International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:4, No:11, 2010

On a Class of Inverse Problems for Degenerate Differential Equations

Fadi Awawdeh, H.M. Jaradat

Abstract—In this paper, we establish existence and uniqueness of solutions for a class of inverse problems of degenerate differential equations. The main tool is the perturbation theory for linear operators.

Keywords—Inverse Problem, Degenerate Differential Equations, Perturbation Theory for Linear Operators

I. Introduction

THe general nature of an inverse problem is to deduce a cause from an effect. Consider a physical system, depending on a collection of parameters, in which one can speak of inputs to the system and outputs from the system. If all of the parameters were known perfectly then for a given input we could predict the output. It may happen, however, that some of the parameters characterizing the system are not known, being inaccessible to direct measurement. If it is important to know what these parameters are, in order that the system be understood as completely as possible, we might try to infer them by observing the outputs from the system corresponding to special inputs. Thus we seek the cause (the system parameters) given the effect (the output of the system for a given input).

An important example is the inverse problem of geophysics, in which we seek to investigate the structure of the interior of the earth. Elastic waves may propagate through the earth in a manner which depends on the material properties of the earth. A concentrated source of energy at the surface of causes waves to penetrate into the earth which are then partially reflected back to the surface. If the material properties of the earth's interior were known completely then we could predict the nature of the reflected wave from knowledge of the source. Since in fact we cannot measure these properties directly we seek to infer them by observing the reflected waves in response to a collection of known sources.

In formulating such problems mathematically, we typically find that the problem amounts to that of determining one or more coefficients in a differential equation, or system of differential equations, given partial knowledge of certain special solutions of the equation(s). In the seismology problem just discussed, the propagation of waves in the earth is governed by the equations of elasticity, a system of partial differential equations in which the material properties of the earth manifest themselves as coefficient functions in the equations. The measurements we can make amount to the knowledge of special solutions of the equations at special points, e.g. those points on the surface of the earth in this example.

Inverse problems in differential equations have this general character. One has a certain definite kind of differential equation (or system of equations) containing one or more unknown (or partially known) coefficient functions. From some limited knowledge about certain special solutions of the equations we seek to determine the unknown coefficient functions. Problems of this type arise in a variety of important applications areas, such as geophysics, optics, quantum mechanics, astronomy, medical imaging and materials testing.

Identification results for some nondegenerate situations are described in [3-4,12-19]. In the last years, inverse problems for degenerate differential equations have given rise to a new research field. In particular, we mention [1-2,8].

Recently, Al Horani [2] proved the existence and uniqueness of the solutions to the following inverse problems for degenerate differential equations by means of the projection method:

Find $v \in C^1([0,\tau];X)$ and $f \in C^1([0,\tau];\mathbb{R})$, $\tau > 0$, such that

$$\begin{aligned} \frac{dMv}{dt} &= Lv(t) + f(t)z, \qquad 0 \le t \le \tau, \\ Mv(0) &= u_0, \\ \phi[Mv(t)] &= g(t), \qquad 0 \le t \le \tau, \end{aligned}$$

where M and L are linear operators in the Banach space X with $D(L) \subset D(M)$, $z, u_0 \in X$, $g \in C^1([0,\tau];\mathbb{R})$ and $\phi \in X^*$, X^* being the dual space to X.

To the best of our knowledge, there is no work reported on inverse problems for degenerate differential equations of the form

$$M\frac{du}{dt} = Lu + Mf(t)z \ , \qquad 0 \le t \le \tau, \tag{1}$$

$$u(0) = u_0, \tag{2}$$

$$\phi[u(t)] = g(t), \qquad 0 \le t \le \tau, \tag{3}$$

where M and L are densely defined linear operators in the Hilbert space X, $z \in X$, $\phi \in X^*$, u_0 is an initial value and u and f are the unknown functions.

The purpose of this paper is to prove the existence and uniqueness of solutions for the inverse problem (1)-(3) using an approach based on the perturbation theory for linear operators [3-4].

F. Awawdeh is with the Department of Mathematics, Hashemite University, Zarqa, Jordan, e-mail: awawdeh@hu.edu.jo..
H.M. Jaradat is with the Department of Mathematics, Al-Al Bayt University, Mafraq, Jordan, e-mail: husseinjaradat@yahoo.com.

II. PRELIMINARIES

In this section, we introduce some facts about the generator of a c_0-semigroup [9-10].

We denote by X a Banach space with norm $\|\cdot\|$ and $A : D(A) \to X$ is the infinitesimal generator of a c_0-semigroup of bounded linear operators $T(t)$, $t \geq 0$, on X. It is well known that A is closed and its domain $D(A)$ equipped with the graph norm

$$\|x\|_A = \|x\| + \|Ax\|$$

becomes a Banach space, which we shall denote by X_A.

If $f \in C^1([0,\tau]; X)$ then the multivalued evolution problem

$$\begin{aligned} &u'(t) \in Au(t) + f(t), \qquad t > 0, \\ &u(0) = u_0, \end{aligned}$$

has a unique solution u on $[0,\tau]$ for every $u_0 \in D(A)$

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)ds.$$

Theorem 1: Let A be a linear operator on X such that $A - \beta$ is maximal dissipative with some real number β, i.e., A satisfies

$$Re(f,u)_X \leq \beta \|u\|_X^2 \qquad \text{for all} \quad f \in Au \tag{4}$$

with the range condition

$$R(\lambda_0 - A) = X \quad \text{ for some } \quad \lambda_0 > \beta. \tag{5}$$

Then, $\rho(A) \supset (\beta, \infty)$ and A satisfies

$$\left\|(\lambda - A)^{-1}\right\|_{(X)} \leq \frac{1}{\lambda - \beta}, \qquad \lambda > \beta.$$

By virtue of Theorem 1, if A is a linear operator on X with a maximal dissipative $A - \beta$, $\beta \in \mathbb{R}$, a semigroup $T(t)$ is generated by A on the whole space X.

We will be interested in the case when $\{L^*(M^*)^{-1}\}^* = M^{-1}L$, where L^*, M^* are the adjoint operators of L and M. The next theorem entails a condition sufficient to this end.

Theorem 2: If M is a bounded linear operator on X and $\rho_{M^*}(L^*) \cap \rho_M(L) \neq \emptyset$, then $\{L^*(M^*)^{-1}\}^* = M^{-1}L$.

The following perturbation result for linear operators will be helpful in the sequel [5-6].

Theorem 3: Let X be a Banach space and let A be the infinitesimal generator of a c_0-semigroup $T(t)$ on X. If $B : X_A \longrightarrow X_A$ is a continuous linear operator, then $A + B$ is the infinitesimal generator of a c_0-semigroup on X.

For more details about perturbation theory one can see [11].

III. MAIN RESULTS

Consider the identification problem (1)-(3) where M and L are densely defined linear operators in the Hilbert space X such that there adjoint operators satisfy $D(L^*) \subset D(M^*)$, $z \in X$, $\phi \in X^*$, u_0 is an initial value and u and f are the unknown functions.

We assume:

$$Re(L^*v, M^*v)_X \leq \beta \|v\|_X^2, \qquad v \in D(L^*); \tag{6}$$

$$\lambda_0 \in \rho_{M^*}(L^*) \qquad \text{for some } \lambda_0 > \beta. \tag{7}$$

Let $T = L^*(M^*)^{-1}$. If $h \in Tu$, then $h = L^*(M^*)^{-1}u = L^*v$ and $M^*v = u$ for some $v \in D(L^*)$; so that, $(h,u)_X = (L^*v, M^*v)_X$, which shows that (4) follows from (6). On the other hand, for any $h \in X$, we have by (7) that $h = (\lambda_0 M^* - L^*)v$ for some $v \in D(L^*)$. If we put $u = M^*v$, then $u \in M^*(D(L^*)) = D(T)$ and $h \in (\lambda_0 - T)u$, i.e. (5). According to Theorem 1, this proves that $T - \beta = L^*(M^*)^{-1} - \beta$ is maximal dissipative in X. As a consequence, its adjoint $\{L^*(M^*)^{-1}\}^* - \beta$ is also maximal dissipative; so that, $\{L^*(M^*)^{-1}\}^*$ is the generator of a c_0-semigroup on X. On the other hand, clearly (1)-(3) is written in the form

$$\frac{du(t)}{dt} \in Au(t) + f(t)z\ , \qquad 0 \leq t \leq \tau, \tag{8}$$

$$u(0) = u_0, \tag{9}$$

$$\phi[u(t)] = g(t), \qquad 0 \leq t \leq \tau, \tag{10}$$

with a coefficient operator $A = M^{-1}L$. Moreover, we are interested in the case when $\{L^*(M^*)^{-1}\}^*$ and $M^{-1}L$ are coincide. If M is a bounded linear operator on X and $\rho_{M^*}(L^*) \cap \rho_M(L) \neq \emptyset$, then by Theorem 2 $\{L^*(M^*)^{-1}\}^* = M^{-1}L$ and so $M^{-1}L$ is the generator of a c_0-semigroup on X.

Similar to the argument used in [3-4], we will use some results from perturbation theory for linear operators for the solvability of the inverse problem (8)-(10).

Applying the linear functional ϕ to both sides of (8) and using (10) we have

$$g'(t) \in \phi[Au(t)] + f(t)\phi[z],$$

and we get for $\phi[z] \neq 0$ that

$$f(t) \in \frac{1}{\phi[z]}(g'(t) - \phi[Au(t)]). \tag{11}$$

Substituting (11) in (8), we obtain

$$u'(t) \in Au(t) + \frac{1}{\phi[z]}(g'(t) - \phi[Au(t)])z. \tag{12}$$

Common practice involves the operator

$$Bx = \frac{-1}{\phi[z]}(\phi[Ax])z, \tag{13}$$

and equation (12) becomes

$$u'(t) \in (A+B)u(t) + \frac{1}{\phi[z]}g'(t)z. \tag{14}$$

Plain calculations show that B is bounded in X_A,

$$\begin{aligned} \|B\|_A &= \sup_{\|x\|_A = 1} \|Bx\| \\ &= \sup_{\|x\|_A = 1} \left\| \frac{-1}{\phi[z]}(\phi[A(x)])z \right\| \\ &\leq \sup_{\|x\|_A = 1} \frac{1}{|\phi[z]|} \|z\| \|\phi\| \|Ax\| \\ &\leq \frac{1}{|\phi[z]|} \|z\| \|\phi\| . \end{aligned}$$

Theorem 3 now implies that $A + B$ is the infinitesimal generator of a semigroup $S(t)$, $t \geq 0$. Since $u_0 \in D(A)$, then the problem (14) with condition (9) has a unique solution $u(t)$

$$u(t) = S(t)u_0 + \frac{1}{\phi[z]}\int_0^t S(t-s)g'(s)zds. \qquad (15)$$

By (11) and (15) $f(t)$ is uniquely determined and the reduced problem (8)-(10) possesses a unique solution (u, f).

Theorem 4: Let M and L be densely defined linear operators in the Hilbert space X such that $D(L^*) \subset D(M^*)$, $z \in X$, $\phi \in X^*$. If (6) and (7) are satisfied and M is a bounded linear operator on X with $\rho_{M^*}(L^*) \cap \rho_M(L) \neq \emptyset$. Then, for any $u_0 \in D(L)$ such that $Lu_0 \in R(M)$, the inverse problem (1)-(3) possesses a unique solution (u, f).

IV. Applications

In this section, we give an example which illustrate our abstract results.

Example 5: Consider

$$\left\{\begin{array}{ll} \frac{\partial m(x)v}{\partial t} = -\frac{\partial v}{\partial x} + f(t)\sin x, & -\infty < x < \infty,\ 0 \leq t \leq T, \\ v(x,0) = u_0(x), & -\infty < x < \infty, \\ {}_0^T v(x,t)dx = t\ , & 0 \leq t \leq T. \end{array}\right. \qquad (16)$$

Here $m(x)$ is the characteristic function of some measurable set $J \subset \mathbb{R}$, i.e.,

$$m(x) = \left\{\begin{array}{ll} 1, & if \quad x \in J \\ 0, & if \quad x \notin J \end{array}\right. ,$$

$u_0(x)$ is an initial function and $v = v(x,t)$ and $f(t)$ are the unknown functions. We consider the problem in the space $X = L^2(\mathbb{R})$. Let M be the multiplication operator by $m(x)$ acting in X. Clearly, M is bounded on X and enjoys the properties $M^* = M$ and $M^2 = M$. Therefore (16) is formulated in X in the form

$$\begin{aligned} M\frac{dv}{dt} &= Lv + Mf(t)z\ , \quad 0 \leq t \leq T, \\ v(0) &= u_0, \\ \phi[v(t)] &= g(t), \qquad 0 \leq t \leq T, \end{aligned}$$

where $L = -\frac{d}{dx}$ with $D(L) = H^1(\mathbb{R})$, L being a closed linear operator in X.

Since $Re(Lv, v)_X = 0$ for all $v \in D(L)$, Condition (6) is satisfied with $\beta = 0$. In the case $J = (-\infty, a) \cup (b, \infty)$, where $a < b$, Condition (7) can be easily verified. For more details the reader can refer to [7].

References

[1] M.H. Al Horani, Projection method for solving degenerate first-order identification problem, J. Math. Anal. Appl. 364(1) (2010) 204-208.
[2] M.H. Al Horani, Favini, A.: An identification problem for first-order degenerate differential equations. J. Optim. Theory Appl. 130 (2006) 41–60.
[3] F. Awawdeh, Perturbation method for abstract second-order inverse problems, Nonlinear Analysis, 72 (2010) 1379-1386.
[4] F. Awawdeh, Ordinary Differential Equations in Banach Spaces with applications, PhD theses, Jordan University, 2006.
[5] W. Desh and W. Schappacher, Some Perturbation Results for Analytic Semigroups. Math. Ann. 281 (1988) 157-162.
[6] W. Desh and W. Schappacher, On Relatively Perturbations of Linear c_0- Semigroups, Annali della Scuola Normale Superiore di Pisa - Classe di Scienze, Sér. 4, 11(2) (1984) 327-341.
[7] A. Favini and A. Yagi, Degenerate Differential Equations in Banach Spaces. Marcel Dekker. Inc. New York. (1999).
[8] J. Goldstein, Semigroups of Linear Operators and Applications, Oxford University Press: New York-Oxford, (1985).
[9] E. Hille and E. Phillips, Functional Analysis and Semigroups, AMS: Providence, RI, (1957).
[10] T. Kato, Perturbation Theory of Linear Operators. Springer: New York-Berlin-Heidelberg, (1976).
[11] A. Lorenzi, An Introduction to Identification Problems via Functional Analysis. Inverse and Ill-Posed Problems Series. VSP, Utrecht (2001).
[12] A. Lorenzi, An inverse problem for a semilinear parabolic equation, Annali di Mathematica Pure e Applicata, 82 (1982) 145-166.
[13] D. Orlovsky, An inverse problem for a second order differential equation in a Banach space, Differential Equations, 25 (1989) 1000-1009.
[14] D. Orlovsky, An inverse problem of determining a parameter of an evolution equation, Differential Equations, 26 (1990) 1614-1621.
[15] D. Orlovsky, Weak and strong solutions of inverse problems for differential equations in a Banach space, Differenial Equations, 27 (1991) 867-874.
[16] G. Pavlov, On the uniqueness of the solution of an abstract inverse problem, Differenial Equations, 24 (1988) 1402-1406.
[17] M. Pilant and W. Rundell, An inverse problem for a nonlinear elliptic differential equation, SIAM J. Mathem. Anal., 18 (6) (1987) 1801-1809.
[18] I. Prilepko, G. Orlovsky and A. Vasin, Methods for Solving Inverse Problems in Mathematical Physics. Marcel Dekker. Inc. New York. (2000).