Boundary-element-based finite element methods for Helmholtz and Maxwell equations on general polyhedral meshes

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Abstract—We present new finite element methods for Helmholtz and Maxwell equations on general three-dimensional polyhedral meshes, based on domain decomposition with boundary elements on the surfaces of the polyhedral volume elements. The methods use the lowest-order polynomial spaces and produce sparse, symmetric linear systems despite the use of boundary elements. Moreover, piecewise constant coefficients are admissible. The resulting approximation on the element surfaces can be extended throughout the domain via representation formulas. Numerical experiments confirm that the convergence behavior on tetrahedral meshes is comparable to that of standard finite element methods, and equally good performance is attained on more general meshes.

Keywords—Boundary elements, finite elements, Helmholtz equation, Maxwell equations.

I. INTRODUCTION

IN industrial applications involving the numerical solution of partial differential equations, refinement of a given mesh may be infeasible due to a prohibitively large number of degrees of freedom. If the mesh consists of elements beyond the simplest types commonly used (e.g. tetrahedra, hexahedra, prisms), then standard finite element methods cannot be applied without refinement. Even if finite element methods are known for the various types of elements in a mesh, the implementation and solution via finite element methods can be quite difficult and expensive for meshes with many different types of elements. Thus there is a need for numerical methods which can be applied robustly to any polyhedral mesh, without refinement or special treatment of each of the various element types. We propose such a method for the Helmholtz and Maxwell equations, including the Laplace equation as a special case (with wavenumber equal to zero).

The mimetic finite difference (MFD) method has been proposed for numerically solving partial differential equations on general polygonal or polyhedral meshes. The MFD method has been proven effective for diffusion problems (see e.g. [3], [23], [24]), and in [22], an MFD method for Maxwell equations in two dimensions is presented, which could be extended to three dimensions. However, this method relies on a logically rectangular mesh structure, and there seems to be no MFD method for Maxwell equations on general meshes.

The discontinuous Galerkin (DG) method appears to be a more promising approach for discretization on general meshes.

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DG methods are known to treat general, non-conforming meshes with ease, and methods for Maxwell equations have been well studied– see e.g. [14], [19], [20] and the references therein. However, the author is unaware of any study of DG methods for Maxwell equations applied to general three-dimensional meshes.

In this paper, we propose a new discretization method based on boundary integral techniques. Boundary element methods have been widely studied and have proven very effective in many types of problems with constant coefficients. Boundary integral representation formulas give the solution in a domain as an expression involving only its traces on the boundary, thereby reducing the problem to the boundary. This dimension reduction makes boundary element methods much faster and more efficient than finite element methods in some cases. However, boundary element methods have some drawbacks in general, such as the limited applicability to problems with constant coefficients and the practical necessity of data-sparse approximations of the dense linear systems that result from discretization. As a result, some very efficient methods have been designed by coupling boundary element and finite element methods in different subdomains as appropriate. For example, in [17], [18], scattering problems are efficiently solved by coupling boundary elements in the unbounded exterior domain, where the coefficient is constant, with finite elements in the interior domain, where the coefficient may vary. In principle, we take a similar approach for acoustic and electromagnetic scattering problems, but in this paper we focus only on developing finite-element-type discretizations for the bounded interior domain. It is a simple matter to couple our method with a boundary element method in the exterior domain, and we will indicate how this can be done.

Given a general polyhedral mesh of the bounded domain, we follow the domain decomposition approach of [21] and apply boundary integral techniques on the surfaces of the polyhedral elements. That is, we treat the elements as subdomains and discretize the element surfaces with boundary element spaces. Strictly speaking, the resulting method is a boundary element domain decomposition, but we refer to it as a *boundary-element-based finite element method* due to its similarity to finite element methods with respect to several fundamental properties. For instance, piecewise constant coefficients are admissible, and the resulting linear system is sparse, with coupling only between degrees of freedom in the same volume element.

The methods developed in this paper can also be viewed

strictly as domain decomposition methods analogous to [21], for subdomains of any size. Such methods were studied in [21] for scalar elliptic partial differential equations, but we extend those results to the Helmholtz and Maxwell equations. In particular, we analyze the Dirichlet-to-Neumann maps and prove that they satisfy generalized Gårding inequalities. Thus the present paper is of both a finite element and domain decomposition nature, depending on one's interpretation. In the theoretical analysis, we can only prove error estimates for a pure domain decomposition setting, with subdomains of fixed size. However, the numerical experiments reported in this paper investigate the method in a finite element setting, demonstrating error behavior comparable to optimal finite element methods.

For a practical implementation, we require a triangular mesh of the *skeleton* comprised of the element surfaces. Alternatively, one could just as easily use a quadrilateral skeleton mesh. Note that hanging nodes are inadmissible, as the method uses a conforming finite dimensional subspace of a trace space on the skeleton. Thus our method allows for general volume elements but requires a triangular or quadrilateral mesh of the skeleton. As standard finite element methods can be applied only after refinement of the volume elements to geometrically simple elements, for some meshes our method involves significantly fewer degrees of freedom than standard finite element methods despite the necessary refinement of the mesh skeleton to triangular or quadrilateral surface elements.

The paper is organized as follows. In section 2, we define some trace operators and relevant Sobolev spaces on the boundary of a Lipschitz polyhedral domain. Sections 3 and 4 have very similar structure, as they describe the boundary-element-based finite element method applied to the Helmholtz and Maxwell equations, respectively. Finally, we report on some numerical experiments in section 5.

II. PRELIMINARIES

The methods proposed in this paper solve for traces of solutions to the Helmholtz or Maxwell equations and therefore require some theory for the trace operators and the Sobolev spaces onto which they map. In this section, we briefly present the necessary theory for Lipschitz polyhedral domains.

On a Lipschitz polyhedral domain Ω and its boundary $\Gamma := \partial\Omega$, the complex-valued Sobolev spaces $H^s(\Omega)$ and $H^t(\Gamma)$ are defined for all s in $\mathbb{R}$ and $|t| \leq 1$ [1], [25], with $L^2 := H^0$. The inner product $< \cdot, \cdot >_\Gamma$ in $L^2(\Gamma)$ is defined by

$$< u, v >_\Gamma := \int_\Gamma \overline{u}v \, dS \qquad \text{for all } u, v \text{ in } L^2(\Gamma).$$

The inner product $(\cdot, \cdot)_\Omega$ in $L^2(\Omega)$ is defined analogously. To formulate a method for solving the Helmholtz equation, we shall utilize the standard Dirichlet and Neumann trace operators $\gamma_0 : H^1_{loc}(\mathbb{R}^3) \to H^{\frac{1}{2}}(\Gamma)$ and $\gamma_1 : H_{loc}(\Delta, \mathbb{R}^3 \setminus \Gamma) \to H^{-\frac{1}{2}}(\Gamma)$ (cf. [13]), where

$$H_{loc}(\Delta, \mathbb{R}^3 \setminus \Gamma) := \{\phi \in H^1_{loc}(\mathbb{R}^3 \setminus \Gamma) : \Delta\phi \in L^2_{loc}(\mathbb{R}^3 \setminus \Gamma)\}.$$

The polyhedral boundary Γ can be decomposed as a union $\Gamma = \cup_{j=1}^{N_\Gamma} \overline{F}_j$ of open planar polygonal faces F_j, $1 \leq j \leq N_\Gamma$.

Let $\mathbf{n}$ denote the unit outward normal vector to Ω, and set $\mathbf{n}_j := \mathbf{n}|_{F_j}$. Further, let e_{ij} denote the open edge between the faces F_i and F_j, i.e. $\overline{e}_{ij} = \overline{F}_i \cap \overline{F}_j$, and τ_{ij} an arbitrary unit vector parallel to the edge e_{ij}. Then $\{\tau_i, \tau_{ij}\}$, with $\tau_i := \tau_{ij} \times \mathbf{n}_i$, is a basis of the plane containing F_i.

Next, we define the appropriate function spaces for the tangential traces involved in solving Maxwell's equations. Theory for these spaces was developed in [5], [6], and we follow the notation therein. First, define the spaces

$$\mathbf{L}^2_{\mathbf{t}}(\Gamma) := \{\mathbf{v} \in L^2(\Gamma)^3 : \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma\},$$

$$\mathbf{H}^{1/2}_-(\Gamma) := \{\mathbf{v} \in \mathbf{L}^2_{\mathbf{t}}(\Gamma) : \mathbf{v}_j \in \mathbf{H}^{1/2}(F_j), 1 \leq j \leq N_\Gamma\},$$

and define the *tangential components trace* mapping $\pi_\tau : \mathcal{D}(\overline{\Omega})^3 \to \mathbf{H}^{1/2}_-(\Gamma)$ and the *tangential trace* mapping $\gamma_t : \mathcal{D}(\overline{\Omega})^3 \to \mathbf{H}^{1/2}_-(\Gamma)$ by $\pi_\tau(\mathbf{v}) := \mathbf{n} \times (\mathbf{v} \times \mathbf{n})|_\Gamma$ and $\gamma_t(\mathbf{v}) := \mathbf{v} \times \mathbf{n}|_\Gamma$. The tangential components trace $\pi_\tau(\mathbf{v})$ of a vector field $\mathbf{v}$ in $\mathbf{H}^1(\Omega)$ is clearly in $\mathbf{H}^{1/2}(F_j)$ for each face F_j, and it was shown in [5] that $\pi_\tau(\mathbf{v})$ is weakly tangentially continuous across each edge e_{ij}. To be precise, the functional $\mathcal{N}^{\|}_{ij}$ defined by

$$\mathcal{N}^{\|}_{ij}(\boldsymbol{\phi}) := \int_{F_i} \int_{F_j} \frac{|\boldsymbol{\phi}_i \cdot \tau_{ij}(\mathbf{x}) - \boldsymbol{\phi}_j \cdot \tau_{ij}(\mathbf{y})|^2}{\|\mathbf{x} - \mathbf{y}\|^3} \, dS_{\mathbf{x}} \, dS_{\mathbf{y}}$$

is finite when applied to $\pi_\tau(\mathbf{v})$. By [5, Proposition 2.6],

$$\mathbf{H}^{1/2}_{\|}(\Gamma) := \{\mathbf{v} \in \mathbf{H}^{1/2}_-(\Gamma) : \mathcal{N}^{\|}_{ij}(\mathbf{v}) < \infty \text{ for all edges } e_{ij}\}$$

is a Hilbert space with the norm

$$\|\mathbf{v}\|^2_{\|,1/2,\Gamma} := \sum_{j=1}^{N_\Gamma} \|\mathbf{v}\|^2_{1/2,\Gamma_j} + \sum_{e_{ij}} \mathcal{N}^{\|}_{ij}(\mathbf{v}).$$

Moreover, $\pi_\tau : \mathbf{H}^1(\Omega) \to \mathbf{H}^{1/2}_{\|}(\Gamma)$ is linear, continuous, and surjective [5, Proposition 2.7]. We denote by $\mathbf{H}^{-1/2}_{\|}(\Gamma)$ the dual space of $\mathbf{H}^{1/2}_{\|}(\Gamma)$, with $\mathbf{L}^2_{\mathbf{t}}(\Gamma)$ as the pivot space, and by $< \cdot, \cdot >_{\|,1/2,\Gamma}$ the duality pairing between $\mathbf{H}^{-1/2}_{\|}(\Gamma)$ and $\mathbf{H}^{1/2}_{\|}(\Gamma)$. The tangential traces of vector fields in $\mathbf{H}(\mathbf{curl}, \Omega)$ are contained in the space

$$\mathbf{H}^{-1/2}_{\|}(\mathrm{div}_\Gamma, \Gamma) := \{\mathbf{v} \in \mathbf{H}^{-1/2}_{\|}(\Gamma) : \mathrm{div}_\Gamma(\mathbf{v}) \in H^{-1/2}(\Gamma)\},$$

where $\mathrm{div}_\Gamma : \mathbf{H}^{-1/2}_{\|}(\Gamma) \to H^{-3/2}(\Gamma)$ is the surface divergence operator defined in [5]. By [5, Theorem 3.9], $\gamma_t : \mathbf{H}(\mathbf{curl}, \Omega) \to \mathbf{H}^{-1/2}_{\|}(\mathrm{div}_\Gamma, \Gamma)$ is linear and continuous. In solving the Maxwell equations, $\mathbf{H}^{-1/2}_{\|}(\mathrm{div}_\Gamma, \Gamma)$ will be the appropriate function space for the tangential traces.

As this paper is concerned with complex-valued Helmholtz and Maxwell systems, the associated bilinear forms are not elliptic. Therefore, we must consider a generalized notion of coercivity suitable for our purposes.

Definition 1: A bilinear form $a : X \times X \to \mathbb{C}$ on a Hilbert space X is said to be *coercive* if it satisfies a generalized Gårding inequality of the form

$$\mathrm{Re} \, (a(u, \Theta u) - c(u, u)) \geq C\|u\|^2_X \qquad \text{for all } u \text{ in } X,$$

where $C > 0$, $c : X \times X \to \mathbb{C}$ is a compact bilinear form, and $\Theta : X \to X$ is an isomorphism.

We remark that such a general definition of coercivity is not necessary for our treatment of the Helmholtz equation. In this case, the isomorphism Θ may be taken to be the identity operator. However, in the Maxwell case, we shall use a bilinear form satisfying this coercivity definition with an isomorphism specified in the proof of Lemma 7. This definition is commonly used in the literature on boundary integral formulations for the Maxwell equations, e.g. [7], [10], [12].

A continuous bilinear form $a : X \times X \to \mathbb{C}$ induces a bounded linear operator $A : X \to X'$ (where X' denotes the dual space), defined by

$$< Av, w >:= a(v, w) \qquad \text{for all } v, w \text{ in } X.$$

The following abstract result (cf. [7, Proposition 3] and [12, Theorem 4.1]) is invoked to prove the unique solvability of all variational problems considered in this paper.

Lemma 1: If a bilinear form $a : X \times X \to \mathbb{C}$ is coercive and the associated operator $A : X \to X'$ is injective, then the inverse $A^{-1} : X' \to X$ exists and is continuous.

III. THE HELMHOLTZ EQUATION

Consider the Helmholtz equation for acoustic scattering, which involves both an interior and exterior domain. Accordingly, let $\Omega^- := \Omega \subset \mathbb{R}^3$ be a bounded Lipschitz polyhedral domain, and denote by $\Omega^+ := \mathbb{R}^3 \setminus \overline{\Omega^-}$ its open complement. Often, we simply write Ω instead of Ω^-. The unit normal vector $\mathbf{n}$ on $\Gamma := \partial \Omega^-$ is defined as pointing from Ω^- into Ω^+. For any trace operator γ, γ^+ and γ^- denote the traces from Ω^+ and Ω^-, respectively. For example, $\gamma_1^\pm \phi = (\nabla \phi|_{\Omega^\pm}) \cdot \mathbf{n}$ for ϕ in $\mathcal{D}(\mathbb{R}^3)$, with the same normal $\mathbf{n}$ in both cases. The jump in a trace operator γ is denoted by $[\gamma\phi]_\Gamma = \gamma^+\phi - \gamma^-\phi$. With this notation, we state the Helmholtz equation with transmission and radiation conditions as

$$\begin{cases} \Delta u + \kappa^2 u = 0 & \text{in } \Omega^- \cup \Omega^+, \\ \gamma_0^+ u^s - \gamma_0^- u = -\gamma_0^+ u^i & \text{on } \Gamma := \partial\Omega^-, \\ \frac{\partial u}{\partial r}(\mathbf{x}) - i\kappa u(\mathbf{x}) = o(r^{-1}) \end{cases} \quad (1)$$

uniformly for $r := |\mathbf{x}| \to \infty$. Here, u^i and u^s denote the incident and scattered waves, respectively. In the exterior domain Ω^+, $u = u^i + u^s$.

For simplicity, in this paper we present a method for solving the interior Dirichlet problem

$$\begin{cases} \Delta u + \kappa^2 u = 0 & \text{in } \Omega^-, \\ \gamma_0^- u = g & \text{on } \Gamma, \end{cases} \quad (2)$$

for some function g in $H^{1/2}(\Gamma)$. The method can easily be extended to solve (1), as we shall indicate in a remark. We assume that $\kappa > 0$ is constant in Ω^+ and piecewise constant in Ω^- and that the differential operator $\Delta + \kappa^2 I : H(\Delta, \Omega^-) \to L^2(\Omega^-)$ has a trivial null space. In the case that κ is constant in Ω^-, this means that κ^2 is bounded away from the interior Dirichlet eigenvalues of $-\Delta$ in Ω^-.

A. Boundary Integral Operators

In this section, we define the necessary boundary integral operators on a general Lipschitz polyhedral domain Ω with boundary Γ and a constant wavenumber $\kappa > 0$. The theory for this section is quite technical and has already been established in the literature, so we only give a survey here of the results necessary for our purposes. For further details, see e.g. [11], [18].

The key to the boundary integral approach is the representation formula (cf. [25, Theorem 6.10])

$$u(\mathbf{x}) = -\int_\Gamma E_\kappa(\mathbf{x} - \mathbf{y}) [\gamma_1 u]_\Gamma \, dS_\mathbf{y}$$
$$+ \int_\Gamma \frac{\partial E_\kappa(\mathbf{x} - \mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} [\gamma_0 u]_\Gamma \, dS_\mathbf{y}, \quad (3)$$

for $\mathbf{x}$ in $\Omega^- \cup \Omega^+$, where

$$E_\kappa(\mathbf{x}) = \frac{e^{i\kappa|\mathbf{x}|}}{4\pi|\mathbf{x}|} \qquad \text{for } \mathbf{x} \neq \mathbf{0},$$

is the fundamental solution of the Helmholtz equation with the constant wavenumber $\kappa > 0$. The representation formula (3) yields a solution to (1) in terms of the Dirichlet and Neumann traces, provided that the wavenumber κ is constant. Defining the scalar single-layer and double-layer potentials

$$\psi_\kappa^{SL} : H^{-\frac{1}{2}}(\Gamma) \to H_{loc}^1(\mathbb{R}^3) \cap H_{loc}(\Delta, \mathbb{R}^3 \setminus \Gamma),$$
$$\psi_\kappa^{DL} : H^{\frac{1}{2}}(\Gamma) \to H_{loc}(\Delta, \mathbb{R}^3 \setminus \Gamma),$$

(see [25, Theorem 6.12]) by

$$\psi_\kappa^{SL}\phi(\mathbf{x}) := \int_\Gamma E_\kappa(\mathbf{x} - \mathbf{y})\phi(\mathbf{y}) \, dS_\mathbf{y}, \qquad \mathbf{x} \text{ in } \Omega^- \cup \Omega^+,$$
$$\psi_\kappa^{DL}\phi(\mathbf{x}) := \int_\Gamma \frac{\partial E_\kappa(\mathbf{x} - \mathbf{y})}{\partial \mathbf{n}(\mathbf{y})}\phi(\mathbf{y}) \, dS_\mathbf{y}, \qquad \mathbf{x} \text{ in } \Omega^- \cup \Omega^+,$$

we may rewrite the representation formula (3) as

$$u = -\psi_\kappa^{SL}([\gamma_1 u]_\Gamma) + \psi_\kappa^{DL}([\gamma_0 u]_\Gamma) \qquad \text{in } \Omega^- \cup \Omega^+. \quad (4)$$

In this paper, we solve the interior Dirichlet problem (2) and therefore employ the representation formula

$$u = \psi_\kappa^{SL}(\gamma_1^- u) - \psi_\kappa^{DL}(\gamma_0^- u) \qquad \text{in } \Omega^-, \quad (5)$$

given in terms of the interior traces only. As in [11], [18], [25], the trace operators $\gamma_0^\pm$ and $\gamma_1^\pm$ may be applied to (5), yielding the the exterior (with the superscript $+$) and interior (with the superscript $-$) Calderon projectors

$$P_H^\pm = \begin{bmatrix} \frac{1}{2}I \pm K & \mp V \\ \mp D & \frac{1}{2}I \mp K' \end{bmatrix} \quad (6)$$

mapping $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ onto itself, where

$V : H^s(\Gamma) \to H^{s+1}(\Gamma), \; -1 \leq s \leq 0, \quad V := \{\gamma_0 \psi_\kappa^{SL}\}_\Gamma,$
$K : H^s(\Gamma) \to H^s(\Gamma), \; 0 \leq s \leq 1, \quad K := \{\gamma_0 \psi_\kappa^{DL}\}_\Gamma,$
$K' : H^s(\Gamma) \to H^s(\Gamma), \; -1 \leq s \leq 0, \quad K' := \{\gamma_1 \psi_\kappa^{SL}\}_\Gamma,$
$D : H^s(\Gamma) \to H^{s-1}(\Gamma), \; 0 \leq s \leq 1, \quad D := -\{\gamma_1 \psi_\kappa^{DL}\}_\Gamma,$

are continuous boundary integral operators (cf. [11], [25]). Here, $\{\}_\Gamma$ denotes the average across Γ, e.g. $\{\gamma_0 \phi\}_\Gamma = \frac{1}{2}(\gamma_0^+\phi + \gamma_0\phi^-)$.

If u is a solution to (1), then the pair of traces $(\gamma_0^\pm u, \gamma_1^\pm u)$ is referred to as *exterior/interior Cauchy data*. The space of all interior Cauchy data for the Helmholtz equation coincides with the range of P_H^-. As a consequence of this fact, the interior Neumann trace of a Helmholtz solution can be expressed in terms of the interior Dirichlet trace via the operator

$$S^- := D + (\frac{1}{2}I + K')V^{-1}(\frac{1}{2}I + K) \qquad (7)$$

from $H^{\frac{1}{2}}(\Gamma)$ to $H^{-\frac{1}{2}}(\Gamma)$, referred to as the interior Dirichlet-to-Neumann map. The existence of the inverse V^{-1} is given next in Proposition 1.

Lemma 2: The boundary integral equation

$$V\eta = (\frac{1}{2}I + K)\xi, \text{ with } (\xi, \eta) \text{ in } H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma), \quad (8)$$

holds if and only if (ξ, η) are interior Cauchy data for a Helmholtz solution.

Proof: If (ξ, η) are interior Cauchy data, then (8) is the first row of $P_H^-(\xi, \eta) = (\xi, \eta)$. Conversely, if (8) holds, then the first component of $P_H^+(\xi, \eta)$ is zero. Since $P_H^+(\xi, \eta)$ is in the range of P_H^+, it is exterior Cauchy data with vanishing Dirichlet trace. It follows from the uniqueness of solutions to the exterior Dirichlet problem [25, Theorem 9.10] that $P_H^+(\xi, \eta) = 0$. Thus (ξ, η) are interior Cauchy data. ∎

Proposition 1: The operator $V : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$ has a bounded inverse $V^{-1} : H^{\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)$.

Proof: By [13, Theorem 2], V is coercive and hence Fredholm of index zero. To verify the injectivity of V, suppose that $V\eta = 0$. By Lemma 2, $(0, \eta)$ are interior Cauchy data. Under the assumption that κ^2 is not an interior Dirichlet eigenvalue of $-\Delta$, we have that $\eta = \gamma_1^- 0 = 0$, which proves the injectivity of V. Therefore, V has a bounded inverse. ∎

For $\mathbf{x}$ in Γ and sufficiently smooth ϕ, we have the following expressions for the action of the aforementioned boundary integral operators:

$$V\phi(\mathbf{x}) = \int_\Gamma E_\kappa(\mathbf{x} - \mathbf{y})\phi(\mathbf{y}) \, dS_\mathbf{y},$$

$$K\phi(\mathbf{x}) = \int_\Gamma \frac{\partial E_\kappa(\mathbf{x} - \mathbf{y})}{\partial n(\mathbf{y})}\phi(\mathbf{y}) \, dS_\mathbf{y},$$

$$K'\phi(\mathbf{x}) = \int_\Gamma \frac{\partial E_\kappa(\mathbf{x} - \mathbf{y})}{\partial n(\mathbf{x})}\phi(\mathbf{y}) \, dS_\mathbf{y},$$

$$D\phi(\mathbf{x}) = -\frac{\partial}{\partial n(\mathbf{x})} \int_\Gamma \frac{\partial E_\kappa(\mathbf{x} - \mathbf{y})}{\partial n(\mathbf{y})}\phi(\mathbf{y}) \, dS_\mathbf{y}.$$

B. Element-wise Domain Decomposition

Given a mesh of the bounded domain Ω^-, we propose a domain decomposition method based on [21] to solve for the Dirichlet trace of the solution to (2) on the skeleton of the mesh. Locally on the surface of each volume element, one can then obtain the Neumann trace from the Dirichlet trace by applying the local Dirichlet-to-Neumann map. The solution inside the volume element is then given by the representation formula (3).

Consider a general conforming polyhedral mesh of the bounded domain Ω^- as a domain decomposition:

$$\overline{\Omega^-} = \cup_{i=1}^p \overline{\Omega}_i, \qquad \Omega_i \cap \Omega_j = \emptyset \qquad \text{for } i \neq j,$$

where each Ω_i is a simply-connected Lipschitz polyhedron. Conformity means that for $i \neq j$, $\overline{\Omega}_i \cap \overline{\Omega}_j$ is empty, a vertex, an edge, or a polygonal face. The case of interest is $\text{diam}(\Omega_i) \approx h$, but for theoretical analysis we only consider the simple case that the subdomains Ω_i are of fixed size, with a family of meshes defined on their surfaces. In order to use boundary integral equations, we require that the mesh resolve any discontinuities in the wavenumber κ; i.e. $\kappa_i := \kappa|_{\Omega_i}$ is constant on Ω_i, for each i. Denote the local boundaries by $\Gamma_i = \partial\Omega_i$ and the mesh skeleton by $\Gamma_S = \cup_{i=1}^p \Gamma_i$. Then $H^{\frac{1}{2}}(\Gamma_S)$ denotes the space of traces of functions in $H^1(\Omega)$ on Γ_S, and

$$\|\phi\|_{H^{1/2}(\Gamma_S)} := \left(\sum_{j=1}^p \|\phi|_{\Gamma_j}\|_{H^{1/2}(\Gamma_j)}^2 \right)^{1/2}$$

defines a norm. We shall apply the results stated in the previous section for the domains Ω_i. Throughout the remainder of the paper, C represents a generic constant independent of h but possibly depending on the wavenumber κ.

Let $\hat{g}$ be an arbitrary extension of g in $H^1(\Omega)$, and set $\tilde{g} := \gamma_0^S \hat{g}$, where γ_0^S denotes the trace on Γ_S. Since the Dirichlet trace on Γ is given as data g, we need only solve for the Dirichlet trace on $\Gamma_S \setminus \Gamma$. Accordingly, we define the test function space

$$W := \{v \text{ in } H^{1/2}(\Gamma_S) : v = 0 \text{ on } \Gamma\}$$

and seek $\tilde{u}$ in W such that $\tilde{u} + \tilde{g}$ is the trace of u on Γ_S. Then we have the following variational problem for $\tilde{u}$:

$$\begin{cases} \text{Find } \tilde{u} \text{ in } W \text{ such that } u = \tilde{u} + \tilde{g} \text{ on } \Gamma_S \text{ and} \\ a^H(\tilde{u}, v) = -a^H(\tilde{g}, v) \qquad \text{for all } v \text{ in } W, \end{cases} \quad (9)$$

where the bilinear form $a^H : W \times W \to \mathbb{C}$ is defined by

$$a^H(v, w) = \sum_{i=1}^p < S_i^- v_i, w_i >_{\Gamma_i} . \qquad (10)$$

Here and throughout the remainder of the paper, ϕ_i denotes the restriction of a scalar or vector-valued function ϕ to Γ_i. Each operator S_i^- is defined by (7), with the boundary integral operators V_i, K_i, and D_i defined on the subdomain Ω_i.

Remark 1: The existence of V_i^{-1} is given by Proposition 1, provided that κ_i is not an interior Dirichlet eigenvalue of $-\Delta$ on Ω_i. In the case that $\text{diam}(\Omega_i) \approx h$ for sufficiently small h, a simple scaling argument shows that this condition holds. Indeed, the eigenvalues of $-\Delta$ on Ω_i equal h^{-2} times the eigenvalues $\lambda_j(\hat{\Omega}_i)$ of $-\Delta$ on $H_0^1(\hat{\Omega}_i)$, where $\hat{\Omega}_i := \{h^{-1}(\mathbf{x} - \mathbf{x}_{i,c}) : \mathbf{x} \in \Omega_i\}$ and $\mathbf{x}_{i,c}$ is the center of mass of Ω_i. For $\kappa_i^2 < h^{-2}\lambda_{\min}(\hat{\Omega}_i)$, $-\Delta - \kappa_i^2 I$ is positive definite on $H_0^1(\Omega_i)$, and V_i has a bounded inverse. In practice, the condition $h < C\kappa_i^{-1}$ is not restrictive, as it is necessary for accurate approximation of the solution.

Remark 2: The scattering problem (1) in $\mathbb{R}^3$ can be solved by including the term $< S^+ v_\Gamma, w_\Gamma >_\Gamma$ in the sum (10), where S^+ is the exterior Dirichlet-to-Neumann map defined on $H^{1/2}(\Gamma)$. Analogous to S^-, an expression for S^+ can be derived from the fact that the range of P_H^+ is the space of all exterior Cauchy data. The space W should then be $H^{1/2}(\Gamma_S)$.

Lemma 3: For each $1 \leq i \leq p$, the local operator S_i^- is coercive. More precisely, there exists a compact bilinear form $c_i : H^{\frac{1}{2}}(\Gamma_i) \times H^{\frac{1}{2}}(\Gamma_i) \to \mathbb{C}$ such that

$$\text{Re}\left(< S_i^- u, u >_{\Gamma_i} - c_i(u, u)\right) \geq C\|u\|^2_{H^{\frac{1}{2}}(\Gamma_i)} \qquad (11)$$

for all u in $H^{\frac{1}{2}}(\Gamma_i)$.

Proof: By [18, Lemma 3.2] (see also [11], [25]), the boundary integral operators $D_i : H^{\frac{1}{2}}(\Gamma_i) \to H^{-\frac{1}{2}}(\Gamma_i)$ and $V_i : H^{-\frac{1}{2}}(\Gamma_i) \to H^{\frac{1}{2}}(\Gamma_i)$ are coercive; specifically, there exist compact operators $T_{D_i} : H^{\frac{1}{2}}(\Gamma_i) \to H^{-\frac{1}{2}}(\Gamma_i)$ and $T_{V_i} : H^{-\frac{1}{2}}(\Gamma_i) \to H^{\frac{1}{2}}(\Gamma_i)$ such that

$$\text{Re} \ < \xi, (V_i + T_{V_i})\xi >_{\Gamma_i} \geq C\|\xi\|^2_{H^{-\frac{1}{2}}(\Gamma_i)},$$

$$\text{Re} \ < (D_i + T_{D_i})\phi, \phi >_{\Gamma_i} \geq C\|\phi\|^2_{H^{\frac{1}{2}}(\Gamma_i)},$$

for all ξ in $H^{-\frac{1}{2}}(\Gamma_i)$ and ϕ in $H^{\frac{1}{2}}(\Gamma_i)$. Moreover, [18, Lemma 3.3] states that K_i' equals the $L^2(\Gamma_i)$-adjoint K_i^* of K_i plus a compact perturbation, i.e. there exists a compact operator $T_{K_i} : H^{-1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i)$ such that

$$< (K_i' + T_{K_i})\xi, \phi >_{\Gamma_i} = < K_i^* \xi, \phi >_{\Gamma_i} \ .$$

Denoting $\hat{T}_{K_i} := T_{K_i} V_i^{-1}(\frac{1}{2}I + K_i)$, we have

$$\text{Re} < (S_i^- + T_{D_i} + \hat{T}_{K_i}\phi, \phi >_{\Gamma_i} = \text{Re} < (D_i + T_{D_i})\phi, \phi >_{\Gamma_i}$$

$$+ \text{Re} < (\frac{1}{2}I + K_i' + T_{K_i})V_i^{-1}(\frac{1}{2}I + K_i)\phi, \phi >_{\Gamma_i}$$

$$\geq C\|\phi\|^2_{H^{\frac{1}{2}}(\Gamma_i)} + \text{Re} < (\frac{1}{2}I + K_i^*)V_i^{-1}(\frac{1}{2}I + K_i)\phi, \phi >_{\Gamma_i}$$

$$= C\|\phi\|^2_{H^{\frac{1}{2}}(\Gamma_i)}$$

$$+ \text{Re} < V_i^{-1}(\frac{1}{2}I + K_i)\phi, V_i V_i^{-1}(\frac{1}{2}I + K_i)\phi >_{\Gamma_i}$$

and hence

$$\text{Re}\left(< (S_i^- + T_{D_i} + \hat{T}_{K_i})\phi, \phi >_{\Gamma_i} \right.$$

$$+ < V_i^{-1}(\frac{1}{2}I + K_i)\phi, T_{V_i}V_i^{-1}(\frac{1}{2}I + K_i)\phi >_{\Gamma_i} \left. \right)$$

$$\geq C\|\phi\|^2_{H^{\frac{1}{2}}(\Gamma_i)},$$

which verifies (11). Thus we have proved the coercivity of S_i^-. ∎

Theorem 1: The variational problem (9) has a unique solution in W.

Proof: We need only verify the hypotheses of Lemma 1. The coercivity of $a^H : W \times W \to \mathbb{C}$ follows immediately from the local coercivity result stated in Lemma 3. To verify the injectivity of the operator A, suppose that $Av = 0$ for some v in W. For each $1 \leq i \leq p$, $v_i := v|_{\Gamma_i}$ is in $H^{1/2}(\Gamma_i)$, and $(v_i, S_i^- v_i)$ are Cauchy data for some local function ϕ_i in $H^1(\Omega_i)$ satisfying

$$\begin{cases} \Delta\phi_i + \kappa^2\phi_i & = 0 & \text{in } \Omega_i, \\ \gamma_0^- \phi_i & = v_i & \text{on } \Gamma_i, \\ \gamma_1^- \phi_i & = S_i^- v_i & \text{on } \Gamma_i. \end{cases}$$

For all w in W, we have

$$\sum_{i=1}^p < \gamma_{1,i}^-\phi_i, w_i >_{\Gamma_i} = \sum_{i=1}^p < S_i^- v_i, w_i >_{\Gamma_i} = a^H(v, w)$$

$$= < Av, w > = 0.$$

Hence the global function ϕ, defined by $\phi = \phi_i$ on Ω_i, is in $H^1(\Omega^-)$ and satisfies the interior Dirichlet problem

$$\begin{cases} \Delta\phi + \kappa^2\phi & = 0 & \text{in } \Omega^-, \\ \gamma_0^- \phi & = 0 & \text{on } \Gamma. \end{cases}$$

Under the assumption that $\Delta + \kappa^2 I$ has a trivial null space, we have $\phi = 0$ and hence $v = 0$. This shows that A is injective and completes the proof of the theorem. ∎

As mentioned in the introduction, the method (9) can be considered as either a finite element method or a domain decomposition method, depending on the choice of the subdomains Ω_j. If one takes the subdomains to be volume elements of size h, then a finite element method is obtained. In this case, the wavenumber may be equal on different subdomains. Alternatively, one may take the subdomains to be maximal with respect to the constant values of the wavenumber. The resulting pure boundary element domain decomposition method would require fewer degrees of freedom, as the skeleton mesh would be significantly smaller. The disadvantage in this would be the large dense matrices on each subdomain, necessitating sophisticated techniques for efficient data-sparse approximation of dense boundary element matrices (see, e.g. [2], [16], [27]). Additionally, the inversion of the single layer potential operator V in the application of the Dirichlet-to-Neumann maps would become expensive. One may indeed find the necessary tools to make this approach feasible in practice, but in our numerical computations we take the "finite element" approach with element-sized subdomains.

C. Discretization

We now discretize the domain decomposition method introduced in the previous section via boundary elements on the skeleton Γ_S. To discretize the solution space W, we further assume a conforming, quasi-uniform triangulation of Γ_S. Thus each element boundary Γ_i is triangulated as $\overline{\Gamma}_i = \cup_{j=1}^{NT_i} \overline{\Delta}_j^i$, where each Δ_j^i is an open planar triangle satisfying diam $\Delta_j^i = O(h)$ and $\Delta_m^i \cap \Delta_n^i = \emptyset$ for $m \neq n$, and NT_i denotes the number of triangles in Γ_i. Note that, in general, the faces of an element Ω_i may be arbitrary polygons.

With this decomposition of the skeleton Γ_S into triangles, we now define boundary element spaces based on polynomial spaces of the lowest possible order, namely

$$S_h^1(\Gamma_S) := \{v \in H^{1/2}(\Gamma_S) : v|_{\Delta_j^i} \in P_1\},$$

$$S_h^0(\Gamma_S) := \{v \in H^{-1/2}(\Gamma_S) : v|_{\Delta_i^j} \in P_0\},$$

for $1 \leq i \leq p$ and $1 \leq j \leq NT_i$. Here, P_k denotes the space of polynomials of degree k. Then we approximate W by the continuous piecewise linear space $W_h := S_h^1(\Gamma_S) \cap W$ and $H^{-1/2}(\Gamma_S)$ by the piecewise constant space $S_h^0(\Gamma_S)$. We assume the standard boundary element approximation estimates (cf. [27, Theorems 10.4, 10.9])

$$\inf_{v_h \in S_h^1(\Gamma_i)} \|v - v_h\|_{H^{\frac{1}{2}}(\Gamma_i)} \leq Ch^{3/2}|v|_{H^2(\Gamma_i)}, \qquad (12)$$

$$\inf_{w_h \in S_h^0(\Gamma_i)} \|w - w_h\|_{H^{-\frac{1}{2}}(\Gamma_i)} \leq Ch^{3/2}|w|_{H^1(\Gamma_i)}, \qquad (13)$$

for all v in $H^2(\Gamma_i)$ and w in $H^1(\Gamma_i)$. Note that $H^2(\Gamma_i)$ is defined as the trace of $H^{5/2}(\Omega_i)$, with the norm

$$\|v\|_{H^2(\Gamma_i)} := \inf_{V \in H^{5/2}(\Omega_i), v=V|_{\Gamma_i}} \|V\|_{H^{5/2}(\Omega_i)}.$$

The following (cf. [12, Theorem 4.1] and [27, Section 8.6]) is a generalization of Cea's lemma for coercive operators and will be used in proving the unique solvability of the linear systems obtained from our proposed method of discretization.

Lemma 4: Let $A : X \to X'$ be a bounded, linear, coercive operator satisfying the uniform inf-sup condition

$$C_0\|v_h\|_X \le \sup_{w_h \in X_h \setminus \{0\}} \frac{|<Av_h, w_h>|}{\|w_h\|_X} \text{ for all } v_h \text{ in } X_h, \tag{14}$$

for some positive constant C_0 independent of h. Then the variational problem

$$<Av_h, w_h> = <f, w_h>, \qquad \text{for all } w_h \text{ in } X_h, \tag{15}$$

has a unique solution v_h in X_h, which satisfies the stability estimate

$$\|v_h\|_X \le C_0^{-1}\|f\|_{X'}$$

and the quasi-optimal error estimate

$$\|v - v_h\|_X \le \left(1 + \frac{\|A\|}{C_0}\right) \inf_{w_h \in X_h} \|v - w_h\|_X.$$

The next lemma automatically yields the uniform inf-sup condition (14) for coercive, injective operators.

Lemma 5: Let $A : X \to X'$ be a bounded, linear, coercive, injective operator, and assume that the family of conforming spaces $X_h \subset X$, with h tending to zero, is dense in X. Then there exist positive constants C_0 and h_0 such that for all $h \le h_0$, the inf-sup condition (14) holds uniformly.

The variational method presented so far in trace spaces on the mesh skeleton does not formally fit the framework of Lemmas 4 and 5. Even the continuous space $H^{\frac{1}{2}}(\Gamma_S)$ depends on the mesh, and we do not have a nested family of discrete subspaces. Moreover, the definition of the bilinear form appears to be mesh-dependent. However, we can apply this theory to our method by considering an equivalent finite-element-type of formulation in a subspace of $H^1(\Omega)$. Indeed, $H^{\frac{1}{2}}(\Gamma_S)$ and $H^1(\Omega)$ are isometric for all $h > 0$, due to the boundedness of the trace operator $\gamma_0^S : H^1(\Omega) \to H^{\frac{1}{2}}(\Gamma_S)$ and an extension operator $\mathcal{E}_S : H^{\frac{1}{2}}(\Gamma_S) \to H^1(\Omega)$. Consequently, the bilinear form $a^H : W \times W \to \mathbb{C}$ equivalently acts on $H_0^1(\Omega) \times H_0^1(\Omega)$.

If ξ and η are in $H^1(\Omega_i)$ and ξ satisfies $\Delta\xi + \kappa_i^2\xi = 0$ in Ω_i, then $S_i^-\gamma_{0,i}\xi = \gamma_{1,i}\xi$, and Green's formula yields

$$\begin{aligned} <S_i^-\gamma_{0,i}\xi, \gamma_{0,i}\eta>_{\Gamma_i} &= <\gamma_{1,i}\xi, \gamma_{0,i}\eta>_{\Gamma_i} \\ &= (\nabla\xi, \nabla\eta)_{\Omega_i} + (\Delta\xi, \eta)_{\Omega_i} \\ &= (\nabla\xi, \nabla\eta)_{\Omega_i} - \kappa_i^2(\xi, \eta)_{\Omega_i}. \end{aligned} \tag{16}$$

Therefore, if ξ and η are in $H^1(\Omega)$ and ξ satisfies $\Delta\xi + \kappa_i^2\xi = 0$ in Ω, then

$$\begin{aligned} a^H(\gamma_0^S\xi, \gamma_0^S\eta) &= \sum_{i=1}^{p}(\nabla\xi, \nabla\eta)_{\Omega_i} - \kappa_i^2(\xi, \eta)_{\Omega_i} \\ &= (\nabla\xi, \nabla\eta)_{\Omega} - (\kappa^2\xi, \eta)_{\Omega}. \end{aligned}$$

Defining the subspace

$$S(\Omega) := \{\xi \text{ in } H(\Delta, \Omega) : \Delta\xi + \kappa^2\xi = 0\}$$

of $H(\Delta, \Omega) := \{\phi \in H^1(\Omega) : \Delta\phi \in L^2(\Omega)\}$ and the bilinear form $a_\Omega^H : S(\Omega) \times S(\Omega) \to \mathbb{C}$ by $a_\Omega^H(\xi, \eta) := (\nabla\xi, \nabla\eta)_\Omega - (\kappa^2\xi, \eta)_\Omega$, we have that $a^H(\gamma_0^S\cdot, \gamma_0^S\cdot)$ and $a_\Omega^H(\cdot, \cdot)$ coincide on $S(\Omega)$ for all $h > 0$. Thus the bilinear form a^H is actually independent of h and the mesh. Therefore, we shall apply Lemmas 4 and 5 with $X = S(\Omega)$.

Formulating the variational method in terms of the bilinear form a_Ω^H and the space $S(\Omega)$ is not only convenient for analysis but also illuminates the finite element nature of the method. Indeed, (9) is equivalent to a finite element method with the commonly used bilinear form a_Ω^H and the space $S(\Omega)$ of PDE-harmonic functions. Moreover, the discrete space used for approximation is a subspace of $S(\Omega)$ and generally consists of PDE-harmonic functions which are not of any simple form such as polynomials. Thus we have a finite element method which can be practically implemented only via boundary integral techniques.

The boundary element space $S_h^1(\Gamma_S)$ induces the finite dimensional subspace

$$S_h := \{u_h \text{ in } S(\Omega) : \gamma_0^S u_h \text{ is in } S_h^1(\Gamma_S) \text{ for } 1 \le i \le p\}.$$

To obtain an approximation estimate for S_h as a subspace of $S(\Omega)$, we begin with the representation formula (5), which holds on each subdomain Ω_i for all functions u in $S(\Omega)$. Approximating the traces $\gamma_0^- u$ and $\gamma_1^- u$ by functions ξ_h in $S_h^1(\Gamma_S)$ and η_h in $S_h^0(\Gamma_S)$, respectively, we define the approximation $u_h := \psi_\kappa^{SL}(\eta_h) - \psi_\kappa^{DL}(\xi_h)$ in Ω_i. The approximation properties (12) and (13) yield

$$\begin{aligned} \|u - &u_h\|_{H^1(\Omega_i)} \\ &\le \|\psi_\kappa^{SL}(\gamma_1^- u - \eta_h)\|_{H^1(\Omega_i)} + \|\psi_\kappa^{DL}(\gamma_0^- u - \xi_h)\|_{H^1(\Omega_i)} \\ &\le \|\psi_\kappa^{SL}\|\|\gamma_1^- u - \eta_h\|_{H^{-\frac{1}{2}}(\Gamma_i)} + \|\psi_\kappa^{DL}\|\|\gamma_0^- u - \xi_h\|_{H^{\frac{1}{2}}(\Gamma_i)} \\ &\le Ch^{3/2}\left(\|\psi_\kappa^{SL}\|\|\gamma_1^- u\|_{H^1(\Gamma_i)} + \|\psi_\kappa^{DL}\|\|\gamma_0^- u\|_{H^2(\Gamma_i)}\right), \end{aligned} \tag{17}$$

where $\|\psi_\kappa^{SL}\|$ and $\|\psi_\kappa^{DL}\|$ are the operator norms of $\psi_\kappa^{SL} : H^{-\frac{1}{2}}(\Gamma_i) \to H^1(\Omega_i)$ and $\psi_\kappa^{DL} : H^{\frac{1}{2}}(\Gamma_i) \to H^1(\Omega_i)$. These operator norms may depend on the size of the subdomain Ω_i, so we can prove an asymptotic error estimate only in the case that $\text{diam }\Omega_i = O(1)$ is independent of h (the minimum diameter of the triangular elements on the surfaces Γ_i). The approximation estimate in Lemma 6 now follows by summing the local estimates (17).

Lemma 6 (Approximation in $S(\Omega)$): For sufficiently small $h > 0$ and subdomains Ω_i of fixed size, there exists a constant $C > 0$ such that

$$\inf_{u_h \in S_h} \|u - u_h\|_{H^1(\Omega)} \le Ch^{3/2}\|u\|_{H^{5/2}(\Omega)}$$

for all u in $S(\Omega) \cap H^{5/2}(\Omega)$.

With the approximating subspace W_h of W, we consider the following discrete variational problem:

$$\begin{cases} \text{Find } \tilde{u}_h \text{ in } W_h \text{ satisfying} \\ a^H(\tilde{u}_h, v_h) = -a^H(\tilde{g}, v_h) \qquad \text{for all } v_h \text{ in } W_h. \end{cases} \tag{18}$$

Note that (18) is equivalent to the following:

$$\begin{cases} \text{Find } \xi_h \text{ in } S_h \text{ satisfying} \\ a_\Omega^H(\xi_h, \eta_h) := a^H(\gamma_0 \xi_h, \gamma_0 \eta_h) = -a^H(\tilde{g}, \gamma_0 \eta_h) \end{cases} \quad (19)$$

for all η_h in S_h.

Theorem 2: Suppose that u is in $H^{5/2}(\Omega)$ and solves (2). For sufficiently small $h > 0$ and subdomains Ω_i of fixed size, the variational problem (18) has a unique solution $\tilde{u}_h$ in W_h, satisfying $\tilde{u}_h + \tilde{g} = \gamma_0^S \xi_h$ with ξ_h in S_h and

$$\|u - \xi_h\|_{H^1(\Omega)} \le Ch^{3/2}\|u\|_{H^{5/2}(\Omega)}.$$

Proof: We have already verified the coercivity and injectivity of the operator A in the proof of Theorem 1, so Lemma 5 implies the inf-sup condition (14) for A. Hence Lemmas 4 and 6 yield the result, when applied to (19). ∎

The composition of operators in the expression (7) for S_i^- precludes a direct Galerkin discretization. Instead, as in [21] we approximate S_i^- by $\tilde{S}_{h,i}^-$, defined by

$$\tilde{S}_{h,i}^- := D_i + (\tfrac{1}{2}I + K_i')V_{h,i}^{-1}\Pi_h^0(\tfrac{1}{2}I + K_i),$$

where Π_h^0 is the $L^2(\Gamma_i)$-orthogonal projector onto $S_h^0(\Gamma_S)$, and $V_{h,i}$ is the Galerkin discretization of V_i. The operator V_i has a bounded inverse for diam Ω_i sufficiently small (see Remark 1), but the invertibility of the discrete operator $V_{h,i}$ can be ensured theoretically only for h sufficiently small (relative to diam Ω_i). The invertibility must be checked in practice; however, in numerical experiments with diam $\Omega_i \approx h$, we have always observed that $V_{h,i}$ is nonsingular. The Galerkin discretization $S_{h,i}^- : W_{h,i} \to W'_{h,i}$ of the approximate Dirichlet-to-Neumann map $\tilde{S}_{h,i}^-$ is given by

$$S_{h,i}^- = D_{h,i} + (\tfrac{1}{2}I_{h,i}^T + K_{h,i}^T)V_{h,i}^{-1}(\tfrac{1}{2}I_{h,i} + K_{h,i}).$$

In practice, we solve the following discrete linear system:

$$\begin{cases} \text{Find } \tilde{u}_h \text{ in } W_h \text{ satisfying} \\ a_h^H(\tilde{u}_h, v_h) = -a_h^H(\tilde{g}, v_h) \qquad \text{for all } v_h \text{ in } W_h, \end{cases} \quad (20)$$

where the discrete bilinear form $a_h^H(\cdot, \cdot)$ is defined by

$$a_h^H(v_h, w_h) := \sum_{i=1}^p <S_{h,i}^- v_h|_{\Gamma_i}, w_h>_{\Gamma_i}.$$

A simple modification of the proof of Lemma 3 shows that the operators $\tilde{S}_{h,i}^-$ are coercive, so the linear system (20) is nonsingular for sufficiently small $h > 0$. However, in order to estimate the error of the solution, we would need to prove the operator error estimate

$$\|(S_i^- - \tilde{S}_{h,i}^-)\phi_i\|_{H^{-\frac{1}{2}}(\Gamma_i)} \le Ch^{3/2}\|\phi_i\|_{H^2(\Gamma_i)} \quad (21)$$

for all ϕ_i in $H^2(\Gamma_i)$. The proof of such an estimate requires spectral bounds for the boundary integral operators K_i and V_i, independent of diam Ω_i. Such bounds are not known to hold, so we can prove the desired error estimates for the approximate system (20) only in the case of a pure domain decomposition method with subdomains of fixed size as h tends to zero. Thus we can assert the error estimate of Theorem 2 only for fixed subdomains. However, numerical experiments in section

V confirm that our method behaves comparably to a standard finite element method on tetrahedral meshes with diam $\Omega_i = O(h)$.

IV. MAXWELL'S EQUATIONS

In this section we present an analogous method and theory for Maxwell's equations. The Maxwell system for electromagnetic scattering with transmission and radiation conditions is given by

$$\begin{cases} \mathbf{curl\,curl\,e} - \kappa^2 \mathbf{e} &= \mathbf{0} & \text{in } \Omega^- \cup \Omega^+, \\ \gamma_t^+ \mathbf{e}^s - \gamma_t^- \mathbf{e} &:= \gamma_t^+ \mathbf{e}^i & \text{on } \Gamma, \\ \lim_{|\mathbf{x}| \to \infty} \mathbf{curl\,e} \times \mathbf{x} - i\kappa|\mathbf{x}|\mathbf{e} &= 0 & \text{uniformly.} \end{cases} \quad (22)$$

As in the previous section for the Helmholtz equation, for convenience we present a method for the interior Dirichlet problem

$$\begin{cases} \mathbf{curl\,curl\,e} - \kappa^2 \mathbf{e} &= \mathbf{0} & \text{in } \Omega^-, \\ \gamma_t^- \mathbf{e} &:= \mathbf{g} & \text{on } \Gamma. \end{cases} \quad (23)$$

We assume that $\kappa > 0$ is constant in Ω^+ and piecewise constant in Ω^- and that the differential operator $\mathbf{curl\,curl} - \kappa^2 I : H(\Delta, \Omega^-) \to L^2(\Omega^-)$ has a trivial null space. In the case that κ is constant in Ω^-, this means that κ^2 is bounded away from the interior Dirichlet eigenvalues of $\mathbf{curl\,curl}$ in Ω^-.

A. Boundary Integral Operators

As in section (III-A), we define the necessary boundary integral operators on the boundary Γ of a general Lipschitz polyhedral domain $\Omega = \Omega^-$, with the intention of taking the domain to be a volume element in a mesh of Ω^-. The wavenumber $\kappa > 0$ is therefore assumed to be constant. For further details, see e.g. [7], [10], [12], [17].

The vectorial single-layer potential operator is defined by

$$\mathbf{\Psi}_\kappa^{SL}\boldsymbol{\mu}(\mathbf{x}) = \int_\Gamma E_\kappa(\mathbf{x} - \mathbf{y})\boldsymbol{\mu}(\mathbf{y})\,dS_\mathbf{y}, \qquad \mathbf{x} \text{ in } \Omega^- \cup \Omega^+,$$

for vector-valued functions $\boldsymbol{\mu}$. The bold letter $\mathbf{\Psi}$ distinguishes this operator from the scalar single-layer potential ψ_κ^{SL} defined in section (III-A).

Following the notation of [12], we define the *electric* and *magnetic* potentials $\mathbf{\Psi}_E, \mathbf{\Psi}_M : \mathbf{H}_\|^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{H}(\mathbf{curl\,curl}, \Omega^-) \times \mathbf{H}_{\mathrm{loc}}(\mathbf{curl\,curl}, \Omega^+)$ by

$$\mathbf{\Psi}_E := \kappa\mathbf{\Psi}_\kappa^{SL} + \kappa^{-1}\mathbf{grad}_\mathbf{x} \circ \mathbf{\Psi}_\kappa^{SL} \circ \mathrm{div}, \qquad \mathbf{\Psi}_M := \mathbf{curl}\mathbf{\Psi}_\kappa^{SL}.$$

Here, $\mathbf{H}(\mathbf{curl\,curl}, \Omega^-) = \{\mathbf{v} \in \mathbf{H}(\mathbf{curl}, \Omega^-) : \mathbf{curl\,curl\,v} \in \mathbf{L}^2(\Omega^-)\}$, and $\mathbf{H}_{\mathrm{loc}}(\mathbf{curl\,curl}, \Omega^+)$ is defined analogously. In terms of these potentials, the Stratton-Chu representation formula for a Maxwell solution $\mathbf{e}$ can be written as

$$\mathbf{e} = -\mathbf{\Psi}_M([\gamma_t\mathbf{e}]_\Gamma) - \mathbf{\Psi}_E([\gamma_N\mathbf{e}]_\Gamma) \qquad \text{in } \Omega^- \cup \Omega^+ \quad (24)$$

(see [7, Theorem 3]), where $\gamma_N\mathbf{v} := \kappa^{-1}\gamma_t\mathbf{curl\,v}$ is the Neumann trace. Taking the Dirichlet and Neumann traces of (24) yields the exterior and interior Calderon projectors

$$\mathbf{P}_M^\pm = \begin{bmatrix} \tfrac{1}{2}\mathbf{I} \pm \mathbf{M} & \pm\mathbf{C} \\ \pm\mathbf{C} & \tfrac{1}{2}\mathbf{I} \pm \mathbf{M} \end{bmatrix}$$

mapping $\mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_{\Gamma},\Gamma)^2$ onto itself (cf. [12]), where the boundary integral operators $\mathbf{C},\mathbf{M}$: $\mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_{\Gamma},\Gamma) \to \mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_{\Gamma},\Gamma)$ are defined by

$$\mathbf{C} := \{\gamma_t\}_{\Gamma} \circ \mathbf{\Psi}_E = \{\gamma_N\}_{\Gamma} \circ \mathbf{\Psi}_M,$$
$$\mathbf{M} := \{\gamma_t\}_{\Gamma} \circ \mathbf{\Psi}_M = \{\gamma_N\}_{\Gamma} \circ \mathbf{\Psi}_E.$$

The action of these operators can be expressed as

$$< \mathbf{Cv},\mathbf{w} >_{\tau,\Gamma} = -\kappa \int_{\Gamma} \int_{\Gamma} E_{\kappa}(\mathbf{x}-\mathbf{y})\mathbf{v}(\mathbf{y}) \cdot \mathbf{w}(\mathbf{x})\, dS_{\mathbf{y}}\, dS_{\mathbf{x}}$$
$$+ \kappa^{-1} \int_{\Gamma} \int_{\Gamma} E_{\kappa}(\mathbf{x}-\mathbf{y})\mathrm{div}_{\Gamma}\mathbf{v}(\mathbf{y})\mathrm{div}_{\Gamma}\mathbf{w}(\mathbf{x})\, dS_{\mathbf{y}}\, dS_{\mathbf{x}},$$
$$< \mathbf{Mv},\mathbf{w} >_{\tau,\Gamma}$$
$$= - \int_{\Gamma} \int_{\Gamma} \mathbf{grad}_{\mathbf{x}}E_{\kappa}(\mathbf{x}-\mathbf{y}) \cdot (\mathbf{v}(\mathbf{y}) \times \mathbf{w}(\mathbf{x}))\, dS_{\mathbf{y}}\, dS_{\mathbf{x}},$$

for tangential vector fields $\mathbf{v},\mathbf{w}$ in $\mathbf{L}^{\infty}(\Gamma)$ (cf. [9]), where the antisymmetric bilinear form $< \cdot,\cdot >_{\tau,\Gamma}$: $\mathbf{L}_{\mathbf{t}}^2(\Gamma) \times \mathbf{L}_{\mathbf{t}}^2(\Gamma) \to \mathbb{C}$ is defined by

$$< \mathbf{v},\mathbf{w} >_{\tau,\Gamma} := \int_{\Gamma} \mathbf{v} \cdot (\mathbf{w} \times \mathbf{n})\, dS.$$

Note that $\mathbf{C}$ and $\mathbf{M}$ are both symmetric with respect to $< \cdot,\cdot >_{\tau,\Gamma}$. The space of all exterior/interior Maxwell Cauchy data $(\gamma_t^{\pm}\mathbf{e},\gamma_N^{\pm}\mathbf{e})$ coincides with the range of $\mathbf{P}_M^{\pm}$. Using this fact and combining the two rows of $\mathbf{P}_M^-$, one obtains the interior Dirichlet-to-Neumann map

$$\mathbf{S}^- := \mathbf{C} + (\tfrac{1}{2}\mathbf{I}+\mathbf{M})\mathbf{C}^{-1}(\tfrac{1}{2}\mathbf{I}-\mathbf{M}),$$

mapping from $\mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_{\Gamma},\Gamma)$ to itself. The existence and continuity of the inverse $\mathbf{C}^{-1}$: $\mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_{\Gamma},\Gamma) \to \mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_{\Gamma},\Gamma)$ is given by [12, Corollary 5.5]. Note that $\mathbf{S}^-$ is symmetric with respect to the bilinear form $< \cdot,\cdot >_{\tau,\Gamma}$. We also remark that the Dirichlet and Neumann traces are in the same space, $\mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_{\Gamma},\Gamma)$, in contrast to the traces for the Helmholtz equation.

B. Element-wise Domain Decomposition

With the same notation for the domain and mesh introduced in section (III-B), we consider the product space $\mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_{\Gamma_S},\Gamma_S)$ of tangential traces of $\mathbf{H}(\mathbf{curl},\Omega^-)$ vector fields on the skeleton Γ_S. Since the tangential traces with respect to two volume elements sharing an interface are of opposite sign, traces on Γ_S are not well defined, except in the product form $(\gamma_{t,1}^-,\ldots,\gamma_{t,p}^-)$. Here, $\gamma_{t,i}^-$ is the tangential trace operator defined on Ω_i. For convenience, we define a skeleton trace operator on Γ_S with its sign determined by an arbitrarily chosen but fixed normal vector field on Γ_S. That is, for all $1 \le i,j \le p$ such that $\Gamma_{ij} := \Gamma_i \cap \Gamma_j \ne \emptyset$, we arbitrarily choose a fixed normal vector $\mathbf{n}_S$ on Γ_{ij}. On Γ, take $\mathbf{n}_S = \mathbf{n}$, so that $\mathbf{n}_S$ is defined on $\Gamma_S = \Gamma \cup_{1 \le i,j \le p} \Gamma_{ij}$. Then

$$\gamma_t^S := (\mathbf{n}_S \cdot \mathbf{n}_i)\gamma_{t,i}^-, \qquad \text{on each } \Gamma_i,$$

uniquely defines a tangential trace on Γ_S. Thus the space $\mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_{\Gamma_S},\Gamma_S)$ is identified with $\gamma_t^S(\mathbf{H}(\mathbf{curl},\Omega^-))$, with the norm

$$\|\mathbf{v}\|_{\mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_{\Gamma_S},\Gamma_S)} := \left(\sum_{i=1}^{p} \|(\mathbf{n}_S \cdot \mathbf{n}_i)\mathbf{v}_i\|_{\mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_{\Gamma_i},\Gamma_i)}^2 \right)^{1/2}.$$

To abbreviate notation, we write $\mathbf{X}_S$ for $\mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_{\Gamma_S},\Gamma_S)$ and $\mathbf{X}_i$ for $\mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_{\Gamma_i},\Gamma_i)$.

Let $\hat{\mathbf{g}}$ be an arbitrary extension of $\mathbf{g}$ in $\mathbf{X}_S$, and set $\tilde{\mathbf{g}} := \gamma_t^S \hat{\mathbf{g}}$. For the Dirichlet problem, we seek $\tilde{\mathbf{u}}$ in

$$\mathbf{W} := \{\mathbf{v} \text{ in } \mathbf{X}_S : \mathbf{v} = \mathbf{0} \text{ on } \Gamma\}$$

such that $\tilde{\mathbf{u}}+\tilde{\mathbf{g}}$ is the trace of $\mathbf{u}$ on Γ_S. A variational problem for $\tilde{\mathbf{u}}$ is

$$\begin{cases} \text{Find } \tilde{\mathbf{u}} \text{ in } \mathbf{W} \text{ such that } \mathbf{u} = \tilde{\mathbf{u}} + \tilde{\mathbf{g}} \text{ on } \Gamma_S \text{ and} \\ a^M(\tilde{\mathbf{u}},\mathbf{v}) = -a^M(\tilde{\mathbf{g}},\mathbf{v}) \qquad \text{for all } \mathbf{v} \text{ in } \mathbf{W}, \end{cases} \quad (25)$$

where the bilinear form $a : \mathbf{W} \times \mathbf{W} \to \mathbb{C}$ is defined by

$$a^M(\mathbf{v},\mathbf{w}) = \sum_{i=1}^{p} < \mathbf{S}_i^- \mathbf{v}_i,\mathbf{w}_i >_{\tau,\Gamma_i} .$$

The invertibility of $\mathbf{C}_i$ for $\mathrm{diam}\,\Omega_i < C\kappa_i^{-1}$ can be verified by a scaling argument involving the eigenvalues of the $\mathbf{curl}\,\mathbf{curl}$ operator, as in Remark 1. Note that the scattering problem (22) can be solved by including the exterior operator $\mathbf{S}^+$ in $a^M(\cdot,\cdot)$ (see Remark 2).

Lemma 7: For each $1 \le i \le p$, the local operator $\mathbf{S}_i^-$ is coercive. That is, there exist an isomorphism $\Theta_i : \mathbf{X}_i \mapsto \mathbf{X}_i$ and a compact bilinear form $c_i : \mathbf{X}_i \times \mathbf{X}_i \to \mathbb{C}$ such that

$$\mathrm{Re}\left(< \mathbf{S}_i^- \mathbf{v},\Theta_i\mathbf{v} >_{\tau,\Gamma_i} -c_i(\mathbf{v},\mathbf{v}) \right) \ge C\|\mathbf{v}\|_{\mathbf{X}_i}^2 \quad (26)$$

for all $\mathbf{v}$ in $\mathbf{X}_i$.

Proof: The proof is analogous to that of Lemma 3. However, we will utilize an isomorphism $\Theta_i : \mathbf{X}_i \mapsto \mathbf{X}_i$ defined as follows. By [8, Theorem 5.5], we have the orthogonal decomposition $\mathbf{X}_i = \nabla_{\Gamma_i}(\mathcal{H}(\Gamma_i)) \oplus \mathbf{curl}_{\Gamma_i}(H^{\frac{1}{2}}(\Gamma_i))$, where

$$\mathcal{H}(\Gamma_i) := \{p \in H^1(\Omega_i)/\mathbb{R} \mid \Delta_{\Gamma_i}p \in H^{-1/2}(\Gamma_i),$$
$$< \Delta_{\Gamma_i}p, 1 >_{1/2,\Gamma_i} = 0\}$$

(see [5], [8] for details and definitions of the operators ∇_{Γ_i} and Δ_{Γ_i}). For any $\mathbf{v}$ in $\mathbf{X}_i$ uniquely decomposed as $\mathbf{v} = \mathbf{v}_{\perp} + \mathbf{v}_0$, with $\mathbf{v}_0$ in $\mathbf{curl}_{\Gamma_i}(H^{\frac{1}{2}}(\Gamma_i))$ and $\mathbf{v}_{\perp}$ in $\nabla_{\Gamma_i}(\mathcal{H}(\Gamma_i))$, we define $\Theta_i\mathbf{v} := \mathbf{v}_{\perp} - \mathbf{v}_0$. Thus Θ_i changes the sign of the divergence-free component, which will be useful as $\mathbf{C}_i$ has two terms of opposite sign.

Now we verify that

$$< \boldsymbol{\xi},\boldsymbol{\eta} >_{\tau,\Gamma_i} = 0 \text{ for all } \boldsymbol{\xi},\boldsymbol{\eta} \text{ in } \mathbf{curl}_{\Gamma_i}(H^{\frac{1}{2}}(\Gamma_i)), \quad (27)$$

$$< \nabla_{\Gamma_i}p,\nabla_{\Gamma_i}q >_{\tau,\Gamma_i} = 0 \text{ for all } p, q \text{ in } \mathcal{H}(\Gamma_i). \quad (28)$$

Green's formula

$$\int_{\Omega_i} \mathbf{curl}\,\mathbf{v} \cdot \mathbf{w} - \mathbf{v} \cdot \mathbf{curl}\,\mathbf{w}\, d\mathbf{x} = < \gamma_t\mathbf{w},\gamma_t\mathbf{v} >_{\tau,\Gamma_i} \quad (29)$$

holds for all $\mathbf{v}$ and $\mathbf{w}$ in $\mathbf{H}(\mathbf{curl}, \Omega_i)$ (see [5], [8]). By definition (see [5]), for u in $H^2(\Omega_i)$ we have $\nabla_{\Gamma_i} u = \pi_\tau(\nabla u)$. Therefore, (29) implies

$$< \nabla_{\Gamma_i} p, \nabla_{\Gamma_i} q >_{\tau, \Gamma_i} = < \pi_\tau \nabla p, \pi_\tau \nabla q >_{\tau, \Gamma_i}$$
$$= < \gamma_t \nabla p, \gamma_t \nabla q >_{\tau, \Gamma_i} = 0$$

for all p, q in $H^2(\Omega_i)$, and a density argument yields (28). A similar argument establishes (27), as $\mathbf{curl}_\Gamma u = \gamma_t(\nabla u)$ for u in $H^2(\Omega_i)$ [5].

By [12, Theorem 5.4], the boundary integral operator $\mathbf{C}_i : \mathbf{X}_i \to \mathbf{X}_i$ is coercive (up to a sign), i.e. there exists a compact operator $T_{\mathbf{C}_i} : \mathbf{X}_i \to \mathbf{X}_i$ such that

$$-\mathrm{Re} < (\mathbf{C}_i + T_{\mathbf{C}_i})\mathbf{v}, \Theta_i \mathbf{v} >_{\tau, \Gamma_i} \geq C\|\mathbf{v}\|_{\mathbf{X}_i}^2, \qquad (30)$$

for all $\mathbf{v}$ in $\mathbf{X}_i$. Hence

$$- \mathrm{Re} < (\mathbf{S}_i^- + T_{\mathbf{C}_i})\mathbf{v}, \Theta_i \mathbf{v} >_{\tau, \Gamma_i}$$
$$= -\mathrm{Re} < (\mathbf{C}_i + T_{\mathbf{C}_i})\mathbf{v}, \Theta_i \mathbf{v} >_{\tau, \Gamma_i}$$
$$- \mathrm{Re} < (\frac{1}{2}\mathbf{I} + \mathbf{M}_i)\mathbf{C}_i^{-1}(\frac{1}{2}\mathbf{I} - \mathbf{M}_i)\mathbf{v}, \Theta_i \mathbf{v} >_{\tau, \Gamma_i}$$
$$\geq C\|\mathbf{v}\|_{\mathbf{X}_i}^2 - \mathrm{Re} < \mathbf{C}_i^{-1}(\frac{1}{2}\mathbf{I} - \mathbf{M}_i)\mathbf{v}, (\frac{1}{2}\mathbf{I} - \mathbf{M}_i)\Theta_i \mathbf{v} >_{\tau, \Gamma_i} .$$

The proof will be complete upon verification of the non-negativity of the last term plus a compact perturbation. Using the identity $\mathbf{S}_i^- = \mathbf{C}_i^{-1}(\frac{1}{2}\mathbf{I} - \mathbf{M}_i)$, the decomposition $\mathbf{v} = \mathbf{v}_\perp + \mathbf{v}_0$, and the symmetry of the bilinear form $< \mathbf{C}_i \cdot, \cdot >_{\tau, \Gamma_i}$, we obtain

$$-\mathrm{Re} < \mathbf{C}_i^{-1}(\frac{1}{2}\mathbf{I} - \mathbf{M}_i)\mathbf{v}, (\frac{1}{2}\mathbf{I} - \mathbf{M}_i)\Theta_i \mathbf{v} >_{\tau, \Gamma_i}$$
$$= \mathrm{Re} < \mathbf{C}_i \mathbf{S}_i^- \mathbf{v}_\perp, \mathbf{S}_i^- \mathbf{v}_\perp >_{\tau, \Gamma_i}$$
$$- \mathrm{Re} < \mathbf{C}_i \mathbf{S}_i^- \mathbf{v}_0, \mathbf{S}_i^- \mathbf{v}_0 >_{\tau, \Gamma_i} .$$

By (28) and the compactness of $< \mathbf{M}_i \cdot, \cdot >_{\tau, \Gamma_i}$ on both $(\mathbf{curl}_{\Gamma_i}(H^{\frac{1}{2}}(\Gamma_i)))^2$ and $(\nabla_{\Gamma_i}(\mathcal{H}(\Gamma_i)))^2$ (cf. [12, Proposition 3.13]),

$$\mathrm{Re} < \mathbf{C}_i \mathbf{S}_i^- \mathbf{v}_\perp, \mathbf{S}_i^- \mathbf{v}_\perp >_{\tau, \Gamma_i}$$
$$= \mathrm{Re} < (\frac{1}{2}\mathbf{I} - \mathbf{M}_i)\mathbf{v}_\perp, \mathbf{C}_i^{-1}(\frac{1}{2}\mathbf{I} - \mathbf{M}_i)\mathbf{v}_\perp >_{\tau, \Gamma_i}$$
$$= -\mathrm{Re} < \mathbf{C}_i \mathbf{C}_i^{-1}(\frac{1}{2}\mathbf{I} - \mathbf{M}_i)\mathbf{v}_\perp, \Theta_i \mathbf{C}_i^{-1}(\frac{1}{2}\mathbf{I} - \mathbf{M}_i)\mathbf{v}_\perp >_{\tau, \Gamma_i}$$
$$+ \text{compact},$$

where *compact* represents a compact bilinear form acting on $(\mathbf{v}, \mathbf{v})$. By (30), the above is non-negative, up to a compact perturbation. Similarly, (27) yields

$$- \mathrm{Re} < \mathbf{C}_i \mathbf{S}_i^- \mathbf{v}_0, \mathbf{S}_i^- \mathbf{v}_0 >_{\tau, \Gamma_i}$$
$$= -\mathrm{Re} < (\frac{1}{2}\mathbf{I} - \mathbf{M}_i)\mathbf{v}_0, \mathbf{C}_i^{-1}(\frac{1}{2}\mathbf{I} - \mathbf{M}_i)\mathbf{v}_0 >_{\tau, \Gamma_i}$$
$$= -\mathrm{Re} < \mathbf{C}_i \mathbf{C}_i^{-1}(\frac{1}{2}\mathbf{I} - \mathbf{M}_i)\mathbf{v}_0, \Theta_i \mathbf{C}_i^{-1}(\frac{1}{2}\mathbf{I} - \mathbf{M}_i)\mathbf{v}_0 >_{\tau, \Gamma_i}$$
$$+ \text{compact}.$$

This completes the proof of (26). ∎

Theorem 3: The variational problem (25) has a unique solution in $\mathbf{W}$.

Proof: The coercivity of $a : \mathbf{W} \times \mathbf{W} \to \mathbb{C}$ follows immediately from the local coercivity result stated in Lemma 7,

and the injectivity of the operator A can be shown in the same way as in the proof of Theorem 1. Therefore, Lemma 1 gives the result. ∎

C. Discretization

As in section (III-C), we assume a triangular mesh of size $O(h)$, with $\overline{\Gamma}_i = \cup_{j=1}^{NT_i} \overline{\Delta}_j^i$ for each $1 \leq i \leq p$. To discretize the space $\mathbf{W}$, one may use any $\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_{\Gamma_S}, \Gamma_S)$-conforming boundary element space, or equivalently, any two-dimensional $\mathbf{H}(\mathrm{div})$-conforming finite element space. Indeed, the only necessary condition for a boundary element space to be $\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_{\Gamma_S}, \Gamma_S)$-conforming is continuity of the normal components across edges in the triangulation of Γ_S, which ensures the boundedness of the surface divergence on Γ_S. Given any straight edge $e := \overline{\Delta_k^i} \cap \overline{\Delta_l^i}$, $k \neq l$, let $\boldsymbol{\nu}_{i,e}^k$ denote the unit normal vector to e in the plane of Δ_k^i, pointing outward from Δ_k^i. For smooth functions $\mathbf{u}$ in $C^\infty(\overline{\Omega}_i)^3$, with trace denoted by $\boldsymbol{\eta} = \gamma_{t,i}\mathbf{u}$, the surface divergence operator div_{Γ_i} is given by

$$\mathrm{div}_{\Gamma_i}\boldsymbol{\eta} := \begin{cases} \mathrm{div}(\boldsymbol{\eta}) & \text{on } \Delta_j^i, \\ ((\boldsymbol{\eta}|_{\Delta_k^i}) \cdot \boldsymbol{\nu}_{i,e}^k + (\boldsymbol{\eta}|_{\Delta_l^i}) \cdot \boldsymbol{\nu}_{i,e}^l)\delta_{kl}^i & \text{on } e \end{cases}$$

(cf. [5], [10]), where δ_{kl}^i is the delta distribution supported on the edge e and div is the ordinary two-dimensional divergence operator acting locally on Δ_j^i. Consequently, for any ϕ in $C^\infty(\Delta_k^i \cup \Delta_l^i)^3$, $\mathrm{div}_{\Gamma_i}\phi$ is in $H^{-1/2}(\Delta_k^i \cup \Delta_l^i \cup e)$ if and only if

$$(\phi|_{\Delta_k^i}) \cdot \boldsymbol{\nu}_{i,e}^k + (\phi|_{\Delta_l^i}) \cdot \boldsymbol{\nu}_{i,e}^l = 0 \qquad \text{on } e.$$

Thus, for each edge we may define a degree of freedom

$$\hat{\lambda}_{e,i}(\phi) := \int_e (\phi|_{\Delta_k^i}) \cdot \boldsymbol{\nu}_{i,e}^k \, ds,$$

where the surface element Δ_k^i is arbitrarily chosen. However, the sign of $\hat{\lambda}_{e,i}$ depends on the choice of Δ_k^i. We remedy this by multiplying by a sign function as follows. Let $\mathbf{n}_i^k$ denote the outward unit normal to Δ_k^i, and let $\mathbf{t}_e$ be a unit tangent vector to e, with arbitrary but fixed orientation. Then

$$\mathrm{sign}\left((\boldsymbol{\nu}_{i,e}^k \times \mathbf{n}_i^k) \cdot \mathbf{t}_e\right) = -\mathrm{sign}\left((\boldsymbol{\nu}_{i,e}^l \times \mathbf{n}_i^l) \cdot \mathbf{t}_e\right).$$

Therefore, if instead of $\hat{\lambda}_{e,i}$ we define the degree of freedom on e to be

$$\lambda_e(\phi) := \qquad\qquad\qquad\qquad\qquad (31)$$
$$\mathrm{sign}\left((\boldsymbol{\nu}_{i,e}^k \times \mathbf{n}_i^k) \cdot \mathbf{t}_e\right)\left(\mathbf{n}_i^k \cdot \mathbf{n}_S(\Delta_k^i)\right) \int_e (\phi|_{\Delta_k^i}) \cdot \boldsymbol{\nu}_{i,e}^k \, ds,$$

then

$$\lambda_e(\gamma_t^S \mathbf{u}) = \mathrm{sign}\left((\boldsymbol{\nu}_{i,e}^k \times \mathbf{n}_i^k) \cdot \mathbf{t}_e\right) \int_e (\gamma_{t,i}\mathbf{u})|_{\Delta_k^i} \cdot \boldsymbol{\nu}_{i,e}^k \, ds.$$

With the basis representation $\mathbf{u} = a(x)\mathbf{n}_i^k + b(x)\boldsymbol{\nu}_{i,e}^k + c(x)\boldsymbol{\nu}_{i,e}^k \times \mathbf{n}_i^k$, we have

$$(\gamma_{t,i}\mathbf{u})|_{\Delta_k^i} = \mathbf{u} \times \mathbf{n}_i^k = b(x)\boldsymbol{\nu}_{i,e}^k \times \mathbf{n}_i^k - c(x)\boldsymbol{\nu}_{i,e}^k.$$

This implies that $(\gamma_{t,i}\mathbf{u})|_{\Delta_k^i} \cdot \boldsymbol{\nu}_{i,e}^k = -c(x)$ and hence

$$
\begin{aligned}
\lambda_e(\gamma_t^S \mathbf{u}) &= -\text{sign}\left((\boldsymbol{\nu}_{i,e}^k \times \mathbf{n}_i^k) \cdot \mathbf{t}_e\right) \int_e c(x)\, ds \\
&= -\text{sign}\left((\boldsymbol{\nu}_{i,e}^k \times \mathbf{n}_i^k) \cdot \mathbf{t}_e\right) \int_e \mathbf{u} \cdot (\boldsymbol{\nu}_{i,e}^k \times \mathbf{n}_i^k)\, de \\
&= -\int_e \mathbf{u} \cdot \mathbf{t}_e\, ds. \qquad (32)
\end{aligned}
$$

Thus λ_e is uniquely defined, i.e. (31) is independent of the choice of the volume element Ω_i and the surface element Δ_k^i.

With the degrees of freedom thus defined, it is clear that the lowest-order Raviart-Thomas space $\mathcal{RT}_0 := \{\mathbf{x} \mapsto \mathbf{a} + \beta\mathbf{x} : \mathbf{a} \in \mathbb{C}^2, \beta \in \mathbb{C}\}$ is appropriate for discretization of the space $\mathbf{W}$. That is, we consider the discrete solution space

$$
\mathbf{W}_h := \{\mathbf{v}_h \text{ in } \mathbf{W} : \mathbf{v}_h|_{\Delta_j^i} \text{ is in } \mathcal{RT}_0 \text{ for all } i, j\}.
$$

Let $\mathcal{P}_h : \mathbf{W} \to \mathbf{W}_h$ denote the orthogonal projector onto $\mathbf{W}_h$ with respect to the inner product in $\mathbf{X}_S$. The following approximation estimate is proved in [4, Lemma 4.9], with the space $\mathbf{H}_{\|}^1(\text{div}_{\Gamma_S}, \Gamma_S)$ defined as

$$
\{\mathbf{v} \in \gamma_t^S(\mathbf{H}^1(\Omega)) \cap \mathbf{H}_-^1(\Gamma_S) : \text{div } \mathbf{v} \in H_-^1(\Gamma_S)\}.
$$

Note that [4] gives approximation estimates of order $h^{1/2+s}$ for s in the range $-1/2 \leq s \leq 1$, but we only state the estimate of order $h^{3/2}$ to avoid defining more spaces.

Lemma 8: For all $\mathbf{v}$ in $\mathbf{H}_{\|}^1(\text{div}_{\Gamma_S}, \Gamma_S)$, we have

$$
\|(\mathbf{I} - \mathcal{P}_h)\mathbf{v}\|_{\mathbf{X}_S} \leq Ch^{3/2}\|\mathbf{v}\|_{\mathbf{H}_{\|}^1(\text{div}_{\Gamma_S}, \Gamma_S)}.
$$

The boundary element space $\mathbf{W}_h$ induces the new finite element subspace

$$
\mathbf{S}_h := \{\mathbf{u}_h \text{ in } \mathbf{S}(\Omega) : \gamma_t^S \mathbf{u}_h|_{\Delta_j^i} \text{ is in } \mathcal{RT}_0 \text{ for all } i, j\}
$$

of

$$
\mathbf{S}(\Omega) := \{\mathbf{u} \text{ in } \mathbf{H}(\mathbf{curl}, \Omega) : \mathbf{curl\,curl\,u} - \kappa^2\mathbf{u} = \mathbf{0}\}.
$$

The following approximation estimate can be proved analogously to Lemma 6.

Lemma 9 (Approximation in $\mathbf{S}(\Omega)$): For sufficiently small $h > 0$ and subdomains Ω_i of fixed size, there exists a constant $C > 0$ such that

$$
\inf_{\mathbf{u}_h \in \mathbf{S}_h} \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{H}(\mathbf{curl}, \Omega)} \leq Ch^{3/2}\|(\gamma_t^S \mathbf{u}, \gamma_N^S \mathbf{u})\|_{\mathbf{H}_{\|}^1(\text{div}_{\Gamma_S}, \Gamma_S)^2}
$$

for all $\mathbf{u}$ in $\mathbf{S}(\Omega)$ with $\gamma_t^S \mathbf{u}$ and $\gamma_N^S \mathbf{u} := \kappa^{-1}\gamma_t^S \mathbf{curl\,u}$ in $\mathbf{H}_{\|}^1(\text{div}_{\Gamma_S}, \Gamma_S)$.

For all $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{H}(\mathbf{curl}, \Omega_i)$ with $\mathbf{curl\,curl\,v} - \kappa^2\mathbf{v} = \mathbf{0}$, Green's formula (29) yields

$$
< \mathbf{S}_i^- \mathbf{v}, \gamma_t \mathbf{w} >_{\tau, \Gamma_i} = \kappa^{-1}\int_{\Omega_i} \mathbf{curl\,w} \cdot \mathbf{curl\,v} - \kappa^2\mathbf{w} \cdot \mathbf{v}\, d\mathbf{x}.
$$

Define the bilinear form $a_\Omega^M : \mathbf{H}(\mathbf{curl}, \Omega) \times \mathbf{H}(\mathbf{curl}, \Omega) \to \mathbb{C}$ by

$$
a_\Omega^M(\mathbf{v}, \mathbf{w}) := (\kappa^{-1}\mathbf{curl\,w}, \mathbf{curl\,v})_\Omega - (\kappa\mathbf{w}, \mathbf{v})_\Omega.
$$

Since $a^M(\gamma_t^S \cdot, \gamma_t^S \cdot)$ and a_Ω^M coincide on $\mathbf{S}(\Omega)$ for all $h > 0$, Lemmas 4 and 5 may be applied, with $X = \mathbf{S}(\Omega)$, to

obtain an error estimate for the following discrete variational problem:

$$
\begin{cases}
\text{Find } \tilde{\mathbf{u}}_h \text{ in } \mathbf{W}_h \text{ such that} \\
a^M(\tilde{\mathbf{u}}_h, \mathbf{v}_h) = -a^M(\tilde{\mathbf{g}}, \mathbf{v}_h) \qquad \text{for all } \mathbf{v}_h \text{ in } \mathbf{W}.
\end{cases} \tag{33}
$$

Theorem 4: Suppose that $\mathbf{e}$ in $\mathbf{H}(\mathbf{curl}, \Omega)$ solves (23), with $\gamma_t^S \mathbf{e}$ and $\gamma_N^S \mathbf{e}$ in $\mathbf{H}_{\|}^1(\text{div}_{\Gamma_S}, \Gamma_S)$. For sufficiently small $h > 0$ and subdomains Ω_i of fixed size, the discrete variational problem (33) has a unique solution $\tilde{\mathbf{u}}_h$ in $\mathbf{W}_h$, satisfying $\tilde{\mathbf{u}}_h + \tilde{\mathbf{g}} = \gamma_t^- \boldsymbol{\xi}_h$ with $\boldsymbol{\xi}_h$ in $\mathbf{S}_h$ and

$$
\|\mathbf{e} - \boldsymbol{\xi}_h\|_{\mathbf{H}(\mathbf{curl}, \Omega)} \leq Ch^{3/2}\|(\gamma_t^S \mathbf{e}, \gamma_N^S \mathbf{e})\|_{\mathbf{H}_{\|}^1(\text{div}_{\Gamma_S}, \Gamma_S)^2}.
$$

In practice, we approximate $\mathbf{S}_i^-$ by the computable discrete operator

$$
\mathbf{S}_{i,h}^- = \mathbf{C}_{i,h} + (\tfrac{1}{2}\mathbf{I}_{i,h} + \mathbf{M}_{i,h})\mathbf{C}_{i,h}^{-1}(\tfrac{1}{2}\mathbf{I}_{i,h} - \mathbf{M}_{i,h})
$$

defined on $\mathbf{W}_h$, where $\mathbf{I}_{i,h}$, $\mathbf{C}_{i,h}$, and $\mathbf{M}_{i,h}$ are the Galerkin discretizations of the operators $\mathbf{I}$, $\mathbf{C}_i$, and $\mathbf{M}_i$, respectively, in the inner product $< \cdot, \cdot >_{\tau, \Gamma_i}$ in $\mathbf{W}_h$. Then we solve the following variational problem:

$$
\begin{cases}
\text{Find } \tilde{\mathbf{u}}_h \text{ in } \mathbf{W}_h \text{ satisfying} \\
a_h^M(\tilde{\mathbf{u}}_h, \mathbf{v}_h) = -a_h^M(\tilde{\mathbf{g}}, \mathbf{v}_h) \text{ for all } \mathbf{v}_h \text{ in } \mathbf{W}_h,
\end{cases} \tag{34}
$$

with

$$
a_h^M(\mathbf{v}_h, \mathbf{w}_h) := \sum_{i=1}^p < \mathbf{S}_{i,h}^- \mathbf{v}_h|_{\Gamma_i}, \mathbf{w}_h >_{\tau, \Gamma_i}.
$$

As discussed at the end of section (III), we cannot prove an error estimate for the solution of the approximate linear system (34). Also, we have no error estimate for the approximate tangential trace computed via (34), but the optimal estimate in the $\mathbf{X}_S$-norm would be of order $O(h^{3/2})$, according to Lemma 8. However, the $\mathbf{X}_S$-norm cannot be computed in practice; instead, we investigate the error behavior in the $\mathbf{L}_{\mathbf{t}}^2(\Gamma_S)$-norm. Considering the inverse inequality given by the following lemma, we expect the optimal error behavior in the $\mathbf{L}_{\mathbf{t}}^2(\Gamma_S)$-norm to be of order $O(h)$.

Lemma 10 (Inverse Inequality): There exists a constant $C > 0$ such that

$$
\|\mathbf{w}_h\|_{\mathbf{L}_{\mathbf{t}}^2(\Gamma_S)} \leq Ch^{-1/2}\|\mathbf{w}_h\|_{\mathbf{X}_S}, \text{ for all } \mathbf{w}_h \text{ in } \mathbf{W}_h.
$$

Proof: For each triangle τ in Γ_S (with diam $\tau = O(h)$), let $\{\mathbf{t}_\tau^{(1)}, \mathbf{t}_\tau^{(2)}\}$ be an orthonormal basis of the plane containing τ and denote by $\lambda_\tau^{(j)}$ the barycentric coordinate functions on τ, for $j = 1, 2, 3$. We define the bubble function $b_\tau := \lambda_\tau^{(1)}\lambda_\tau^{(2)}\lambda_\tau^{(3)}$ and the projection operator $Q_h^B : L^2(\Gamma_S) \to \oplus_\tau\{pb_\tau : p \in P_1(\tau)\}$ by

$$
\int_\tau pQ_h^B\phi\, dS_\mathbf{x} = \int_\tau p\phi\, dS_\mathbf{x}, \text{ for all } \phi \text{ in } L^2(\Gamma_S), p \text{ in } P_1(\tau).
$$

If $Q_h^B \phi = p_\phi b_\tau$, then mapping to a reference triangle $\hat{\tau}$ yields

$$\|p_\phi\|_{L^\infty(\tau)}^2 = \|\hat{p}_\phi\|_{L^\infty(\hat{\tau})}^2 \le C \int_{\hat{\tau}} \hat{p}_\phi^2 \hat{b}_\tau \, dS_{\hat{\mathbf{x}}}$$

$$\le Ch^{-2} \int_\tau p_\phi^2 b_\tau \, dS_{\mathbf{x}} = Ch^{-2} \int_\tau p_\phi Q_h^B \phi \, dS_{\mathbf{x}}$$

$$= Ch^{-2} \int_\tau p_\phi \phi \, dS_{\mathbf{x}} \le Ch^{-2} \|p_\phi\|_{L^\infty(\tau)} \int_\tau \phi \, dS_{\mathbf{x}}$$

$$\le Ch^{-1} \|p_\phi\|_{L^\infty(\tau)} \|\phi\|_{L^2(\tau)}$$

and thus

$$\|p_\phi\|_{L^\infty(\tau)} \le Ch^{-1} \|\phi\|_{L^2(\tau)}, \qquad \|p_\phi\|_{L^2(\tau)} \le C\|\phi\|_{L^2(\tau)}. \tag{35}$$

The standard inverse estimate $\|\nabla p\|_{L^2(\tau)} \le Ch^{-1}\|p\|_{L^2(\tau)}$, for p in $P_1(\tau)$, yields

$$\|\nabla Q_h^B \phi\|_{L^2(\tau)} \le \|\nabla p_\phi\|_{L^2(\tau)} + \|p_\phi\|_{L^\infty(\tau)} \|\nabla b_\tau\|_{L^2(\tau)}$$
$$\le Ch^{-1} \left(\|p_\phi\|_{L^2(\tau)} + \|\phi\|_{L^2(\tau)} \|\nabla b_\tau\|_{L^2(\tau)} \right)$$
$$\le Ch^{-1} \|\phi\|_{L^2(\tau)}.$$

Clearly $\|Q_h^B \phi\|_{L^2(\tau)} \le C\|\phi\|_{L^2(\tau)}$, so interpolation gives the estimate

$$\|Q_h^B \phi\|_{H^{1/2}(\Gamma_S)} \le Ch^{-1/2} \|\phi\|_{L^2(\Gamma_S)}, \text{ for all } \phi \text{ in } L^2(\Gamma_S). \tag{36}$$

Now we set $B_{\mathbf{t}}(\tau) := \text{span}\{b_\tau \mathbf{t}_\tau^{(1)}, b_\tau \mathbf{t}_\tau^{(2)}\}$ and define the projection $Q_{\mathbf{t},h}^B : \mathbf{L}_{\mathbf{t}}^2(\Gamma_S) \to \oplus_\tau B_{\mathbf{t}}(\tau)$ locally on each triangle τ by

$$Q_{\mathbf{t},h}^B \mathbf{v} := Q_h^B(\mathbf{v} \cdot \mathbf{t}_\tau^{(1)}) \mathbf{t}_\tau^{(1)} + Q_h^B(\mathbf{v} \cdot \mathbf{t}_\tau^{(2)}) \mathbf{t}_\tau^{(2)}, \text{ for } \mathbf{v} \text{ in } \mathbf{L}_{\mathbf{t}}^2(\Gamma_S).$$

Next, we verify the boundedness of the operator $Q_{\mathbf{t},h}^B : \mathbf{L}_{\mathbf{t}}^2(\Gamma_S) \to \mathbf{H}_{\|}^{1/2}(\Gamma_S)$. Clearly $Q_{\mathbf{t},h}^B \mathbf{v}|_\tau$ is smooth and in $\mathbf{H}^{1/2}(\tau)$ on each triangle τ, with the estimate

$$\|Q_{\mathbf{t},h}^B \mathbf{v}\|_{\mathbf{H}^{1/2}(\tau)}$$
$$\le C\|Q_h^B(\mathbf{v} \cdot \mathbf{t}_\tau^{(1)})\|_{H^{1/2}(\tau)} + \|Q_h^B(\mathbf{v} \cdot \mathbf{t}_\tau^{(2)})\|_{H^{1/2}(\tau)}$$
$$\le Ch^{-1/2} \|\mathbf{v}\|_{\mathbf{L}_{\mathbf{t}}^2(\tau)},$$

by (36). It remains only to estimate the functionals $\mathcal{N}_{ij}^{\|}(Q_{\mathbf{t},h}^B \mathbf{v})$ in the norm $\|Q_{\mathbf{t},h}^B \mathbf{v}\|_{\|,1/2,\Gamma_S}$. For any two triangles τ_1 and τ_2 intersecting on an edge e, we may assume without loss of generality that $\lambda_{\tau_1}^{(1)}$ vanishes on the edge e. Then

$$\int_{\tau_1} \int_{\tau_2} \frac{|b_{\tau_1}(\mathbf{x})|^2}{\|\mathbf{x} - \mathbf{y}\|^3} \, dS_{\mathbf{x}} \, dS_{\mathbf{y}}$$

$$\le C \int_{\tau_1} \int_{\tau_2} \frac{|h^{-1}\|\mathbf{x} - \mathbf{y}\| \lambda_{\tau_1}^{(2)}(\mathbf{x}) \lambda_{\tau_1}^{(3)}(\mathbf{x})|^2}{\|\mathbf{x} - \mathbf{y}\|^3} \, dS_{\mathbf{x}} \, dS_{\mathbf{y}}$$

$$\le Ch^{-2} \int_{\tau_1} \int_{\tau_2} \|\mathbf{x} - \mathbf{y}\|^{-1} \, dS_{\mathbf{x}} \, dS_{\mathbf{y}}$$

$$\le Ch \int_{\hat{\tau}_1} \int_{\hat{\tau}_2} \|\hat{\mathbf{x}} - \hat{\mathbf{y}}\|^{-1} \, d\hat{\mathbf{x}} \, d\hat{\mathbf{y}} \le Ch,$$

where $\hat{\tau}_1$ and $\hat{\tau}_2$ are the reference triangles with the vertices $\{(-1,0),(0,0),(0,1)\}$ and $\{(0,1),(0,0),(1,0)\}$, respectively. Combining this estimate with (35) results in

$$\mathcal{N}_{ij}^{\|}(Q_{\mathbf{t},h}^B \mathbf{v}) \le Ch^{-1} \|\mathbf{v}\|_{\mathbf{L}_{\mathbf{t}}^2(F_i \cup F_j)}^2$$

for all triangular faces F_i and F_j intersecting along an edge. Thus

$$\|Q_{\mathbf{t},h}^B \mathbf{v}\|_{\|,1/2,\Gamma_S} \le Ch^{-1/2} \|\mathbf{v}\|_{\mathbf{L}_{\mathbf{t}}^2(\Gamma_S)}.$$

For any $\mathbf{w}_h$ in $\mathbf{W}_h$, we have

$$\|\mathbf{w}_h\|_{\mathbf{L}_{\mathbf{t}}^2(\Gamma_S)} = \sup_{\mathbf{v} \in \mathbf{L}_{\mathbf{t}}^2(\Gamma_S)} \frac{(\mathbf{w}_h, \mathbf{v})_{\mathbf{L}_{\mathbf{t}}^2(\Gamma_S)}}{\|\mathbf{v}\|_{\mathbf{L}_{\mathbf{t}}^2(\Gamma_S)}}$$

$$= \sup_{\mathbf{v} \in \mathbf{L}_{\mathbf{t}}^2(\Gamma_S)} \frac{(\mathbf{w}_h, Q_{\mathbf{t},h}^B \mathbf{v})_{\mathbf{L}_{\mathbf{t}}^2(\Gamma_S)}}{\|\mathbf{v}\|_{\mathbf{L}_{\mathbf{t}}^2(\Gamma_S)}}$$

$$\le \|\mathbf{w}_h\|_{\|,-1/2,\Gamma_S} \sup_{\mathbf{v} \in \mathbf{L}_{\mathbf{t}}^2(\Gamma_S)} \frac{\|Q_{\mathbf{t},h}^B \mathbf{v}\|_{\|,1/2,\Gamma_S}}{\|\mathbf{v}\|_{\mathbf{L}_{\mathbf{t}}^2(\Gamma_S)}}$$

$$\le Ch^{-1/2} \|\mathbf{w}_h\|_{\|,-1/2,\Gamma_S}.$$

∎

V. NUMERICAL EXPERIMENTS

As the aim of this paper is mainly to introduce the methods and theoretically analyze the convergence behavior, the numerical experiments reported in this section manifest certain computational aspects which are lacking. First, a preconditioner has yet to be proposed, resulting in very slow convergence of iterative solvers. Second, we have not computed approximate solutions throughout the domain Ω verifying the predicted convergence rates in $H^1(\Omega)$ (Helmholtz case) or $\mathbf{H}(\mathbf{curl}, \Omega)$ (Maxwell case). Our methods solve for Dirichlet traces on the mesh skeleton, which in theory should yield approximate Neumann traces via the Dirichlet-to-Neumann maps. These two traces should then provide an approximate solution in Ω via the representation formula. However, we have not addressed these computational issues.

In our experiments, we simply take a nodal interpolant of the Dirichlet traces to obtain an approximation to the discrete solution in Ω. Obviously, this approximation does not demonstrate the theoretically predicted error behavior of the methods, but it is optimal in the corresponding standard finite element spaces. In the Helmholtz case, the nodal interpolant of the discrete solution in the continuous piecewise linear finite element space approximates the exact solution with the optimal order $O(h^2)$ in the $\mathbf{L}^2(\Omega)$-norm, and in the Maxwell case we observe the optimal $O(h)$ convergence rate with respect to the $\mathbf{L}^2(\Omega)$-norm in the first-order Nédélec space. Thus, our methods are as accurate as the corresponding standard finite element method on simple meshes, and they are also applicable to general meshes.

Although we use a triangular mesh of the skeleton, it is also possible to use a quadrilateral mesh or a mesh with mixed triangular and quadrilateral elements. A triangular or quadrilateral mesh of the skeleton facilitates the application of quadrature techniques, as in [15]. It is worth noting that the approximate linear system and its solution depend on the accuracy of the quadrature rules used in computing the necessary singular integrals. In order to make the computation of the linear system as fast as possible, we have used the minimum degree of quadrature accuracy required for obtaining reasonably good results. With higher-order quadratures, slightly better numerical results could be obtained.

TABLE I
HELMHOLTZ PROBLEM ON UNIFORM TETRAHEDRAL MESHES
OF THE UNIT CUBE $\Omega = (0,1)^3$, $\kappa = 1$.

h	nodes	GMRES	Error(Γ_S)	$\|u - \Pi_h^1 u_h\|_{L^2(\Omega)}$
1/4	71	11	4.0050	0.40578
1/8	429	25	1.4263	0.13109
1/16	2969	51	0.43766	0.038895
1/32	22065	98	0.11046	0.0097991
1/64	170081	186	0.021596	0.0019040
	$O(h^{-3})$	$O(h^{-1})$	$O(h^2)$	$O(h^2)$

TABLE II
HELMHOLTZ PROBLEM ON UNSTRUCTURED MESHES
OF THE UNIT CUBE $\Omega = (0,1)^3$, $\kappa = 1$.

h	nodes	tetra.	hex.	GMRES	Error(Γ_S)
0.22807	71	144	24	14	4.0781
0.10526	429	1152	192	28	1.4273
0.051648	2969	8904	1692	52	0.42889
0.025536	22065	69496	14404	97	0.11023
0.012676	170081	551392	117520	184	0.021997
	$O(h^{-3})$	$O(h^{-3})$	$O(h^{-3})$	$O(h^{-1})$	$O(h^2)$

TABLE III
HELMHOLTZ PROBLEM ON UNIFORM TETRAHEDRAL MESHES
OF THE UNIT CUBE $\Omega = (0,1)^3$, $\kappa = 10$.

h	nodes	GMRES	Error(Γ_S)	$\|u - \Pi_h^1 u_h\|_{L^2(\Omega)}$
1/4	71	12	6.2358	0.72021
1/8	429	37	1.6662	0.16422
1/16	2969	80	0.48067	0.044746
1/32	22065	163	0.11966	0.011046
1/64	170081	322	0.026442	0.0025347
	$O(h^{-3})$	$O(h^{-1})$	$O(h^2)$	$O(h^2)$

TABLE IV
MAXWELL PROBLEM ON UNIFORM TETRAHEDRAL MESHES
OF THE UNIT CUBE $\Omega = (0,1)^3$, $\kappa = 1$.

h	edges	GMRES	Error(Γ_S)	$\|\mathbf{u} - \Pi_h^{ND} \mathbf{u}_h\|_{\mathbf{L}^2(\Omega)}$
1/8	2156	242	0.10960	9.7059 E-3
1/16	16024	555	0.057978	5.1002 E-3
1/32	123440	1223	0.030625	2.6784 E-3
1/64	968800	2993	0.017036	1.4803 E-3
	$O(h^{-3})$	$O(h^{-1})$	$O(h)$	$O(h)$

TABLE V
MAXWELL PROBLEM ON UNIFORM TETRAHEDRAL MESHES
OF THE UNIT CUBE $\Omega = (0,1)^3$, $\kappa = 10$.

h	edges	GMRES	Error(Γ_S)	$\|\mathbf{u} - \Pi_h^{ND} \mathbf{u}_h\|_{\mathbf{L}^2(\Omega)}$
1/4	310	92	0.94874	0.64590
1/8	2156	424	0.40711	0.26524
1/16	16024	1421	0.23313	0.14949
1/32	123440	31776	0.13441	0.085339
	$O(h^{-3})$		$O(h)$	$O(h)$

A. Helmholtz Equation

Table (I) reports some computations done for the Helmholtz equation on the unit cube $\Omega = (0,1)^3$, with a non-nested sequence of uniform tetrahedral meshes. The wavenumber κ is taken to be 1, and the exact solution is

$$u(\mathbf{x}) = E_\kappa(\mathbf{x} - (1.01, 0, 0)).$$

The discrete solution for the trace is $u_h = \tilde{u}_h + \tilde{g}$, where $\tilde{u}_h$ is the solution of (20). As explained above, we approximate the solution in Ω by extending u_h via nodal interpolation. Thus $\Pi_h^1 u_h$ denotes the unique continuous piecewise linear function in Ω which agrees with u_h at all nodes of the mesh.

We solve the linear system (20) by the unpreconditioned GMRES method, and the iteration counts grow in proportion to $1/h$ when $\kappa = 1$. Table (I) lists the relative error

$$\text{Error}(\Gamma_S) := \|u - u_h\|_{L^2(\Gamma_S)} / \|u\|_{L^2(\Gamma_S)}$$

of the Dirichlet trace on the skeleton, although there is no theoretical estimate of it. Considering the fact that the skeleton grows in proportion to $1/h$, the order of convergence is somewhat ambiguous. However, we observe that the relative $L^2(\Gamma_S)$-norm of the error $u - u_h$ is of order $O(h^2)$. Also, $\|u - \Pi_h^1 u_h\|_{L^2(\Omega)}$ is of the optimal order $O(h^2)$ in the space of piecewise linear finite element functions.

The results in table (II) are for unstructured mixed meshes, obtained from the uniform tetrahedral meshes by perturbing the coordinates of the vertices by random numbers in $[-h/10, h/10]$ and forming hexahedra from some of the tetrahedra. The columns titled "tetra." and "hex." list the numbers of tetrahedral and hexahedral volume elements, respectively. We only report the relative error on the skeleton, which is almost identical to Table (I).

Table (III) shows similar results for uniform tetrahedral meshes with $\kappa = 10$, in which case κ^2 is larger than the minimal Dirichlet eigenvalue of $-\Delta$ on the unit cube. Our method applies immediately to the Laplace equation simply by taking $\kappa = 0$. Numerical experiments with $\kappa = 0$ are almost identical to Table (I), where $\kappa = 1$.

B. Maxwell Equations

Table (IV) lists some results for the Maxwell equations, with the same meshes as in table (I). We take the exact solution

$$\mathbf{u}(\mathbf{x}) = \nabla \times (E_\kappa(\mathbf{x} - (1.5, 0, 0))\mathbf{x}).$$

The discrete solution for the trace is $\mathbf{u}_h = \tilde{\mathbf{u}}_h + \tilde{\mathbf{g}}$, where $\tilde{\mathbf{u}}_h$ is the solution of (34). The extension $\Pi_h^{ND} \mathbf{u}_h$ denotes the unique function in the lowest-order Nédélec space on Ω (see e.g. [26]) satisfying

$$- \int_e \Pi_h^{ND} \mathbf{u}_h \cdot \mathbf{t}_e \, ds = \lambda_e(\mathbf{u}_h)$$

for all edges e of the mesh (cf. (32)). As in the Helmholtz case, the convergence behavior is optimal. Indeed, one cannot expect better than $O(h)$ convergence in $\mathbf{L}^2(\Gamma_S)$ (considering Lemma 10) or in $\mathbf{L}^2(\Omega)$ for the Π_h^{ND} interpolant. The relative error

$$\text{Error}(\Gamma_S) := \|\gamma_t^S(\mathbf{u} - \mathbf{u}_h)\|_{\mathbf{L}^2(\Gamma_S)} / \|\gamma_t^S \mathbf{u}\|_{\mathbf{L}^2(\Gamma_S)}$$

of the Dirichlet trace on the skeleton is of order $O(h)$. Table (V) shows that for $\kappa = 10$, the linear system has a much larger condition number.

The computation of the system matrix is significantly slower than for standard finite element methods. However, this example with a simplicial mesh is the worst-case scenario for the

numerical integration required for our method. In computing double integrals over the surfaces of volume elements, the integration is singular or nearly singular on pairs of triangular surface elements which coincide or share a common edge. With larger, more complex volume elements, the numerical integration would also involve some pairs of triangles which are sufficiently far apart to allow for much faster, lower-order quadrature rules. Thus any comparison of setup times would be strongly biased in favor of standard finite element methods.

Although the methods of this paper can theoretically be applied with piecewise constant κ, we do not report numerical experiments for such cases, since an exact solution is unknown. Experiments with varying κ could only show a growth in the iteration numbers, as a preconditioner has not yet been proposed.

A very important consideration for numerical performance, out of the scope of this paper, is the optimal choice of the element size. In the context of boundary-element-based finite element methods, elements and subdomains are theoretically interchangeable, so elements may be of any size as long as they resolve jumps in the wavenumber. Increasing the element size would decrease the size of the mesh skeleton and therefore the number of unknowns in the discrete system. Also, the numerical integration would involve fewer singular integrals, reducing much of the computational bottleneck. The negative computational effects of increasing the element size are that the dense local matrices become larger and the matrix inversion involved in their computation becomes more expensive. At some point, special boundary element techniques to obtain data-sparse matrices would become necessary. However, it seems plausible that the numerical performance could be improved by balancing the reduction in the system size with the cost of larger element matrices.

VI. Conclusion

We have introduced new finite element methods which can be applied to general polyhedral meshes, with piecewise constant coefficients. The error behavior has been shown theoretically to be quasi-optimal for the pure domain decomposition case with subdomains of fixed size. Although we did not perform numerical experiments demonstrating the theoretical convergence rates, we have confirmed that the $L^2(\Omega)$ convergence rates for both the Helmholtz and Maxwell methods are of the same order as for standard finite element methods on simple meshes. Thus the methods have good convergence rates and are more generally applicable than conventional methods.

It is evident from the previous section that this work is only the beginning, as several critical computational issues need to be addressed. To obtain better accuracy, techniques should be developed to utilize the representation formulas to compute an approximate solution in Ω. Perhaps more important in practice is the need for a preconditioner that is robust with respect to the size of the wavenumber and its jumps. Also, it would be interesting to investigate the advantages in performance that might be attained by taking larger elements.

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