Controller synthesis of switched positive systems with bounded time-varying delays

Xinhui Wang, Xiuyong Ding

Abstract—This paper addresses the controller synthesis problem of discrete-time switched positive systems with bounded time-varying delays. Based on the switched copositive Lyapunov function approach, some necessary and sufficient conditions for the existence of state-feedback controller are presented as a set of linear programming and linear matrix inequality problems, hence easy to be verified. Another advantage is that the state-feedback law is independent on time-varying delays and initial conditions. A numerical example is provided to illustrate the effectiveness and feasibility of the developed controller.

Keywords—Switched copositive Lyapunov functions, Positive linear systems, Switched systems, Time-varying delays, Stabilization.

I. INTRODUCTION

A Dynamical system is called *positive* if for any non-negative initial condition, the corresponding solution of system is also nonnegative. In the real world, positive systems play an important role in the modeling of dynamical phenomena whose variables are restricted to be nonnegative[1], [2]. This model class is used in many areas such as absolute temperature, level of liquid in tanks and concentrations of chemicals, ecology, etc.. These features make analysis and synthesis of positive systems a challenging and interesting job [3], [4], [5], [6], [7], [8], [9].

Recently, the importance of linear switched positive systems (LSPSs) has been highlighted by many researchers because of finding broad application in communication systems [5], formation flying [6], and other areas. It should be noted that, although positive systems had been many recent studies in the control engineering and mathematics literature, there are still many open questions relating to LSPSs. Thus, this observation has led to great interest in the stability of such systems under arbitrary switching regimes. A key result in this connection is that stability of such systems is equivalent to the existence of a common Lyapunov function[10]. Generally speaking, three classes of Lyapunov function naturally suggest themselves for LSPSs: common quadratic Lyapunov functions, common diagonal Lyapunov functions, and common linear copositive Lyapunov functions. For continuous time-invariant LSPSs, the authors of [11] and independently Dvid Angeli, posed a conjecture that the existence of common quadratic Lyapunov function can be determined by testing the Hurwitz-stability of an associated convex set of matrices. Gurvits, Shorten and

X. Wang is with the School of Management, Sichuan University, Chengdu, 610064, P. R. China. E-mail: wxh865@yahoo.cn.

X. Ding is with the School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu, 611731, P. R. China. E-mail: dingxingzhi@yahoo.cn.

Mason [12] proved that this conjecture is true for pairs of second order systems and is false in general. In the paper [13], a necessary and sufficient was derived for the existence of common diagonal Lyapunov function for the systems with irreducible system matrices. It is well known that traditional Lyapunov functions may give conservative stability conditions for LSPSs as they fail to take account that the trajectories are naturally constrained to the positive orthant. Therefore, it is natural to adopt common linear copositive Lyapunov functions which is both necessary and sufficient for the stability of LSPSs [14]. Moreover, work discussed in [15] provided a method for determining whether or not a given LSPSs is stable. Such approach is based upon determining verifiable conditions for a common linear copositive Lyapunov function. For the discrete time-invariant LSPSs, switched copositive Lyapunov function (SCLF) method is proposed in [16], some necessary and sufficient conditions for the existence of such a function has been established.

Up to now, the studies on LSPSs with delays have not been reported and note that SCLFs have less conservative than other Lyapunov functions. For these consideration, this paper aims to solve the stabilization problem for LSPSs with bounded time-varying delays by means of SCLF. The organization of this paper is as follows. Section II gives the mathematical background and notations necessary. Section III is dedicated to stabilization analysis of discrete-time LSPSs with bounded time-varying delays by SCLF approach. A numerical example is presented to illustrate the validity of the designed algorithms in Section IV , and some concluding remarks are presented in Section V.

II. NOTATION AND BACKGROUND

Throughout, $\mathbb{R}(\mathbb{R}_{0,+}, \mathbb{R}_+)$ denotes the set of all real (non-negative, positive) numbers, $\mathbb{R}^n(\mathbb{R}_{0,+}^n, \mathbb{R}_+^n)$ stands for the $n-$dimensional real (nonnegative, positive) vector space and $\mathbb{R}^{n \times m}(\mathbb{R}_{0,+}^{n \times m})$ is the space of $m \times n$ matrices with real (nonnegative) entries. For A in $\mathbb{R}^{n \times n}$, $A \succeq 0 (\preceq 0)$ means that all elements of matrix A are nonnegative (nonpositive) and $A \succ 0 (\prec 0)$ means that all elements of matrix A are positive (negative). The notion $A > 0 (< 0)$ means that matrix A is a symmetric positive (negative) definite matrix. Meanwhile, we write $A^T(A^{-1})$ for the transpose (inverse) of matrix A. Let $\mathbb{N} = \{1, 2, 3, \cdots\}$ and $\mathbb{N}_0 = \{0\} \bigcup \mathbb{N}$. $\|x\|$ denotes the norm of vector x.

A function $f : \mathbb{R}_{0,+} \to \mathbb{R}_{0,+}$ is said to be of class $\mathcal{K}$ if and only if it is continuous, strictly increasing, zero at zero, and satisfying $f(t) \to +\infty$ as $t \to +\infty$. Also, when referring to

the switched linear systems, stability shall be used to denote asymptotic stability under arbitrary switching signals.

Consider the delayed system

$$x(k+1) = A^{(0)}x(k) + \sum_{l=1}^{p} A^{(l)}x(k - h^{(l)}(k)),$$
$$k \in \mathbb{N}_0,$$
$$x(k) = \varphi(k) \succeq 0, \ k = -h, \cdots, 0, \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector. $A^{(l)} \in \mathbb{R}^{n \times n}$, $h^{(l)}(k) \in \mathbb{N}_0$ are bounded time-varying delays, $l \in \{0, \cdots, p\}$, $h = \max_{l \in \{1, \cdots, p\}} \{\sup h^{(l)}(k) | k \in \mathbb{N}_0\}$ is a constant in $\mathbb{N}_{0,+}$, $\varphi : \{-h, \cdots, 0\} \to \mathbb{R}_{0,+}$ is the vector-valued initial function.

Definition 1: System (1) is said to be positive if and only if for any initial condition $x(0) \succeq 0$, the corresponding trajectory $x(k) \succeq 0$ holds for all $k \in \mathbb{N}_0$.

Lemma 1: [17] System (1) is positive if and only if $A^{(l)} \succeq 0$ for all $l \in \{0, \cdots, p\}$.

Lemma 2: [18]The equilibrium zero of

$$x(k+1) = f(x(k), k), k \in \mathbb{N}_0 \tag{2}$$

is globally uniformly asymptotically stable if there is a positive definite, decrescent, and radially unbounded function $V : \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}$ such that $\triangle V(k, x(k)) = V(k+1, x(k+1)) - V(k, x(k))$ is negative definite along the solution of (2).

Remark 1: In Lemma 2, a function $V(k, x(k))$ is definite, decrescent, and radially unbound, which means that there exist two functions α and β of $\mathcal{K}$ such that $\alpha(\|x(k)\|) \leq V(k, x(k)) \leq \beta(\|x(k)\|)$. A function $V(k, x(k))$ is called negative definite if and only there exists a function γ of class $\mathcal{K}$ such that $V(k, x(k)) \leq -\gamma(\|x(k)\|)$.

III. MAIN RESULTS

In this section, we shall investigate the stabilization of the discrete-time switched system with bounded time-varying delays given by

$$x(k+1) = A^{(0)}(k)x(k) +$$
$$\sum_{l=1}^{p} A^{(l)}(k)x(k - h^{(l)}(k)) + u(k), \tag{3}$$
$$x(0) = \varphi(k) \succeq 0, \ k = -h, \cdots, 0,$$

where $x(k) \in \mathbb{R}^n$ is the state vector and $u(k) \in \mathbb{R}^n$ is the state-feedback, not sign restricted. $k \in \mathbb{N}_0$, $A^{(l)}(k) \in \{A_1^{(l)}, \cdots, A_m^{(l)}\}$, and $A_i^{(l)} \in \mathbb{R}^{n \times n}$. $h^{(l)}(k) \in \mathbb{N}_0$ are the delays satisfying $0 \leq h^{(l)}(k) \leq h^{(l)}$ with constant $h^{(l)} \in \mathbb{N}_0$ and $l \in \mathcal{P} = \{0, \cdots, p\}$. $i \in \mathcal{I} = \{1, \cdots, m\}$ is the index set, m is the number of the subsystems, $\varphi : \{-h, \cdots, 0\} \to \mathbb{R}_{0,+}$ is the vector-valued initial function, $h = \max\{h^{(l)} | l \in \mathcal{P}\}$, and write $\mathcal{H} = \{0, \cdots, h\}$.

The aim of this paper is to solve the controller synthesis problem by means of SCLF which guarantees the stability of the system with bounded time-varying delays (3). Especially, the control law must be designed in such way that the resulting governed system is positive and stable. In this paper, we are interested in designing a state-feedback controller

$$u(k) = F(k) \sum_{l=0}^{p} x(k - h^{(l)}(k)), \tag{4}$$

where $F(k) \in \{F_1, \cdots, F_m\}, F_i \in \mathbb{R}^{n \times n}$.

With regard to the previous preliminary results, the problem reduces to look for a state-feedback law $u(k)$ as (4) has to be determined to satisfy the following problem.

- The system (3) is positive, i.e., $A_i^{(l)} + F_i \succeq 0$.
- The system (3) is stable, i.e., find a SCLF whose difference is negative definite.

Now, substituting (4) into system (3) yields the following closed-loop system

$$x(k+1) = \widetilde{A}^{(0)}(k)x(k) + \sum_{l=1}^{p} \widetilde{A}^{(l)}(k)x(k - h^{(l)}(k)),$$
$$x(0) = \varphi(k) \succeq 0, \ k = -h, \cdots, 0, \tag{5}$$

where $\widetilde{A}^{(l)}(k) = A^{(l)}(k) + F(k), l \in \mathcal{P}$.

Furthermore, let $\bar{x}(k) = [x^T(k), x^T(k-1), \cdots, x^T(k-h)]^T \in \mathbb{R}^{n(h+1)}$ and

$$\bar{A}(k) = \begin{bmatrix} \bar{A}^{(0)}(k) & \cdots & \bar{A}^{(h-1)}(k) & \bar{A}^{(h)}(k) \\ I_n & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & I_n & 0 \end{bmatrix} \tag{6}$$

with

$$\bar{A}^{(g)}(k) = \begin{cases} \widetilde{A}^{(l)}(k), & h^{(l)}(k) = g \\ 0, & h^{(l)}(k) \neq g \end{cases}, \forall g \in \mathcal{H}, l \in \mathcal{P}. \tag{7}$$

Then, the switched system without delays equivalent to the switched system (5) has the form

$$\bar{x}(k+1) = \bar{A}(k)\bar{x}(k). \tag{8}$$

Now, it is easy to see that positivity and stability of (8) implies that (5) is positive and stability. Furthermore, define the indicator function with $i \in \mathcal{I}$ as

$$\xi_i(k) = \begin{cases} 1, & \text{when the } i\text{th mode of (5) is activated} \\ 0, & \text{otherwise} \end{cases} \tag{9}$$

Then, the system (8) can also be rewritten as

$$\bar{x}(k+1) = \sum_{i=1}^{m} \xi_i(k)\bar{A}_i\bar{x}(k), \tag{10}$$

where

$$\bar{A}_i = \begin{bmatrix} \bar{A}_i^{(0)} & \cdots & \bar{A}_i^{(h-1)} & \bar{A}_i^{(h)} \\ I_n & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & I_n & 0 \end{bmatrix} \in \mathbb{R}^{n(h+1) \times n(h+1)} \tag{11}$$

with

$$\bar{A}_i^{(g)} = \begin{cases} \widetilde{A}_i^{(l)}, & h^{(l)}(k) = g \\ 0, & h^{(l)}(k) \neq g \end{cases}, \forall g \in \mathcal{H}. \tag{12}$$

In the case of switched systems as (8), this corresponds to the SCLF defined as

$$V(k) = \bar{x}^T(k) \sum_{i=1}^{m} \xi_i(k)\lambda_i = \bar{x}^T(k)\lambda(k), \tag{13}$$

where $\lambda(k) = \sum_{i=1}^{m} \xi_i(k)\lambda_i \in \mathbb{R}_+^{n(h+1)}$.

According to the structure of the system (8) and (10), we thus write

$$\lambda(k) = [\lambda^{(0)T}(k), \cdots, \lambda^{(h)T}(k)]^T,$$
$$\lambda_i = [\lambda_i^{(0)T}, \cdots, \lambda_i^{(h)T}]^T \in \mathbb{R}_+^{n(h+1)}, \quad (14)$$
$$\lambda_i^{(g)} = [\lambda_{i1}^{(g)}, \cdots, \lambda_{in}^{(g)}]^T \in \mathbb{R}_+^n.$$

Moreover, it is easy to show that the SCLF (13) is positive definite, decrescent, and radially unbounded since

$$\alpha\|\bar{x}(k)\| \le V(k, x(k)) \le \beta\|\bar{x}(k)\|$$

with $\alpha = \min_{\forall(i,s,g)}\{\lambda_{is}^{(g)}\}, \beta = \max_{\forall(i,s,g)}\{\lambda_{is}^{(g)}\}, i \in \mathcal{I}, s \in \mathcal{N} = \{1, \cdots, n\}$, and $g \in \mathcal{H}$.

The following Theorem ensures that the state-feedback controller is easily fixed, such controller guarantees that the switched closed-loop system (5) is positive and stable.

Theorem 1: For the switched close-loop system (5), the following statements are equivalent:

(i) There exist a state-feedback controller given by (4) and a SCLF of the form (13) whose difference is negative definite, such that system (5) is positive and stable.

(ii) *(LP problem)* For all $(g, l, i, j, u, v) \in \mathcal{H} \times \mathcal{P} \times \mathcal{I} \times \mathcal{I} \times \mathcal{N} \times \mathcal{N}$, there exist $m \times (h+1)$ vectors $\lambda_i^{(g)} \in \mathbb{R}_+^n$ and $m \times m \times n$ vectors $\boldsymbol{k}_{ijv} = [k_{ijv1}, \cdots, k_{ijvn}] \in \mathbb{R}^n$, satisfying

$$a_{iuv}^{(l)}\lambda_{ju}^{(0)} + k_{ijvu} \ge 0,$$
$$A_i^{(l)T}\lambda_j^{(0)} + \lambda_j^{(l+1)} - \lambda_i^{(l)}$$
$$+ \sum_{v=1}^{n} \boldsymbol{k}_{ijv} \prec 0, \ g = h^{(l)}(k), \quad (15)$$
$$\lambda_j^{(g+1)} - \lambda_i^{(g)} \prec 0, \ g \ne h^{(l)}(k)$$

with $A_i^{(l)} = [a_{iuv}^{(l)}]$. Moreover, the gain matrices F_i can be calculated as follows:

$$F_i = [\boldsymbol{f}_{i1}, \cdots, \boldsymbol{f}_{in}] = \left[\frac{\boldsymbol{k}_{ij1}^T}{\lambda_{j1}^{(0)}}, \cdots, \frac{\boldsymbol{k}_{ijn}^T}{\lambda_{jn}^{(0)}}\right]^T. \quad (16)$$

(iii) *(LMI problem)* For all $(g, l, i, j, u, v) \in \mathcal{H} \times \mathcal{P} \times \mathcal{I} \times \mathcal{I} \times \mathcal{N} \times \mathcal{N}$, there exist $m \times (h+1)$ matrices $Q_i^{(g)} = diag\left(q_{i1}^{(g)}, q_{i2}^{(g)}, \cdots, q_{in}^{(g)}\right) > 0$ and $m \times m$ matrices $D_{ij} = [\boldsymbol{d}_{ij1}, \cdots, \boldsymbol{d}_{ijn}] = [d_{ijvu}] \in \mathbb{R}^{n \times n}$ such that

$$a_{iuv}^{(l)}\lambda_{ju}^{(0)} + d_{ijvu} \ge 0,$$
$$\Phi_{ij}^{(l)} < 0, \ g = h^{(l)}(k), \quad (17)$$
$$Q_j^{(g+1)} - Q_i^{(g)} < 0, \ g \ne h^{(l)}(k),$$

where

$$\Phi_{ij}^{(l)} = diag\left(\phi_{ij1}^{(l)}, \phi_{ij2}^{(l)}, \cdots, \phi_{ijn}^{(l)}\right),$$
$$\phi_{iju}^{(l)} = \boldsymbol{a}_{iu}^{(l)T}\boldsymbol{q}_j^{(0)} + q_{ju}^{(l+1)} - q_{iu}^{(l)} + \sum_{v=1}^{n} d_{ijvu}$$

with $\boldsymbol{q}_j^{(0)} = [q_{j1}^{(0)}, q_{j2}^{(0)}, \cdots, q_{jn}^{(0)}]^T$ and $\boldsymbol{a}_{iu}^{(l)}$ is the uth column vector of matrix $A_i^{(l)}$. In this case, the gain matrices F_i can be fixed as follows:

$$F_i = [\boldsymbol{f}_{i1}, \cdots, \boldsymbol{f}_{in}] = \left[\frac{\boldsymbol{d}_{ij1}^T}{q_{j1}^{(0)}}, \cdots, \frac{\boldsymbol{d}_{ijn}^T}{q_{jn}^{(0)}}\right]^T. \quad (18)$$

Proof: (i)⇒(ii): Suppose that the statement (i) is satisfied.

On the one hand, as system (5) is positive, it follows from Lemma 1 that

$$A_i^{(l)} + F_i \succeq 0. \quad (19)$$

By setting $f_{iuv} = k_{ijvu}/\lambda_{jv}^{(0)}$, we have

$$A_i^{(l)} + F_i \succeq 0 \Rightarrow a_{iuv}^{(l)} + f_{iuv} \ge 0$$
$$\Rightarrow a_{iuv}^{(l)} + \frac{k_{ijvu}}{\lambda_{jv}^{(0)}} \ge 0 \quad (20)$$
$$\Rightarrow a_{iuv}^{(l)}\lambda_{ju}^{(0)} + k_{ijvu} \ge 0.$$

On the other hand, as the SCLF (13) is negative definite, one can get from (8) that

$$\triangle V(k) = V(k+1) - V(k)$$
$$= \bar{x}^T(k+1)\lambda(k+1) + \bar{x}^T(k)\lambda(k) \quad (21)$$
$$= \bar{x}^T(k)(\bar{A}^T(k)\lambda(k+1) - \lambda(k)) < 0.$$

Moreover, from the positivity of system (5), we know system (8) is positive . Therefor, for nonzero $\bar{x}^T(k) \in \mathbb{R}_+^{n(h+1)}$, (21) implies that

$$\bar{A}^T(k)\lambda(k+1) - \lambda(k) \prec 0.$$

Substitution of (6) and (14) yields

$$\begin{bmatrix} \bar{A}^{(0)T}(k) & I_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \bar{A}^{(h-1)T}(k) & 0 & \cdots & I_n \\ \bar{A}^{(h)T}(k) & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \lambda^{(0)}(k+1) \\ \vdots \\ \lambda^{(h-1)}(k+1) \\ \lambda^{(h)}(k+1) \end{bmatrix}$$
$$- \begin{bmatrix} \lambda^{(0)}(k) \\ \vdots \\ \lambda^{(h-1)}(k) \\ \lambda^{(h)}(k) \end{bmatrix} \prec 0. \quad (22)$$

For all $g \in \mathcal{H}$, let $\lambda^{(h+1)}(k+1) = 0$, it follows from (22) that

$$\bar{A}^{(g)T}(k)\lambda^{(0)}(k+1) + \lambda^{(g+1)}(k+1) - \lambda^{(g)}(k) \prec 0. \quad (23)$$

As this has to be satisfied for arbitrary switching signals, it implies that (23) has to hold for any special configuration $\xi_i(k) = 1, \xi_{s\ne i}(k) = 0, \xi_j(k+1) = 1, \xi_{s\ne j}(k+1) = 0, s \in \{1, \cdots, n\}$ and for all $\bar{x}^T(k) \in \mathbb{R}_+^{n(h+1)}$. Then, combining (12) with (14), we obtain

$$\widetilde{A}_i^{(l)T}\lambda_j^{(0)} + \lambda_j^{(l+1)} - \lambda_i^{(l)} \prec 0, \ g = h^{(l)}(k),$$
$$\lambda_j^{(g+1)} - \lambda_i^{(g)} \prec 0, \ g \ne h^{(l)}(k)$$

for all $(g, l, i, j) \in \mathcal{H} \times \mathcal{P} \times \mathcal{I} \times \mathcal{I}$. Furthermore, note that $\widetilde{A}_i^{(l)T} = A_i^{(l)} + F_i$, it follows that

$$A_i^{(l)}\lambda_j^{(0)} + \lambda_j^{(l+1)} - \lambda_i^{(l)} + F_i\lambda_j^{(0)} \prec 0, \ g = h^{(l)}(k),$$
$$\lambda_j^{(g+1)} - \lambda_i^{(g)} \prec 0, \ g \ne h^{(l)}(k).$$

Setting $\mathbf{f}_{iv} = \mathbf{k}_{ijv}/\lambda_j^{(0)}, v \in \mathcal{N}$ and taking into account (20), we thus know that statement (ii) holds.

(ii)$\Rightarrow$ (i): Assume that (ii) is satisfied. Firstly, since $a_{iuv}^{(l)}\lambda_{ju}^{(0)} + k_{ijvu} \geq 0$, together with (16), one can deduce that

$$a_{iuv}^{(l)}\lambda_{ju}^{(0)} + k_{ijvu} \geq 0 \Rightarrow a_{iuv}^{(l)} + \frac{k_{ijvu}}{\lambda_{jv}^{(0)}} \geq 0$$
$$\Rightarrow a_{iuv}^{(l)} + f_{iuv} \geq 0$$
$$\Rightarrow A_i^{(l)} + F_i \succeq 0.$$

This immediately implies that the system (5) is positive. Next, we shall show that system (5) is stable.

For all $(i,j) \in \mathcal{I} \times \mathcal{I}$, by using (9), it is easy to show that

$$\sum_{i=1}^m \xi_i(k) = \sum_{j=1}^m \xi_j(k+1) = 1, \ \sum_{i=1}^m \xi_i(k) \sum_{j=1}^m \xi_j(k+1) = 1.$$

Taking into account (15), we first consider $g \neq h^{(l)}(k)$. From (16) we thus find

$$\sum_{i=1}^m \xi_i(k) \sum_{j=1}^m \xi_j(k+1) \times$$
$$\left(A_i^{(l)T}\lambda_j^{(0)} + \lambda_j^{(l+1)} - \lambda_i^{(l)} + \sum_{v=1}^n \mathbf{k}_{ijv} \right)$$
$$= \sum_{i=1}^m \xi_i(k) \sum_{j=1}^m \xi_j(k+1) \times$$
$$\left[\left(A_i^{(l)T} + F_i^T \right)\lambda_j^{(0)} + \lambda_j^{(l+1)} - \lambda_i^{(l)} \right]$$
$$= \sum_{i=1}^m \xi_i(k) \left[\widetilde{A}_i^{(l)T} \left(\sum_{j=1}^m \xi_j(k+1)\lambda_j^{(0)} \right) \right.$$
$$\left. + \left(\sum_{j=1}^m \xi_j(k+1)\lambda_j^{(l+1)} \right) - \lambda_i^{(l)} \right]$$
$$= \sum_{i=1}^m \xi_i(k) \left[\widetilde{A}_i^{(l)T}\lambda^{(0)}(k+1) + \lambda^{(l+1)}(k+1) - \lambda_i^{(l)} \right]$$
$$= \widetilde{A}^{(l)T}(k)\lambda^{(0)}(k+1) + \lambda^{(l+1)}(k+1) - \lambda^{(l)}(k)$$
$$\prec 0 \qquad .$$

Similarly, for $g = h^{(l)}(k)$, it follows that

$$\sum_{i=1}^m \xi_i(k) \sum_{j=1}^m \xi_j(k+1) \left(\lambda_j^{(g+1)} - \lambda_i^{(g)} \right)$$
$$= \sum_{i=1}^m \xi_i(k) \left[\left(\sum_{j=1}^m \xi_j(k+1)\lambda_j^{(g+1)} \right) - \lambda_i^{(g)} \right]$$
$$= \sum_{i=1}^m \xi_i(k) \left[\lambda^{(g+1)}(k+1) - \lambda_i^{(g)} \right]$$
$$= \lambda^{(g+1)}(k+1) - \lambda^{(g)}(k)$$
$$\prec 0.$$

Based on the argument above, we have shown that

$$\widetilde{A}^{(l)T}(k)\lambda^{(0)}(k+1) + \lambda^{(l+1)}(k+1)$$
$$- \lambda^{(l)}(k) \prec 0, g = h^{(l)}(k), \qquad (24)$$
$$\lambda^{(g+1)}(k+1) - \lambda^{(g)}(k) \prec 0, g \neq h^{(l)}(k)$$

for all $(g,l) \in \mathcal{H} \times \mathcal{P}$.
Furthermore, by using (7) and (14), (24) is equivalent to

$$\bar{A}^T(k)\lambda(k+1) - \lambda(k) \prec 0. \qquad (25)$$

This corresponds to the SCLF defined as

$$V(k) = \bar{x}^T(k)\lambda(k). \qquad (26)$$

As we have shown that the system (5) is positive, for nonzero $\bar{x}^T(k) \in \mathbb{R}_+^{n(h+1)}$, it is easy to check from (25) and (26) that

$$\bar{x}^T(k) \left(\bar{A}^T(k)\lambda(k+1) - \lambda(k) \right)$$
$$= \bar{x}^T(k)\bar{A}^T(k)\lambda(k+1) - \lambda(k)$$
$$= \bar{x}^T(k+1)\lambda(k+1) - \lambda(k) = \triangle V(k)$$
$$< 0.$$

Note that (15), it follows that $\triangle V(k) \leq -\gamma\|\bar{x}(k)\|$ with

$$\gamma = -\max\left\{ \theta_{ijs}^{(g)}|\forall(g,i,j,s) \in \mathcal{H} \times \mathcal{I} \times \mathcal{I} \times \mathcal{N} \right\},$$

where $\theta_{ijs}^{(g)}$ is the sth element of $\bar{A}_i^{(g)T}\lambda_j^{(0)} + \lambda_j^{(g+1)} - \lambda_i^{(g)}$ and $\bar{A}_i^{(g)}$ is defined as (10). Therefore, the system (8) is stable, which implies the stability of closed-loop system (5).

(ii)$\Leftrightarrow$(iii): The equivalence between statement (ii) and (iii) obviously holds by setting $\lambda_{iv}^{(g)} = q_{iv}^{(g)}$ and $\mathbf{k}_{ijv} = \mathbf{d}_{ijv}, v \in \mathcal{N}$. ∎

Remark 2: Observe that selecting $h^{(l)} \equiv u(k) \equiv 0$ in Theorem 1 reproduces the main result given in [12].

Remark 3: Note that the conditions of the state-feedback control law in Theorem 1 are presented as a set of linear programming (LP) problems and linear matrix inequality (LMI) problems. Therefore, these conditions are not only checkable but also numerical. Indeed, Statement (ii) can be solved by linear programming optimal toolbox. Statement (iii) may be verified by linear matrix inequality toolbox.

Remark 4: We stress that the stabilization of (5) is independent on delays and initial conditions. In other words, the magnitude of the delays and the choice of the initial condition does not affect the design of the controller for system (5) and are completely determined by the system matrices.

Remark 5: In Theorem 1, if we make a stronger assumption that $\lambda_1^{(g)} = \cdots = \lambda_m^{(g)}$ for all $g \in \mathcal{H}$, then the SCLF of the form (13) is reduced to the common copositive Lyapunov function. This is shown in the following corollary, which is straightforward from Theorem 1.

Corollary 1: If there exist $h+1$ vectors $\lambda^{(g)}$ satisfying and $m \times n$ vectors $\boldsymbol{k}_{iv} = [k_{iv1}, \cdots, k_{ivn}] \in \mathbb{R}^n$, satisfying

$$a_{iuv}^{(l)}\lambda_u^{(0)} + k_{ivu} \geq 0,$$
$$A_i^{(l)T}\lambda^{(0)} + \lambda^{(l+1)} - \lambda^{(l)} + \sum_{v=1}^n \boldsymbol{k}_{iv} \prec 0, \ g = h^{(l)}(k),$$
$$\lambda^{(g+1)} - \lambda^{(g)} \prec 0, \ g \neq h^{(l)}(k)$$

with $A_i^{(l)} = [a_{iuv}^{(l)}]$. Then the switched system (5) is positive and stable. Moreover, the gain matrices F_i can be calculated as follows:

$$F_i = [\boldsymbol{f}_{i1}, \boldsymbol{f}_{i2}, \cdots, \boldsymbol{f}_{in}] = \left[\frac{\boldsymbol{k}_{i1}^T}{\lambda_1^{(0)}}, \frac{\boldsymbol{k}_{i2}^T}{\lambda_2^{(0)}}, \cdots, \frac{\boldsymbol{k}_{in}^T}{\lambda_n^{(0)}} \right]^T.$$

In the following Corollary, we will consider the system (3) with $u(k) \equiv 0$ to be positive, and provide the following stability criteria.

Corollary 2: For all $(g, l, i, j, s) \in \mathcal{H} \times \mathcal{P} \times \mathcal{I} \times \mathcal{I} \times \mathcal{N}$, if the system (3) with $u(k) \equiv 0$ satisfies $A_i^{(l)} \succeq 0$, then the following statements are equivalent:

(i) There exists a SCLF of the form (13) whose difference is negative definite, proving the stability of (3).

(ii) *(LP problem)* There exist $m \times (h+1)$ vectors $\lambda_i^{(g)} \in \mathbb{R}_+^n$, satisfying

$$A_i^{(l)T} \lambda_j^{(0)} + \lambda_j^{(l+1)} - \lambda_i^{(l)} \prec 0, \ g = h^{(l)}(k),$$
$$\lambda_j^{(g+1)} - \lambda_i^{(g)} \prec 0, \ g \neq h^{(l)}(k).$$

(iii) *(LMI problem)* There exist $m \times (h+1)$ matrices $Q_i^{(g)} = diag\left(q_{i1}^{(g)}, q_{i2}^{(g)}, \cdots, q_{in}^{(g)} \right) > 0$ such that

$$\Phi_{ij}^{(l)} < 0, \ g = h^{(l)}(k),$$
$$Q_j^{(g+1)} - Q_i^{(g)} < 0, \ g \neq h^{(l)}(k),$$

where

$$\Phi_{ij}^{(l)} = diag\left(\phi_{ij1}^{(l)}, \phi_{ij2}^{(l)}, \cdots, \phi_{ijn}^{(l)} \right),$$
$$\phi_{ijs}^{(l)} = \boldsymbol{a}_{is}^{(l)T} \boldsymbol{q}_j^{(0)} + q_{js}^{(l+1)} - q_{is}^{(l)}$$

with $\boldsymbol{q}_j^{(0)} = [q_{i1}^{(0)}, q_{i2}^{(0)}, \cdots, q_{in}^{(0)}]^T$ and $\boldsymbol{a}_{is}^{(l)}$ is the sth column vector of $A_i^{(l)}$.

IV. EXAMPLE

In this section, an example is given to verify technically feasibility and operability of the developed results.

Example 1: Consider the switched closed-loop system with bounded time-varying delays (5) given by

$$x(k+1) = (A_i^{(0)} + F_i)x(k)$$
$$+ (A_i^{(1)} + F_i)x(k - h^{(1)}(k)), \quad (27)$$
$$x(0) = \varphi(k) \succeq 0, \quad k = -1, 0,$$

where $x(k) \in \mathbb{R}^2$, $i = 1, 2$, $0 \leq h^{(1)}(k) \leq 1$ and

$$A_1^{(0)} = \begin{bmatrix} 0.1 & -0.2 \\ -0.12 & -0.2 \end{bmatrix}, \quad A_1^{(1)} = \begin{bmatrix} -0.1 & -0.1 \\ -0.2 & 0.08 \end{bmatrix},$$
$$A_2^{(0)} = \begin{bmatrix} -0.3 & -0.1 \\ 0.3 & 0.1 \end{bmatrix}, \quad A_2^{(1)} = \begin{bmatrix} -0.4 & 0.1 \\ 0.2 & -0.2 \end{bmatrix}.$$

Obviously, the state of system (27) may be nonpositive for all $x(0) \succeq 0$. Using state-feedback we want to stabilize the system and enforce the state to be positive and stable. Applying Theorem 1, one feasible solution of the LP problem provides

$$\lambda_1^{(0)} = [0.9155 \ 0.7332]^T, \lambda_1^{(1)} = [0.3515 \ 0.5433]^T,$$
$$\lambda_2^{(0)} = [0.9454 \ 0.8726]^T, \lambda_2^{(1)} = [0.3828 \ 0.4296]^T,$$

and

$$\boldsymbol{k}_{111} = [0.1945 \ 0.2999]^T, \boldsymbol{k}_{112} = [0.2705 \ 0.2132]^T,$$
$$\boldsymbol{k}_{121} = [0.2008 \ 0.3097]^T, \boldsymbol{k}_{122} = [0.3219 \ 0.2538]^T,$$
$$\boldsymbol{k}_{211} = [0.3823 \ 0.1802]^T, \boldsymbol{k}_{212} = [0.1587 \ 0.1623]^T,$$
$$\boldsymbol{k}_{221} = [0.3948 \ 0.1861]^T, \boldsymbol{k}_{222} = [0.1889 \ 0.1932]^T.$$

By using this solution, together with (16), a controller can be obtained as

$$F_1 = \begin{bmatrix} 0.2124 & 0.3276 \\ 0.3689 & 0.2908 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0.4176 & 0.1968 \\ 0.2165 & 0.2214 \end{bmatrix}.$$

With this controller the closed-loop system (27) is positive and stable. See Fig. 1, Fig. 2, and Fig. 3. Where the initial randomly generated in $[25 \ 30] \times [25 \ 30]$, the time delays randomly takes value in $\{0, 1\}$, and the switching signal is also randomly generated. Fig. 1 show the response of the state variables. The mark '*' in the Fig. 2 describes the state change. Fig. 3 connects all the sampling points into a continuous trajectory.
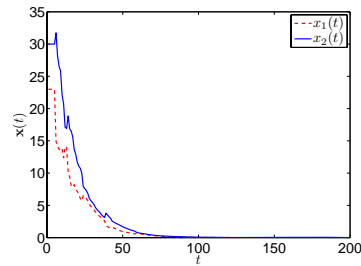

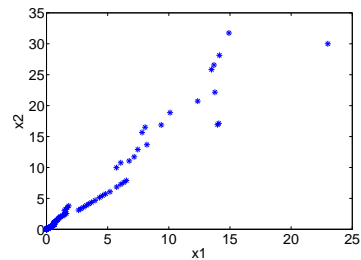
Fig. 1. The state variables of system (27).
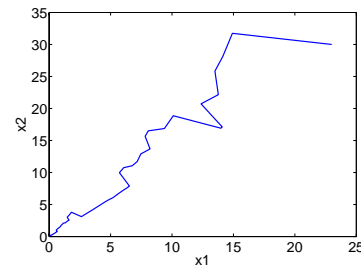

Fig. 2. The trajectory of system (27).

Fig. 3. The trajectory of system (27).

V. Conclusions

By means of SCLF, two necessary and sufficient conditions for the existence of state-feedback controller have been presented for switched linear positive systems with bounded time-varying delays. The advantages of the results lies in two aspects. First, all the results are formulated as linear programming or linear matrix inequality problems, hence easy to be solved. Second, the existence of controller is not affected by the size of delays and the choice of initial condition. Such advantages have been shown in the numerical example.

Acknowledgment

References

[1] A. Berman, M. Neumann, and R. Stern, *Nonnegative matrices in dynamic systems*. Wiley, New York, 1989.

[2] T. Kaczorek, *Positive 1D and 2D Systems*. Springer- Verlag, London, 2002.

[3] L. Benvenuti, A. D. Santis, and L. Farina, *Positive Systems*. Springer-Verlag, Berlin, Germany, 2003.

[4] M. Busłicz, T. Kaczorek, "Robust stability of positive discrete-time interval systems with time-delays," *Bulletin of the Polish Academy of Sciences*, vol. 52(2), pp. 99–102, 2005.

[5] R. Shorten, D. Leith, J. Foy, and R. Kilduff, "Towards an analysis and design framework for congestion control in communication networks," *Proceedings of the 12th Yale Workshop on Adaptive and Learning Systems*, Yale University, New Haven, CT, USA, 2003.

[6] A. Jadbabaie, J. Lin, and A. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Transaction on Automatic Control*, vol. 48(6), pp. 1099–1103, 2003.

[7] T. Kaczorek, "The choice of the forms of lyapunov functions for a positive 2d roesser model," *International Journal of Applied Mathematics and Computer Science*, vol. 17(4), pp. 471– 475, 2007.

[8] T. Kaczorek, "A realzation problem for positive continuous-time systems with reduced numbers of delays," *International Journal of Applied Mathematics and Computer Science*. vol. 16(3), pp. 325–331, 2006.

[9] M. Rami, F. Tadeo, and A. Benzaouia, "Control of constrained positive discrete systems," *Proceedings of the American Control Conference*, New York City, USA, pp. 5851–5857, 2007.

[10] R. Shorten, F. Wirth, O. Mason, K. Wulff, and C. King, "Stability theory for switched and hybrid systems," *Linear Algebra and its Application*, vol. 49(4), pp. 545–592, 2007.

[11] O. Mason, R. Shorten, "A conjecture on the existence of common quadratic lyapunov functions for positive linear systems," *Proceedings of the American Control Conference*, New York City, USA, pp. 4469–4470, 2003.

[12] L. Gurvits, R. Shorten, and O. Mason, "On the stability of switched positive linear systems," *IEEE Transaction on Automatic Control* vol. 52(6), pp. 1099–1103, 2007.

[13] O. Mason, R. Shorten, "On the simultaneous diagonal stability of pair of positive linear systems," *Linear Algebra and its Application*, vol. 413(1), pp. 13–23, 2006.

[14] L. Farina, S. Rinaldi, *Positive linear systems: theory and applications*. Wiley, New York, 2000.

[15] O. Mason, R. Shorten, "On linear copositive lyapunov functions and the stability of switched positive linear systems," *IEEE Transaction on Automatic Control*, vol. 52(7), pp. 1346–1349, 2007.

[16] X. Liu, "Stability analysis of switched positive systems: A switched linear copositive lyapunov function method," *IEEE Transaction on Circuits and Systtems II*, vol. 56(5), pp. 414– 418, 2009.

[17] X. Liu, "Stability analysis of positive systems with bounded time-varying delays," *IEEE Transaction on Circuits and Systtems II*, vol. 56(7), pp. 600–604, 2009.

[18] M. Vidyasagar, *Nonlinear Systems Analysis*. Prentice- Hall, Upper Saddle River, NJ, 1993.

Xinhui Wang was born in Sichuan Province, China, in 1980. He obtained his M.S. degree in applied mathematics from the University of Electronic Science and Technology of China (UESTC), Chengdu, in 2009. Currently, He is pursuing his Ph.D. degree with SiChuan University . His research interests include the application of cybernetics, supply chain management.

Xiuyong Ding was born in Sichuan Province, China, in 1981. He obtained his M.S. degree in applied mathematics from the University of Electronic Science and Technology of China (UESTC), Chengdu, in 2009. Currently, He is pursuing his Ph.D. degree with UESTC. His research interests include the application of positive systems, switched systems, and fuzzy systems.