International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:5, No:10, 2011

Development Partitioning Intervalwise Block Method for Solving Ordinary Differential Equations

K.H.Khairul Anuar, K.I.Othman, F.Ishak, Z.B.Ibrahim, Z.Majid

Abstract**—**Solving Ordinary Differential Equations (ODEs) by using Partitioning Block Intervalwise (PBI) technique is our aim in this paper. The PBI technique is based on Block Adams Method and Backward Differentiation Formula (BDF). Block Adams Method only use the simple iteration for solving while BDF requires Newton-like iteration involving Jacobian matrix of ODEs which consumes a considerable amount of computational effort. Therefore, PBI is developed in order to reduce the cost of iteration within acceptable maximum error

Keywords**—**Adam Block Method, BDF, Ordinary Differential Equations, Partitioning Block Intervalwise

I. INTRODUCTION

LET's consider the linear system of first order ODEs as follows:

$$y' = Ay + \phi(x), \quad y(a) = \eta, \quad a \le x \le b \tag{1}$$

where

$$y^T = (y_1, y_2, ..., y_s), \quad \eta^T = (\eta_1, \eta_2, ..., \eta_s)$$

Initially, the partitioning technique was developed by Enright and Kamel [1] and then had been extended by Watkins and Hansonsmith [5] which developed automatically partitioning into stiff systems and non-stiff systems. Other researchers that studied partitioning technique were Hall and Suleiman [2] and Suleiman and Baok [4]. Recently, Khairil et.al [3] has developed intervalwise partitioning technique using 2-point block method formula for solving ODEs. The development of partitioning technique in this paper is based on Khairil et.al [3] paper but we use 3-point block method formula.

II. 3 POINT BLOCK BDF AND 3 POINT ADAMS BLOCK METHOD

In 3-point block method, 3 new values y_{n+1}, y_{n+2} and y_{n+3} are generated simultaneously at each step using previous block. We use point $x_{n-3}, x_{n-2}, x_{n-1}, x_n, x_{n+1}, x_{n+2}$ and x_{n+3} and substitute into Lagrange polynomial and be as follow:

$$\begin{aligned}
P(x) = {} & \frac{(X - X_{n+2}).......(X - X_{n-3})}{(X_{n+3} - X_{n+2}).....(X_{n+3} - X_{n-3})} Y_{n+3} \\
& + \frac{(X - X_{n+3})......(X - X_{n-3})}{(X_{n+2} - X_{n+3}).......(X_{n+2} - X_{n-3})} Y_{n+2} \\
& + \frac{(X - X_{n+3}).......(X - X_{n-3})}{(X_{n+1} - X_{n+3})......(X_{n+1} - X_{n-3})} Y_{n+1} \\
& + \frac{(X - X_{n+3}).......(X - X_{n-3})}{(X_n - X_{n+3}).......(X_n - X_{n-3})} Y_n \\
& + \frac{(X - X_{n+3})......(X - X_{n-3})}{(X_{n-1} - X_{n+3}).......(X_{n-1} - X_{n-3})} Y_{n-1} \\
& + \frac{(X - X_{n+3})......(X - X_{n-3})}{(X_{n-2} - X_{n+3})......(X_{n-2} - X_{n-3})} Y_{n-2} \\
& + \frac{(X - X_{n+3})....(X - X_{n-2})}{(X_{n-3} - X_{n+3}).......(X_{n-3} - X_{n-2})} Y_{n-3}
\end{aligned} \tag{2}$$

Eventually, 3-point block BDF with step size ratio $r = 1. r = \frac{1}{2}$ and $r = 2$ are using for solving stiff system as follow:

$r = 1$

$$\begin{aligned}
Y_{n+1} &= -\frac{1}{35}Y_{n-3} + \frac{8}{35}Y_{n-2} - \frac{6}{7}Y_{n-1} + \frac{16}{7}Y_n \\
&\quad - \frac{24}{35}Y_{n+2} + \frac{2}{35}Y_{n+3} + \frac{12}{7}hf_{n+1} \\
Y_{n+2} &= \frac{2}{77}Y_{n-3} - \frac{15}{77}Y_{n-2} + \frac{50}{77}Y_{n-1} - \frac{100}{77}Y_n \\
&\quad + \frac{150}{77}Y_{n+1} - \frac{10}{77}Y_{n+3} + \frac{60}{77}hf_{n+2} \\
Y_{n+3} &= -\frac{10}{147}Y_{n-3} + \frac{24}{49}Y_{n-2} - \frac{75}{49}Y_{n-1} + \frac{400}{147}Y_n \\
&\quad - \frac{150}{49}Y_{n+1} + \frac{120}{49}Y_{n+2} + \frac{20}{49}hf_{n+3}
\end{aligned} \tag{3}$$

.

K.H.Khairul Anuar is with Universiti Teknologi Mara (UiTM) Malaysia (e-mail:khairulhamidi87@ gmail.com).
K.I. Othman is with Universiti Teknologi Mara (UiTM) Malaysia.(e-mail: khairil17@yahoo.com).
F.Ishak is with Universiti Teknologi Mara (UiTM) Malaysia.
Z.B.Ibrahim is with the Universiti Putra Malaysia (UPM)
Z.Majid is with Universiti Putra Malaysia (UPM).

$r = \frac{1}{2}$

$$Y_{n+1} = -\frac{4}{21}Y_{n-3} + \frac{75}{64}Y_{n-2} - \frac{20}{7}Y_{n-1} + \frac{25}{8}Y_n - \frac{15}{56}Y_{n+2} + \frac{25}{1344}Y_{n+3} + \frac{15}{16}hf_{n+1}$$
$$Y_{n+2} = \frac{320}{957}Y_{n-3} - \frac{1225}{638}Y_{n-2} + \frac{1344}{319}Y_{n-1} - \frac{1225}{319}Y_n + \frac{735}{319}Y_{n+1} - \frac{175}{1914}Y_{n+3} + \frac{210}{319}hf_{n+2}$$
$$Y_{n+3} = -\frac{3584}{3265}Y_{n-3} + \frac{3969}{653}Y_{n-2} - \frac{41472}{3265}Y_{n-1} + \frac{7056}{653}Y_n - \frac{15876}{3265}Y_{n+1} + \frac{9072}{3265}Y_{n+2} + \frac{252}{653}hf_{n+3} \quad (4)$$

$r = 2$

$$Y_{n+1} = -\frac{25}{3552}Y_{n-3} + \frac{21}{296}Y_{n-2} - \frac{245}{592}Y_{n-1} + \frac{1225}{296}Y_n - \frac{3675}{1184}Y_{n+2} + \frac{35}{111}Y_{n+3} + \frac{210}{37}hf_{n+1}$$
$$Y_{n+2} = \frac{1}{525}Y_{n-3} - \frac{16}{875}Y_{n-2} + \frac{12}{125}Y_{n-1} - \frac{16}{25}Y_n + \frac{1536}{875}Y_{n+1} - \frac{512}{2625}Y_{n+3} + \frac{24}{25}hf_{n+2}$$
$$Y_{n+3} = -\frac{175}{46112}Y_{n-3} + \frac{405}{11528}Y_{n-2} - \frac{3969}{23056}Y_{n-1} + \frac{11025}{11528}Y_n - \frac{2835}{1441}Y_{n+1} + \frac{99225}{46112}Y_{n+2} + \frac{630}{1441}hf_{n+3} \quad (5)$$

Meanwhile, for solving non stiff system, Zanariah [6] has derived the 3-point Adams Block Method Formula with step size ratio $r = 1.r = \frac{1}{2}$ and $r = 2$ as follow:

$r = 1$

$$Y_{n+1} = Y_n + \frac{h}{60480}(271 f_{n+3} - 2760 f_{n+2} + 30819 f_{n+1} + 37504 f_n - 6771 f_{n-1} + 1608 f_{n-2} - 191 f_{n-3})$$
$$Y_{n+2} = Y_n + \frac{h}{3780}(-37 f_{n+3} + 1398 f_{n+2} + 4863 f_{n+1} + 1328 f_n + 33 f_{n-1} - 30 f_{n-2} + 5 f_{n-3})$$
$$Y_{n+3} = Y_n + \frac{h}{2240}(685 f_{n+3} + 3240 f_{n+2} + 1161 f_{n+1} + 2176 f_n - 729 f_{n-1} + 216 f_{n-2} - 29 f_{n-3}) \quad (6)$$

$r = \frac{1}{2}$

$$Y_{n+1} = Y_n + \frac{h}{317520}(631 f_{n+3} - 7794 f_{n+2} + 133560 f_{n+1} + 313026 f_n - 175680 f_{n-1} + 63441 f_{n-2} - 9664 f_{n-3})$$
$$Y_{n+2} = Y_n + \frac{h}{39690}(-341 f_{n+3} + 14274 f_{n+2} + 52794 f_{n+1} + 6594 f_n + 11520 f_{n-1} - 6741 f_{n-2} + 1280 f_{n-3})$$
$$Y_{n+3} = Y_n + \frac{h}{11760}(3469 f_{n+3} + 18090 f_{n+2} + 1512 f_{n+1} + 30534 f_n - 29376 f_{n-1} + 13419 f_{n-2} - 2368 f_{n-3}) \quad (7)$$

$r = 2$

$$Y_{n+1} = Y_n + \frac{h}{2540160}(24160 f_{n+3} - 200277 f_{n+2} + 1527840 f_{n+1} + 1229718 f_n - 48132 f_{n-1} + 7578 f_{n-2} - 727 f_{n-3})$$
$$Y_{n+2} = Y_n + \frac{h}{158760}(-1600 f_{n+3} + 59031 f_{n+2} + 203328 f_{n+1} + 57246 f_n - 504 f_{n-1} + 18 f_{n-2} + f_{n-3})$$
$$Y_{n+3} = Y_n + \frac{h}{94080}(30112 f_{n+3} + 127197 f_{n+2} + 73440 f_{n+1} + 54642 f_n - 3780 f_{n-1} + 702 f_{n-2} - 73 f_{n-3}) \quad (8)$$

III. Implementation Of PBI

In Partitioning Intervalwise Block technique, primarily the whole system of ODEs (1) was treated as non stiff system of ODEs and solving ODEs by using Adams Block Method. Once it indicates instability due to step failure, there may be presence of stiffness. Therefore, the system will be tested for stiffness by calculating the trace of Jacobian, $\left(\frac{\partial f}{\partial y}\right)$. If the trace is negative, the whole system is switched to stiff system and solve by using BDF method. If the trace is positive, the whole system will be solved by using Adams Block Method with the half step size. Khairil et.al [3] has described that the error algorithm is control as follow:-

i. If the error control is less than tolerance limit, the step size *h* is doubled to gain computation speed.
ii. In case of step failure, the step size *h* is halved and the step is repeated.

International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:5, No:10, 2011

IV. NUMERICAL RESULTS

The PBI technique will be tested with some test problems in order to validate the efficiency.

Problem 1:

$$y_1' = -2y_1 + y_2 + 2\sin x \qquad y_1(0) = 2 \qquad 0 \le x \le 5$$
$$y_2' = 998y_1 - 999y_2 + 999(\cos - \sin x) \qquad y_2(0) = 3$$
$$\textit{Solution}: \quad y_1(x) = 2e^{-x} + \sin x$$
$$y_2(x) = 2e^{-x} + \cos x$$

Problem 2:

$$y_1' = -20y_1 - 0.25y_2 - 19.75y_3 \qquad y_1(0) = 1 \qquad 0 \le x \le 5$$
$$y_2' = 20y_1 - 20.25y_2 + 0.25y_3 \qquad y_2(0) = 0$$
$$y_3' = 20y_1 - 19.75y_2 - 0.25y_3 \qquad y_3(0) = -1$$
$$\textit{Solution}: \quad y_1(x) = 0.5(e^{-0.5} + e^{-20x}(\cos 20x + \sin 20x))$$
$$y_2(x) = 0.5(e^{-0.5x} - e^{-20x}(\cos 20x - \sin 20x))$$
$$y_3(x) = -0.5(e^{-0.5x} + e^{-20x}(\cos 20x - \sin 20x))$$

Table I and Table II show the numerical results for tested problems.

The notation in the table is defined as follow:

TOL	Tolerance
MAXE	Maximum Error
TIME	Time taken to execute the problem in seconds
IFST	Total number of failure step
IST	Total number of step accepted
TS	Total number of steps taken
Ode15s	BDFs in Matlab
PBI	Partitioning Block Intervalwise

TABLE I
NUMERICAL RESULT FOR PROBLEM 1

TOL	METHOD	IFST	IST	TS	MAXE	TIME
10^{-3}	ODE15s	1	28	29	$2.4e-03$	0.7938
	PBI	1	21	22	$1.25e-04$	0.0035
10^{-4}	ODE15s	0	38	38	$2.41e-04$	2.5669
	PBI	2	28	30	$3.16e-05$	0.0038
10^{-5}	ODE15s	1	54	55	$5.30e-05$	52.4463
	PBI	3	25	28	$3.78e-07$	0.0036

TABLE II
NUMERICAL RESULT FOR PROBLEM 2

TOL	METHOD	IFST	IST	TS	MAXE	TIME
10^{-3}	ODE15s	0	64	64	0.9011	1.0482
	PBI	1	37	38	$8.49e-03$	0.0040
10^{-4}	ODE15s	0	89	89	0.9004	2.5669
	PBI	1	41	42	$3.88e-03$	0.0040
10^{-5}	ODE15s	0	122	122	0.9004	444.5928
	PBI	3	25	28	$1.71e-06$	0.0044

V. CONCLUSION

The numerical results from Table I and Table II show that partitioning technique gives better accuracy and execution time compare Ode15s. As conclusion, the PBI is more efficient and applicable to solve ODEs.

REFERENCES

[1] D.S. Watkins, R.W. Hansonsmith (1983). The Numerical Solution of Separably Stiff System by Precise Partitioning, ACM Trans Math Software 9. 293-301.
[2] G. Hall, M.B. Suleiman (1985). A single Code for the Solution of Stiff and Nonstiff ODEs, SIAM J. Stat. Comput. 6(3) 684-697.
[3] K.I. Othman, Z.B. Ibrahim, M. Suleiman, Z. Majid(2007). Automatic Intervalwise Block Partitioning using Adams type Method and Backward Differentiation Formula for Solving ODEs, Applied Maths and Comput 188. 1642-1646.
[4] M.B. Suleiman, S. Baok (1992). Using Nonconvergence of Iteration to Partitions ODEs, Appl. Math. Comput 49. 111-139.
[5] W.H. Enright, M.S. Kamel (1979), Automatic Partitioning of Stiff System and Exploiting the Resulting Structure, ACM Trans. Math. Sofware 5 (4) 374-385.
[6] Z. Majid (2003). Parallel Block Methods For Solving Ordinary Differential Equations, PhD thesis, University Putra Malaysia.