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Another Approach of Similarity Solution in Reversed Stagnation-point Flow

Vai Kuong Sin, *Member, ASME; Fellow, MIEME,* and Chon Kit Chio

Abstract—In this paper, the two-dimensional reversed stagnation-point flow is solved by means of an anlytic approach. There are similarity solutions in case the similarity equation and the boundary condition are modified. Finite analytic method are applied to obtain the similarity velocity function.

Keywords—reversed stagnation-point flow, similarity solutions, asymptotic solution

I. Introduction

The full Navier-Stokes equations are difficult or impossible to obtain an exact solution in almost every real situation because of the analytic difficulties associated with the nonlinearity due to convective acceleration. The existence of exact solutions are fundamental not only in their own right as solutions of particular flows, but also are agreeable in accuracy checks for numerical solutions.

In some simplified cases, such as two-dimensional stagnation point flows, by introducing coordinate variable transformation, the number of independent variables is reduced by one or more. The governing equations can be simplified to the non-linear ordinary differential equations and are analytic solvable. The classic problems of two-dimensional stagnation-point flows can be analyzed exactly by Hiemenz [1], one of Prandtl's first students. These are exact solutions for flow directed perpendicular to an infinite flat plate. Howarth [2] extended the two-dimensional and axisymmetric flows to three dimensions, which is based on boundary layer approximation in the direction normal to the plane. Reversed stagnation-point flow against an impermeable flat wall does not exist in two dimensions, but Davey [3] showed that certain reverse flows in three in three dimension are possible.

The aim of this study is to investigate the steady viscous reversed stagnation-point flow, which is started impulsively in motion with a constant velocity away form near the stagnation point. A similarity solution of full Navier-Stokes equations is solved by applying numerical method. Studies of the reversed stagnation-point flow have been considered during the last few years, as this flow can be applied in different important applications that occur in oil recovery industry.

II. Analytical Analysis

A. Governing equation

The viscous fluid flows in a rectangular Catesian coordinates (x, y, z), Fig. 2, which illustrates the motion of external flow

V. K. Sin is with the Department of Electromechanical Engineering, University of Macau, Macao SAR, China, E-mail: (vksin@umac.mo).
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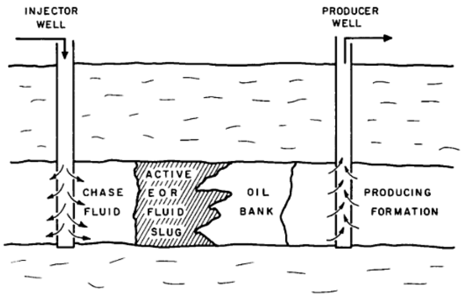

Fig. 1. Oil recovery industry

directly moves perpendicular out of an infinite flat plane wall. The origin is the so-called stagnation point and z is the normal to the plane.

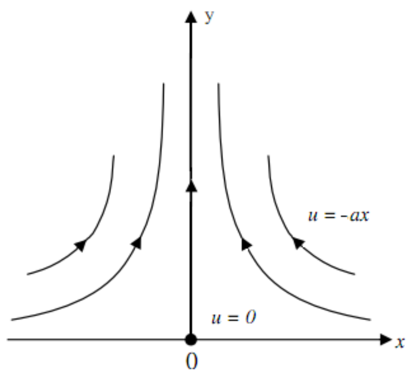

Fig. 2. Coordinate system of reversed stagnation-point flow

By conservation of mass principle with constant physical properties , the equation of continuity is:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

We consider the two-dimensional reversed stagnation-point flow in unsteady state and the flow is bounded by an infinite plane $y = 0$, the fluid remains at rest when time $t < 0$. At $t = 0$, it starts impulsively in motion which is determined by the stream function

$$\psi = -\alpha xy \tag{2}$$

At large distances far above the planar boundary, the existence of the potential flow implies an inviscid boundary condition. It is given by

$$u = -\alpha x \quad (3a)$$

$$v = V_0 \quad (3b)$$

where u and v are the components of flow velocity, A is a constant proportional to V_0/L, V_0 is the external flow velocity removing from the plane and L is the characteristic length. We have $u = 0$ at $x = 0$ and $v = 0$ at $y = 0$, but the no-slip boundary at wall $(y = 0)$ cannot be satisfied.

For a viscid fluid the stream function, since the flow motion is determined by only two factors, the kinematic viscosity ν and α, we consider the following modified stream function

$$\psi = -\sqrt{A\nu}xf(\eta,\tau) \quad (4a)$$

$$\eta = \sqrt{\frac{A}{\nu}}y \quad (4b)$$

$$\tau = At \quad (4c)$$

where η is the non-dimensional distance from wall and τ is the non-dimensional time. Noting that the stream function automatically satisfies equation of continuity (1) . The Navier-Stokes equations [4] governing the unsteady flow with constant physical properties are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) \quad (5a)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}) \quad (5b)$$

where u and v are the velocity components along x and y axes, and ρ is the density.

Substituting u and v into the governing equations results a simplified partial differential equation. From the definition of stream function, we have

$$u = \frac{\partial \psi}{\partial y} = -Axf_\eta \quad (6a)$$

$$v = -\frac{\partial \psi}{\partial x} = \sqrt{A\nu}f \quad (6b)$$

The governing equations can be simplified by a similarity transformation when several independent variables appear in specific combinations, in flow geometries involving infinite or semi-infinite surfaces. This leads to rescaling, or the introduction of dimensionless variables, converting the original partial differential equations into a partial differential equation.

$$-A^2xf_{\eta\tau} + A^2x(f_\eta)^2 - A^2xff_{\eta\eta} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - A^2xf_{\eta\eta\eta} \quad (7a)$$

$$A\sqrt{A\nu}f_\tau + A\sqrt{A\nu}ff_\eta = -\frac{1}{\rho}\frac{\partial p}{\partial y} + A\sqrt{A\nu}f_{\eta\eta} \quad (7b)$$

The pressure gradient in Eq. (7a) can be again reduced by a further differentiation Eq. (7b) with respect to x. That is

$$\frac{\partial^2 p}{\partial x \partial y} = 0 \quad (8)$$

and Eq. (7a) reduces to

$$[f_{\eta\tau} - (f_\eta)^2 + ff_{\eta\eta} - f_{\eta\eta\eta}]_\eta = 0. \quad (9)$$

The initial and boundary conditions are

$$f(\eta,0) \equiv \eta \qquad (\eta \neq 0) \quad (10a)$$

$$f(0,\tau) = f_\eta(0,\tau) = 0 \quad (t \neq 0) \quad (10b)$$

$$f(\infty,\tau) \sim \eta \quad (10c)$$

The last condition reduces the above differential equation (9) to the form

$$f_{\eta\tau} - (f_\eta)^2 + ff_{\eta\eta} - f_{\eta\eta\eta} + 1 = 0, \quad (11)$$

with the boundary conditions

$$f(0,\tau) = f_\eta(0,\tau) = 0 \quad (12a)$$

$$f_\eta(\infty,\tau) = 1. \quad (12b)$$

Equation (11) is the similarity equation of the full Navier-Stokes equations at two-dimension reversed stagnation point. The coordinates x and y are vanished, leaving only a dimensionless variable η. Under the boundary conditions $f_\eta(\infty,\tau) = 1$, when the flow is in steady state such that $f_{\eta\tau} \equiv 0$, the differential equation has no solution.

B. Insolubility

When $f_{\eta\tau} \equiv 0$, the ODE reduces to

$$f''' - ff'' + (f')^2 - 1 = 0 \quad (13a)$$

$$f(0) = f'(0) = 0 \quad (13b)$$

$$f'(\infty) = 1 \quad (13c)$$

It is proven that all of the solutions, however, do not satisfy the boundary conditions.

Lemma 1: No solution $f'(\eta)$ exists which has stationary value of 1 for finite η.
Proof. Rearrange Eq. (13) yields

$$f''' = 1 - (f')^2 + ff'' \quad (14)$$

Suppose for $\eta = \eta_0$, we have $f'(\eta_0) = 1$ and $f''(\eta_0) = 0$. Afterwards, it follows from the derivatives of Eq. (14) that f''' and all higher derivatives are zero when $\eta = \eta_0$. Considering a variable transformation

$$\lambda(\eta) = f'(\eta)$$

$$\lambda(\eta_0) = 1 \quad (15)$$

Expand the function into Taylor's series near η_0, we have

$$\begin{aligned} f'(\eta) = \lambda(\eta) &= \sum_{n=0}^{\infty} \frac{\lambda^{(n)}(\eta_0)}{n!}(\eta - \eta_0)^n \\ &= \lambda(\eta_0) + \sum_{n=1}^{\infty} \frac{\lambda^{(n)}(\eta_0)}{n!}(\eta - \eta_0)^n \\ &= 1 + \sum_{n=1}^{\infty} \frac{\lambda^{(n)}(\eta_0)}{n!}(\eta - \eta_0)^n \\ &\equiv 1 \end{aligned}$$

Hence, the boundary condition $f'(0) = 0$ is thus not satisfied and the Lemma is proved.

Lemma 2: When f' has a stationary value, if $|f'| < 1$ it is a minimum and if $|f'| > 1$ it is a maximum.

Proof: From Eq. (14), when f' has a stationary value, it means $f''=0$ and Eq. (14) becomes

$$f''' = 1-(f')^2 \tag{16}$$

If $|f'|<1$, $f'''>0$ and it is minima. Else if $|f'|>1$, $f'''<0$ and it is maxima. Eventually, the lemma is proved.

Lemma 3: If $f''(\eta)$ vanishes for $\eta=\eta_1,\eta_2$ ... with $\eta_1<\eta_2<$ ..., then the sequence $f'(\eta_i)$ does not tend to 1 as $\eta_i\to\infty$.

Proof: Consider a region where (η_1,η_2) is far away from the origin. Multiply f'' to Eq. (14) and integrate it between η_1 and η_2 with respect to η.

$$f''f''' = f''-(f')^2F''+f(f'')^2$$

$$\int_{\eta_1}^{\eta_2} f''f'''d\eta = \int_{\eta_1}^{\eta_2}[f''-(f')^2f''+f(f'')^2]d\eta$$

$$\frac{1}{2}[(f'')^2]_{\eta_1}^{\eta_2} = [f'-\frac{1}{3}(f')^3]_{\eta_1}^{\eta_2}+\int_{\eta_1}^{\eta_2} f(f'')^2d\eta$$

When $f'(\infty)\to 1$, it is required that $f''(\eta_1)=f''(\eta_2)=0$ and thus

$$[f'-\frac{1}{3}(f')^3]_{\eta_1}^{\eta_2} = -L$$

wheras $L=\int_{\eta_1}^{\eta_2} f(f'')^2d\eta$ is always positive and we can obtain

$$[f'-\frac{1}{3}(f')^3]_{\eta_1}^{\eta_2} < 0$$

$$f'(\eta_2)-\frac{1}{3}[f'(\eta_2)]^3 < f'(\eta_1)-\frac{1}{3}[f'(\eta_1)]^3$$

$$[f'(\eta_2)]^3-3f'(\eta_2) > [f'(\eta_1)]^3-3f'(\eta_1)$$

Consider $G=f'^3-3F'$ as a function of f', then

$$G' = 3f'^2-3$$

As $f'=1$, then $G'(1)=0$, which makes G a minimum. We do not have $f'(\eta_i)=1$ as $\eta_i\to\infty$

Theorem 1: Given any $f'(\eta)\to 1$ as $\eta\to\infty$, no solution of Eq. (13) exists.

Proof : When $|f'|<1$, since $f'\to 1$ as $\eta\to\infty$, then f'' must be greater than zero. Hence, recall from Eq. (14),

$$f''' = 1-(f')^2+ff''>0.$$

for all $\eta>\eta_0$. After integrating $f'''(\eta)>0$ from η_0 to $\eta>\eta_0$, we have

$$f''(\eta) > f''(\eta_0) = K > 0.$$

Another integration from η_0 to $\eta>\eta_0$ yields

$$f'(\eta) > f'(\eta_0)+K(\eta-\eta_0).$$

By Lemma 2, $f'(\eta)$ has at most one stationary value because one cannot have two consecutive stationary values which are both minima. Since $f''(\eta)>0$, when $\eta\to\infty$, $f'(\eta)\to\infty$. It violates that $f'(\eta)\to 1$. A similar argument shows that a solution cannot approach to 1 when $|f'|>1$.

III. Asymptotic solution

When τ is relatively small, Proudman and Johnson [5] first considered the early stages of the diffusion of the initial vortex sheet at $y=0$. They suggested that, when the flow is near the plane region, the viscous forces are dominant, and the viscous term in the governing Navier-Stokes equations is important only near the boundary. On the contrary, the viscous forces were neglected away from the wall. The convection terms dominate the motion of external flow in considering the inviscid equation in the fluid. According to their solution, the general features of the predicted streamline are sketched in Fig. (3).

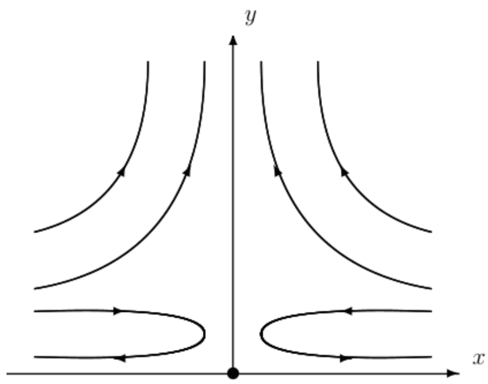

Fig. 3. Streamlines of reversed stagnation-point flow

We therefore consider the similarity of the inviscid equation

$$f_{\eta\tau}-(f_\eta)^2+ff_{\eta\eta}+1=0. \tag{17}$$

Proudman and Johnson obtained a similarity solution of (17) is in the form

$$f(\eta,\tau) = e^{\tau}F(\gamma) \tag{18}$$

and the further integration provides an exact solution

$$F(\gamma) = \gamma-\frac{2}{c}(1-e^{-c\gamma}) \tag{19}$$

where c is a constant of integration; the improved numerical evaluations of Robins and Howarth [6] estimate the value of c to be approximately 3.51. This solution describes the flow in the outer region.

IV. Solving ODEs with MATLAB

In the previous section, it was proven that Eq. (13) does not satisfy the boundary condition $f(\infty)=1$. Consider another situation that we differentiate the similarity equation (13) respect to η twice, the similarity equation becomes

$$f^v+f'f''+(f'')^2-ff^{iv}-f''f'''=0 \tag{20}$$

where it is assumed that

$$f''(\infty)=f'''(\infty)=0, \tag{21}$$

to ensure that the fluid flows smoothly far away from the wall. Also, in order to satisfy the no-slip condition at $y=0$, a boundary condition is required that

$$f(0)=f'(0)=0 \tag{22}$$

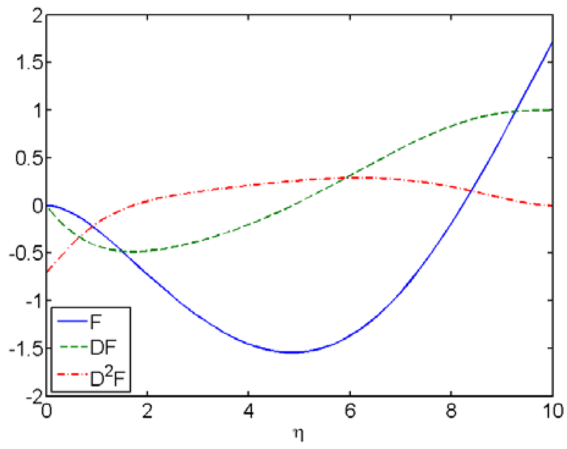

Fig. 4. Similarity solutions of reversed stagnation-point flow

Eq.(20) is a fourth-order nonlinear ordinary differential equation. It does not have exact analytic solution, and thus it is necessary to apply approximation and numerical techniques to solve it.

It is convenient when solving an ODE system numerically to describe the problem in terms of a system of first-order equations in MATLAB[7]. For example when solving an n^{th}-order problem numerically is common practice to reduce the equation to a system of n first-order equations. Then, by defining $y_1 = f$, $y_2 = f'$, $y_3 = f''$, $y_4 = f'''$, $y_5 = f^{iv}$, the ODE reduces to the form

$$\frac{d\mathbf{y}}{dx} = \begin{bmatrix} y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_1y_5 + y_3y_4 - y_2y_4 + y_3^2 \end{bmatrix} \qquad (23)$$

The first task is to reduce the equation above to a system of first order equations and define in MATLAB a function to return these. The relevant MATLAB expression for Eq. (23) would be:

Listing 1. MATLAB expression
```
function dydx = ode(~,y)
dydx=[y(2);
      y(3);
      y(4);
      y(5);
      y(1)*y(5)+y(3)*y(4)
      -y(2)*y(4)-y(3)*y(3)];
end
```

Later, we need to rewrite the boundary conditions to correspond to this form of the problem. For two-point boundary value conditions of the form $bc(y(a), y(b))$,

Listing 2. MATLAB expression
```
function res = bc(ya,yb)
res=[ya(1);
     ya(2);
     yb(2)-1;
     yb(3)
     yb(4)];
end
```

The next step is to create an initial guess for the form of the solution. Here we select the asymptotic solution at $\tau = 3$ as our initial guess.

Listing 3. MATLAB expression
```
function y=guess(x)
c=3.51;
t=3;
y(1)=exp(t)*(x*exp(-t)-2/c*
        (1-exp(-c*x*exp(-t))));
if x==0
    y(2)=0;
else
    y(2)=1-2*exp(-c*x*exp(-t));
end
y(3)=2*c*exp(-c*x*exp(-t)-t);
y(4)=-2*c^2*exp(-c*x*exp(-t)-2*t);
y(5)=2*c^3*exp(-c*x*exp(-t)-3*t);
end
```

We can solve boundary value problems for ordinary differential equations by the ode solver *bvp4c*. Similarly, we then rewrite them to correspond to this form of the problem.

Listing 4. MATLAB expression
```
function main
  solinit = bvpinit(linspace(0,10,10),
            @guess(x));
  sol = bvp4c(@ode,@bc, solinit);
```

The similarity solution for two-dimensional stagnation-point flow is shown in Fig.(4).

V. Conclusion

This study provides that the similarity solution of reversed stagnation-point flow does not exist, because the governing equation does not satisfy the boundary conditions. On the other side, under the external boundary condition that $f''(\infty) = f'''(\infty = 0$, the numerical solution of the reversed stagnation-point flow is provided.

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Vai Kuong Sin is with the Department of Electromechanical Engineering, University of Macau, Macao SAR, China, E-mail: vksin@umac.mo. His current research interests include scientific computing, numerical simulation, and computational fluid dynamics.

Chio Chon Kit is with the Department of Electromechanical Engineering, University of Macau, Macao SAR, China, E-mail: mb054275@umac.mo.