMA)

Reliability analysis in PMA is formulated as the inverse of reliability analysis in RIA. The first-order probabilistic performance measure $G_{p,\text{target}}$ is obtained from a nonlinear optimization problem in U-space as

$$
\begin{aligned}
&Min. &&G(\mathbf{u})\\
&S.t. &&\|\mathbf{u}\| = \sqrt{\mathbf{u}^T \mathbf{u}} = \beta_{\text{target}}
\end{aligned}
\qquad (13)
$$

where the optimum point on the target reliability surface is identified as the MPP $u^*_{\beta=\beta_{\text{target}}}$ with a prescribed reliability target $\beta_t = \left\| u^*_{\beta=\beta_{\text{target}}} \right\|$. In iterative optimization process, unlike RIA, only the direction vector $u^*_{\beta=\beta_{\text{target}}} / \left\| u^*_{\beta=\beta_{\text{target}}} \right\|$ needs to be determined by exploring the spherical equality constraint $\|\mathbf{u}\| = \beta_{\text{target}}$ in Eq. (13). Rather than a general optimization algorithm, the advanced mean value (AMV), conjugate mean value (CMV), and hybrid mean value (HMV) methods are commonly used to solve the problem in Eq. (13), since they do not require a line search [5], [8]. In this paper, the HMV method, which adaptively employs the AMV and CMV methods, is used to solve the inverse problem in PMA.

IV. NUMERICAL EXAMPLES

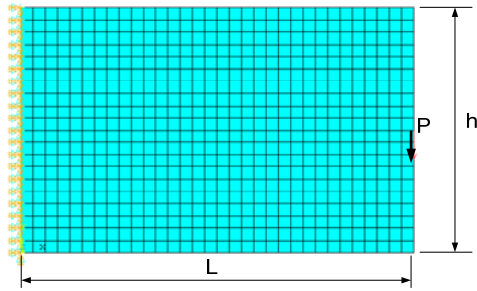The cantilever beam shown in Fig. 1 is under plane stress conditions. The left-hand side of the beam is fixed and a
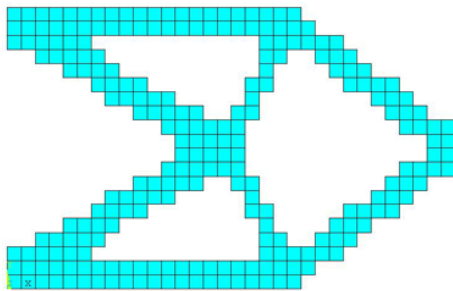
Fig. 1 2-D design domain

Fig. 3 RBTO with 1 random variable (RIA)

Fig. 2 optimal topology of deterministic topology optimization

Fig. 4 RBTO with 1 random variable (PMA)

vertical load of $P = 3kN$ is applied at the middle of the free end. The dimensions of the beam are $L = 0.16m$, $h = 0.10m$ and the thickness $t = 0.001m$. The Young's modulus $E = 207GPa$ and Poisson's ration $v = 0.3$ are assumed. The design domain is divided into 32×20 quadrilateral elements. ESO is used as topology optimization technique and the removal ratio is 2% of the initial elements [10]. RIA and PMA are used in order to compute the probability constraints, as they are both representative analysis methods of probabilistic constraints in RBTO

The uncertain variables have 10% standard deviation of the mean value and are assumed to be normally distributed. The target reliability index, β_{target}, is 3.

From the definition of given problem the RBTO problem formulated as follow

$$
\begin{aligned}
&Min. \quad Volume \\
&S.t. \quad P[G \le 0] \le P_{\text{target}} = 0.135\%
\end{aligned} \qquad (14)
$$

where limit state function is defined as $G = 0.0075 - \delta_{\max}$. In probabilistic constraint, $P_{\text{target}} = 0.135\%$ mean the probability of failure at $\beta_{\text{target}} = 3$, relation between the probability of failure and reliability index is represented as $P_f = \Phi(-\beta)$. On the other word, the probability of safety must be larger than 99.865%.

The limit state function is approximate using Monte Carlo Simulation and Central Composite Design at each iteration in order to evaluate the probabilistic constraints. Fig. 2 shows the optimal topology using the DTO when removal ration is 2%.
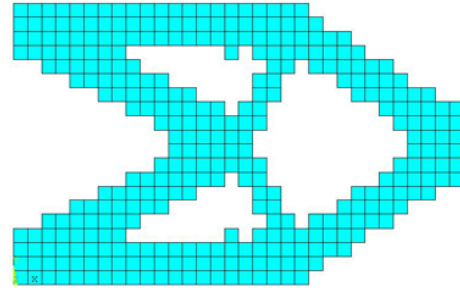
TABLE I
COMPARISON BETWEEN DTO AND RBTO FOR 1 UNCERTAINTY

	Volume (%)	Deflection	Reliability index
DTO	37.18%	0.000752	0.0163
RIA	51.5%	0.00054	2.9397
PMA	51.87%	0.000539	3.0000

A. RBTO with 0ne Uncertainty parameter

The uncertainty variable is Young's modulus. It has 10% standard deviation of the mean value and are assumed to be normally distributed. The optimal results of RIA and PMA are shown in Fig. 3 and Fig. 4 respectively. In Table 1, the optimization results obtained from each approach are summarized. The second column shows the objective function, i.e., total volume(%) and the last column the reliability index. The objective of the DTO is smaller than the results of RBTO. However, DTO has poor reliability index, $\beta = 0.0163$, which means that the optimum of DTO has about a 50% failure probability. When the reliability in considered into this design, the volume used is more than the DTO required to satisfy the probabilistic constraint. This is because the feasible region becomes smaller due to the distribution of the uncertain variable. RBTO results show that the proposed method achieves the target reliability index.

B. RBTO with Three Uncertainty parameters

This example has the same design domain as the first example. However, two more uncertain variables, external load and thickness, are considered as uncertain variables. Again, all
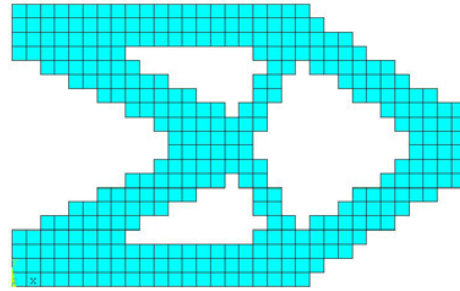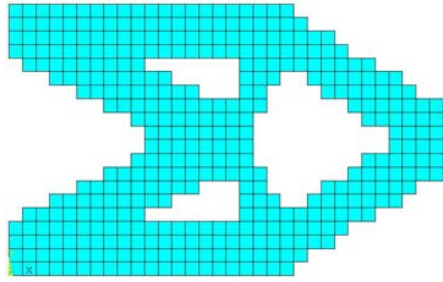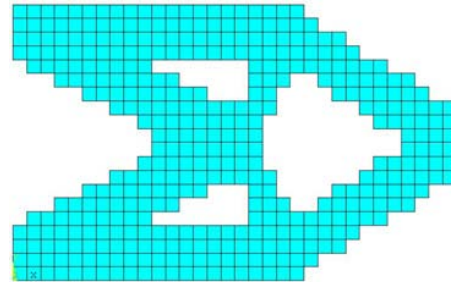
Fig. 5 RBTO with 3 random variables (RIA)

TABLE II
COMPARISON BETWEEN DTO AND RBTO FOR 3 UNCERTAINTIES

	Volume (%)	Deflection	Reliability index
DTO	37.18%	0.000752	0.0156
RIA	60.6%	0.000464	2.9338
PMA	60.6%	0.000464	3.0000

uncertain variables have 10% standard deviation standard deviation of the mean value and are assumed to be normally distributed. The maximum allowable deflection is $0.00075m$ and the target reliability index, β_{target}, 3.0. The formulation of RBTO is the same as in Eq. (14). The optimal results of RIA and PMA are shown in Fig. 5 and Fig. 6 respectively. In Table II, the optimization results obtained from each approach are summarized. The optimum topology shows in Fig. 5 and Fig. 6 are different from previous results. Because of the increased number of uncertain variables, a more robust solution is obtained in order to satisfy the target reliability index. Also, the used volumes are much larger than in the one uncertain variable case. From Table II, we can find that the results of RBTO have a larger volume than DTO. A DTO model using more volume can be more reliable.

V. CONCLUSION

This paper performed the reliability-based topology optimization (RBTO) using evolutionary structural optimization (ESO). RBTO method is implemented by reliability index based approach (RIA) and performance based approach (PMA). To calculate the probability of constraints, the advanced first order reliability method (AFORM) is used. The limit state function is approximated using Monte Carlo simulation and central composite design.

RBTO gave results that are more reliable with respect to uncertainties. In Table I and Table II, RBTO can give the requested solution under the condition of the second uncertainties. Due to the characteristic of ESO, difference of optimal topology between DTO and RBTO is only the volume. Their results of optimal topology are not different.

In order to effectively apply RBTO to ESO, it is found that the evaluation of the sensitivity number in ESO, must be improved by a way to consider the uncertainties of random parameters. At present we are researching about the improvement of calculating the sensitivity number in ESO.

Fig. 6 RBTO with 3 random variables (PMA)

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