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A Note on Toeplitz Matrices

Hsuan-Chu Li

***Abstract*—**In this note, we demonstrate explicit LU factorizations of Toeplitz matrices for some small sizes. Furthermore, we obtain the inverse of referred Toeplitz matrices by appling the above-mentioned results.

***Keywords*—**Toeplitz matrices, LU factorization, inverse of a matrix.

I. Introduction

LET T_n be a Toeplitz matrix with size n, i.e.

$$T_n := T_{\{n;a_{-(n-1)},a_{-(n-2)},\cdots,a_{n-2},a_{n-1}\}} = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & \cdots & a_{1-n} \\ a_1 & a_0 & a_{-1} & \cdots & a_{2-n} \\ a_2 & a_1 & a_0 & \cdots & a_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0 \end{bmatrix}$$

where $a_{-(n-1)}, a_{-(n-2)}, \cdots, a_{n-2}, a_{n-1}$ are complex numbers [3, p.1189]. In recent papers, many authors discussed some properties of Toeplitz matrices. Trench[5] firstly established the inversion of a Toeplitz matrix by a low number of its columns and the entries of the original Toeplitz matrix. Gohberg and Semencul [1], Gohberg and Krupnik [4] and Heinig and Rost [2] expressed the inverse of a Toeplitz matrix in diferent forms.

In this work, we try to obtain the explict LU factorization of a Toeplitz matrix with size $n = 2,3,4,5$ respectively. By appling these results, we can easily calculated the determinants of the referred Toeplitz matrix by multipling the main diagonals of the corresponding upper triangular matrix. Finally, we seek to establish the inverses of Toeplitz matrices by using the following elementary fact :

For any $n \times n$ matrix A, let $\tilde{A}_{ij}$ be the matrix obtained from A by deleting the *i*-th row and the *j*-th column. Then

$$A^{-1} = Transpose\ of\ \frac{(-1)^{i+j}\det(\tilde{A}_{ij})}{\det(A)}.$$

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II. Main Result and an Example

Now we are in a position to state the main theorem.

Theorem: For $n = 2,3,4,5$, T_n can be factored as $T_n = L_n U_n$, where $L_n = [L_n(i,j)]$ is a lower triangular matrix with unit main diagonal and $U_n = [U_n(i,j)]$ is an upper triangular matrix, whose entries are defined as follows:

$n = 2$:

$T_2 = L_2 U_2$, where $T_2 = \begin{bmatrix} a_0 & a_{-1} \\ a_1 & a_0 \end{bmatrix}$, $L_2 = \begin{bmatrix} 1 & 0 \\ \frac{a_1}{a_0} & 1 \end{bmatrix}$ and

$$U_2 = \begin{bmatrix} a_0 & a_{-1} \\ 0 & \frac{a_0^2 - a_1 a_{-1}}{a_0} \end{bmatrix}.$$

$n = 3$:

$T_3 = L_3 U_3$, where $T_3 = \begin{bmatrix} a_0 & a_{-1} & a_{-2} \\ a_1 & a_0 & a_{-1} \\ a_2 & a_1 & a_0 \end{bmatrix}$;

$$L_3 = \begin{bmatrix} 1 & 0 & 0 \\ \frac{a_1}{a_0} & 1 & 0 \\ \frac{a_2}{a_0} & \frac{a_1 a_0 - a_2 a_{-1}}{a_0^2 - a_1 a_{-1}} & 1 \end{bmatrix};$$

$$U_3 = \begin{bmatrix} a_0 & a_{-1} & a_{-2} \\ 0 & \frac{a_0^2 - a_1 a_{-1}}{a_0} & \frac{a_0 a_{-1} - a_1 a_{-2}}{a_0} \\ 0 & 0 & U_3(3,3) \end{bmatrix} \text{ and}$$

$$U_3(3,3) = \frac{(a_0^2 - a_1 a_{-1})(a_0^2 - a_2 a_{-2}) - (a_1 a_0 - a_2 a_{-1})(a_0 a_{-1} - a_1 a_{-2})}{a_0(a_0^2 - a_1 a_{-1})}.$$

$n = 4$:

$T_4 = L_4 U_4$, where $T_4 = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} \\ a_1 & a_0 & a_{-1} & a_{-2} \\ a_2 & a_1 & a_0 & a_{-1} \\ a_3 & a_2 & a_1 & a_0 \end{bmatrix}$;

$$L_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{a_1}{a_0} & 1 & 0 & 0 \\ \frac{a_2}{a_0} & \frac{a_1 a_0 - a_2 a_{-1}}{a_0^2 - a_1 a_{-1}} & 1 & 0 \\ \frac{a_3}{a_0} & \frac{a_2 a_0 - a_3 a_{-1}}{a_0^2 - a_1 a_{-1}} & L_4(4,3) & 1 \end{bmatrix}$$; where

$$L_4(4,3) = \frac{(a_0^2 - a_1a_{-1})(a_1a_0 - a_3a_{-2}) - (a_2a_0 - a_3a_{-1})(a_0a_{-1} - a_1a_{-2})}{(a_0^2 - a_1a_{-1})(a_0^2 - a_2a_{-2}) - (a_1a_0 - a_2a_{-1})(a_0a_{-1} - a_1a_{-2})};$$

$$U_4 = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} \\ 0 & \frac{a_0^2 - a_1a_{-1}}{a_0} & \frac{a_0a_{-1} - a_1a_{-2}}{a_0} & \frac{a_0a_{-2} - a_1a_{-3}}{a_0} \\ 0 & 0 & U_4(3,3) & U_4(3,4) \\ 0 & 0 & 0 & U_4(4,4) \end{bmatrix};$$

$$U_4(3,3) = \frac{(a_0^2 - a_1a_{-1})(a_0^2 - a_2a_{-2}) - (a_1a_0 - a_2a_{-1})(a_0a_{-1} - a_1a_{-2})}{a_0(a_0^2 - a_1a_{-1})};$$

$$U_4(3,4) = \frac{(a_0^2 - a_1a_{-1})(a_0^2 - a_2a_{-3}) - (a_1a_0 - a_2a_{-1})(a_0a_{-2} - a_1a_{-3})}{a_0(a_0^2 - a_1a_{-1})};$$

$$\begin{aligned} U_4(4,4) = \{ & [(a_0^2 - a_1a_{-1})(a_0^2 - a_2a_{-2}) - (a_1a_0 - a_2a_{-1})(a_0a_{-1} - a_1a_{-2})] \\ \times & [(a_0^2 - a_1a_{-1})(a_0^2 - a_3a_{-3}) - (a_2a_0 - a_3a_{-1})(a_0a_{-2} - a_1a_{-3})] \\ - & [(a_0^2 - a_1a_{-1})(a_1a_0 - a_3a_{-2}) - (a_2a_0 - a_3a_{-1})(a_0a_{-1} - a_1a_{-2})] \\ \times & [(a_0^2 - a_1a_{-1})(a_0a_{-1} - a_2a_{-3}) - (a_1a_0 - a_2a_{-1})(a_0a_{-2} - a_1a_{-3})]\} \\ \div \{ & a_0(a_0^2 - a_1a_{-1})[(a_0^2 - a_1a_{-1})(a_0^2 - a_2a_{-2}) - (a_1a_0 - a_2a_{-1})(a_0a_{-1} - a_1a_{-2})]\} \end{aligned}$$

$n = 5$:

$T_5 = L_5 U_5$, where $T_5 = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & a_{-4} \\ a_1 & a_0 & a_{-1} & a_{-2} & a_{-3} \\ a_2 & a_1 & a_0 & a_{-1} & a_{-2} \\ a_3 & a_2 & a_1 & a_0 & a_{-1} \\ a_4 & a_3 & a_2 & a_1 & a_0 \end{bmatrix}$;

$$L_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ L_5(2,1) & 1 & 0 & 0 & 0 \\ L_5(3,1) & L_5(3,2) & 1 & 0 & 0 \\ L_5(4,1) & L_5(4,2) & L_5(4,3) & 1 & 0 \\ L_5(5,1) & L_5(5,2) & L_5(5,3) & L_5(5,4) & 1 \end{bmatrix};$$

where

$L_5(2,1) = \frac{a_1}{a_0}$;

$L_5(3,1) = \frac{a_2}{a_0}$;

$L_5(4,1) = \frac{a_3}{a_0}$;

$L_5(5,1) = \frac{a_4}{a_0}$;

$L_5(3,2) = \frac{a_1a_0 - a_2a_{-1}}{a_0^2 - a_1a_{-1}}$;

$L_5(4,2) = \frac{a_2a_0 - a_3a_{-1}}{a_0^2 - a_1a_{-1}}$;

$L_5(5,2) = \frac{a_3a_0 - a_4a_{-1}}{a_0^2 - a_1a_{-1}}$;

$$L_5(4,3) = \frac{(a_0^2 - a_1a_{-1})(a_1a_0 - a_3a_{-2}) - (a_2a_0 - a_3a_{-1})(a_0a_{-1} - a_1a_{-2})}{(a_0^2 - a_1a_{-1})(a_0^2 - a_2a_{-2}) - (a_1a_0 - a_2a_{-1})(a_0a_{-1} - a_1a_{-2})};$$

$$L_5(5,3) = \frac{(a_0^2 - a_1a_{-1})(a_2a_0 - a_4a_{-2}) - (a_3a_0 - a_4a_{-1})(a_0a_{-1} - a_1a_{-2})}{(a_0^2 - a_1a_{-1})(a_0^2 - a_2a_{-2}) - (a_1a_0 - a_2a_{-1})(a_0a_{-1} - a_1a_{-2})};$$

$$\begin{aligned} L_5(5,4) = \{ & [(a_0^2 - a_1a_{-1})(a_0^2 - a_2a_{-2}) - (a_1a_0 - a_2a_{-1})(a_0a_{-1} - a_1a_{-2})] \\ \times & [(a_0^2 - a_1a_{-1})(a_0^2 - a_4a_{-3}) - (a_3a_0 - a_4a_{-1})(a_0a_{-2} - a_1a_{-3})] \\ - & [(a_0^2 - a_1a_{-1})(a_2a_0 - a_4a_{-2}) - (a_3a_0 - a_4a_{-1})(a_0a_{-1} - a_1a_{-2})] \\ \times & [(a_0^2 - a_1a_{-1})(a_0a_{-1} - a_2a_{-3}) - (a_1a_0 - a_2a_{-1})(a_0a_{-2} - a_1a_{-3})]\} \\ \div \{ & [(a_0^2 - a_1a_{-1})(a_0^2 - a_2a_{-2}) - (a_1a_0 - a_2a_{-1})(a_0a_{-1} - a_1a_{-2})] \\ \times & [(a_0^2 - a_1a_{-1})(a_0^2 - a_3a_{-3}) - (a_2a_0 - a_3a_{-1})(a_0a_{-2} - a_1a_{-3})] \\ - & [(a_0^2 - a_1a_{-1})(a_1a_0 - a_3a_{-2}) - (a_2a_0 - a_3a_{-1})(a_0a_{-1} - a_1a_{-2})] \\ \times & [(a_0^2 - a_1a_{-1})(a_0a_{-1} - a_2a_{-3}) - (a_1a_0 - a_2a_{-1})(a_0a_{-2} - a_1a_{-3})]\}; \end{aligned}$$

$$U_5 = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & a_{-4} \\ 0 & U_5(2,2) & U_5(2,3) & U_5(2,4) & U_5(2,5) \\ 0 & 0 & U_5(3,3) & U_5(3,4) & U_5(3,5) \\ 0 & 0 & 0 & U_5(4,4) & U_5(4,5) \\ 0 & 0 & 0 & 0 & U_5(5,5) \end{bmatrix};$$

where

$U_5(2,2) = \frac{a_0^2 - a_1a_{-1}}{a_0}$;

$U_5(2,3) = \frac{a_0a_{-1} - a_1a_{-2}}{a_0}$;

$U_5(2,4) = \frac{a_0a_{-2} - a_1a_{-3}}{a_0}$;

$$U_5(2,5)=\frac{a_0a_{-3}-a_1a_{-4}}{a_0};$$

$$U_5(3,3)=\frac{(a_0^2-a_1a_{-1})(a_0^2-a_2a_{-2})-(a_1a_0-a_2a_{-1})(a_0a_{-1}-a_1a_{-2})}{a_0(a_0^2-a_1a_{-1})};$$

$$U_5(3,4)=\frac{(a_0^2-a_1a_{-1})(a_0^2-a_2a_{-3})-(a_1a_0-a_2a_{-1})(a_0a_{-2}-a_1a_{-3})}{a_0(a_0^2-a_1a_{-1})};$$

$$U_5(3,5)=\frac{(a_0^2-a_1a_{-1})(a_0a_{-2}-a_2a_{-4})-(a_1a_0-a_2a_{-1})(a_0a_{-3}-a_1a_{-4})}{a_0(a_0^2-a_1a_{-1})};$$

$$U_5(4,4)=\{[(a_0^2-a_1a_{-1})(a_0^2-a_2a_{-2})-(a_1a_0-a_2a_{-1})(a_0a_{-1}-a_1a_{-2})]$$
$$\times[(a_0^2-a_1a_{-1})(a_0^2-a_3a_{-3})-(a_2a_0-a_3a_{-1})(a_0a_{-2}-a_1a_{-3})]$$
$$-[(a_0^2-a_1a_{-1})(a_1a_0-a_3a_{-2})-(a_2a_0-a_3a_{-1})(a_0a_{-1}-a_1a_{-2})]$$
$$\times[(a_0^2-a_1a_{-1})(a_0a_{-1}-a_2a_{-3})-(a_1a_0-a_2a_{-1})(a_0a_{-2}-a_1a_{-3})]\}$$
$$\div\{a_0(a_0^2-a_1a_{-1})[(a_0^2-a_1a_{-1})(a_0^2-a_2a_{-2})-(a_1a_0-a_2a_{-1})(a_0a_{-1}-a_1a_{-2})]\};$$

$$U_5(4,5)=\{[(a_0^2-a_1a_{-1})(a_0^2-a_2a_{-2})-(a_1a_0-a_2a_{-1})(a_0a_{-1}-a_1a_{-2})]$$
$$\times[(a_0^2-a_1a_{-1})(a_0a_{-1}-a_3a_{-4})-(a_2a_0-a_3a_{-1})(a_0a_{-3}-a_1a_{-4})]$$
$$-[(a_0^2-a_1a_{-1})(a_1a_0-a_3a_{-2})-(a_2a_0-a_3a_{-1})(a_0a_{-1}-a_1a_{-2})]$$
$$\times[(a_0^2-a_1a_{-1})(a_0a_{-2}-a_2a_{-4})-(a_1a_0-a_2a_{-1})(a_0a_{-3}-a_1a_{-4})]\}$$
$$\div\{a_0(a_0^2-a_1a_{-1})[(a_0^2-a_1a_{-1})(a_0^2-a_2a_{-2})-(a_1a_0-a_2a_{-1})(a_0a_{-1}-a_1a_{-2})]\}$$

$$U_5(5,5)=\{\{[(a_0^2-a_1a_{-1})(a_0^2-a_2a_{-2})-(a_1a_0-a_2a_{-1})(a_0a_{-1}-a_1a_{-2})]$$
$$\times[(a_0^2-a_1a_{-1})(a_0^2-a_3a_{-3})-(a_2a_0-a_3a_{-1})(a_0a_{-2}-a_1a_{-3})]$$
$$-[(a_0^2-a_1a_{-1})(a_1a_0-a_3a_{-2})-(a_2a_0-a_3a_{-1})(a_0a_{-1}-a_1a_{-2})]$$
$$\times[(a_0^2-a_1a_{-1})(a_0a_{-1}-a_2a_{-3})-(a_1a_0-a_2a_{-1})(a_0a_{-2}-a_1a_{-3})]\}$$
$$\times\{[(a_0^2-a_1a_{-1})(a_0^2-a_2a_{-2})-(a_1a_0-a_2a_{-1})(a_0a_{-1}-a_1a_{-2})]$$
$$\times[(a_0^2-a_1a_{-1})(a_0^2-a_4a_{-4})-(a_3a_0-a_4a_{-1})(a_0a_{-3}-a_1a_{-4})]$$
$$-[(a_0^2-a_1a_{-1})(a_2a_0-a_4a_{-2})-(a_3a_0-a_4a_{-1})(a_0a_{-1}-a_1a_{-2})]$$
$$\times[(a_0^2-a_1a_{-1})(a_0a_{-2}-a_2a_{-4})-(a_1a_0-a_2a_{-1})(a_0a_{-3}-a_1a_{-4})]\}$$
$$-\{[(a_0^2-a_1a_{-1})(a_0^2-a_2a_{-2})-(a_1a_0-a_2a_{-1})(a_0a_{-1}-a_1a_{-2})]$$
$$\times[(a_0^2-a_1a_{-1})(a_1a_0-a_4a_{-3})-(a_3a_0-a_4a_{-1})(a_0a_{-2}-a_1a_{-3})]$$
$$-[(a_0^2-a_1a_{-1})(a_2a_0-a_4a_{-2})-(a_3a_0-a_4a_{-1})(a_0a_{-1}-a_1a_{-2})]$$
$$\times[(a_0^2-a_1a_{-1})(a_0a_{-1}-a_2a_{-3})-(a_1a_0-a_2a_{-1})(a_0a_{-2}-a_1a_{-3})]\}$$
$$\times\{[(a_0^2-a_1a_{-1})(a_0^2-a_2a_{-2})-(a_1a_0-a_2a_{-1})(a_0a_{-1}-a_1a_{-2})]$$
$$\times[(a_0^2-a_1a_{-1})(a_0a_{-1}-a_3a_{-4})-(a_2a_0-a_3a_{-1})(a_0a_{-3}-a_1a_{-4})]$$
$$-[(a_0^2-a_1a_{-1})(a_1a_0-a_3a_{-2})-(a_2a_0-a_3a_{-1})(a_0a_{-1}-a_1a_{-2})]$$
$$\times[(a_0^2-a_1a_{-1})(a_0a_{-2}-a_2a_{-4})-(a_1a_0-a_2a_{-1})(a_0a_{-3}-a_1a_{-4})]\}\}$$
$$\div\{\{a_0(a_0^2-a_1a_{-1})[(a_0^2-a_1a_{-1})(a_0^2-a_2a_{-2})-(a_1a_0-a_2a_{-1})(a_0a_{-1}-a_1a_{-2})]]$$
$$\times\{[(a_0^2-a_1a_{-1})(a_0^2-a_2a_{-2})-(a_1a_0-a_2a_{-1})(a_0a_{-1}-a_1a_{-2})]$$
$$\times[(a_0^2-a_1a_{-1})(a_0^2-a_3a_{-3})-(a_2a_0-a_3a_{-1})(a_0a_{-2}-a_1a_{-3})]$$
$$-[(a_0^2-a_1a_{-1})(a_1a_0-a_3a_{-2})-(a_2a_0-a_3a_{-1})(a_0a_{-1}-a_1a_{-2})]$$
$$\times[(a_0^2-a_1a_{-1})(a_0a_{-1}-a_2a_{-3})-(a_1a_0-a_2a_{-1})(a_0a_{-2}-a_1a_{-3})]\}\}$$

Proof. By direct calculations and are omitted.

Corollary: The determinants of T_2, T_3, T_4, T_5 are listed as follows:

$$\det T_2=\prod_{i=1}^{2}U_2(i,i)=a_0^2-a_1a_{-1};$$

$$\det T_3=\prod_{i=1}^{3}U_3(i,i)=\frac{(a_0^2-a_1a_{-1})(a_0^2-a_2a_{-2})-(a_1a_0-a_2a_{-1})(a_0a_{-1}-a_1a_{-2})}{a_0};$$

$$\det T_4=\prod_{i=1}^{4}U_4(i,i)$$
$$=\{[(a_0^2-a_1a_{-1})(a_0^2-a_2a_{-2})-(a_1a_0-a_2a_{-1})(a_0a_{-1}-a_1a_{-2})]$$
$$\times[(a_0^2-a_1a_{-1})(a_0^2-a_3a_{-3})-(a_2a_0-a_3a_{-1})(a_0a_{-2}-a_1a_{-3})]$$
$$-[(a_0^2-a_1a_{-1})(a_1a_0-a_3a_{-2})-(a_2a_0-a_3a_{-1})(a_0a_{-1}-a_1a_{-2})]$$
$$\times[(a_0^2-a_1a_{-1})(a_0a_{-1}-a_2a_{-3})-(a_1a_0-a_2a_{-1})(a_0a_{-2}-a_1a_{-3})]\}$$
$$\div\{a_0^2(a_0^2-a_1a_{-1})\};$$

$$\det T_5=\prod_{i=1}^{5}U_5(i,i)$$
$$=\{\{[(a_0^2-a_1a_{-1})(a_0^2-a_2a_{-2})-(a_1a_0-a_2a_{-1})(a_0a_{-1}-a_1a_{-2})]$$
$$\times[(a_0^2-a_1a_{-1})(a_0^2-a_3a_{-3})-(a_2a_0-a_3a_{-1})(a_0a_{-2}-a_1a_{-3})]$$
$$-[(a_0^2-a_1a_{-1})(a_1a_0-a_3a_{-2})-(a_2a_0-a_3a_{-1})(a_0a_{-1}-a_1a_{-2})]$$
$$\times[(a_0^2-a_1a_{-1})(a_0a_{-1}-a_2a_{-3})-(a_1a_0-a_2a_{-1})(a_0a_{-2}-a_1a_{-3})]\}$$
$$\times\{[(a_0^2-a_1a_{-1})(a_0^2-a_2a_{-2})-(a_1a_0-a_2a_{-1})(a_0a_{-1}-a_1a_{-2})]$$
$$\times[(a_0^2-a_1a_{-1})(a_0^2-a_4a_{-4})-(a_3a_0-a_4a_{-1})(a_0a_{-3}-a_1a_{-4})]$$
$$-[(a_0^2-a_1a_{-1})(a_2a_0-a_4a_{-2})-(a_3a_0-a_4a_{-1})(a_0a_{-1}-a_1a_{-2})]$$
$$\times[(a_0^2-a_1a_{-1})(a_0a_{-2}-a_2a_{-4})-(a_1a_0-a_2a_{-1})(a_0a_{-3}-a_1a_{-4})]\}$$
$$-\{[(a_0^2-a_1a_{-1})(a_0^2-a_2a_{-2})-(a_1a_0-a_2a_{-1})(a_0a_{-1}-a_1a_{-2})]$$
$$\times[(a_0^2-a_1a_{-1})(a_1a_0-a_4a_{-3})-(a_3a_0-a_4a_{-1})(a_0a_{-2}-a_1a_{-3})]$$
$$-[(a_0^2-a_1a_{-1})(a_2a_0-a_4a_{-2})-(a_3a_0-a_4a_{-1})(a_0a_{-1}-a_1a_{-2})]$$
$$\times[(a_0^2-a_1a_{-1})(a_0a_{-1}-a_2a_{-3})-(a_1a_0-a_2a_{-1})(a_0a_{-2}-a_1a_{-3})]\}$$
$$\times\{[(a_0^2-a_1a_{-1})(a_0^2-a_2a_{-2})-(a_1a_0-a_2a_{-1})(a_0a_{-1}-a_1a_{-2})]$$
$$\times[(a_0^2-a_1a_{-1})(a_0a_{-1}-a_3a_{-4})-(a_2a_0-a_3a_{-1})(a_0a_{-3}-a_1a_{-4})]$$
$$-[(a_0^2-a_1a_{-1})(a_1a_0-a_3a_{-2})-(a_2a_0-a_3a_{-1})(a_0a_{-1}-a_1a_{-2})]$$

$$\times[(a_0^2-a_1a_{-1})(a_0a_{-2}-a_2a_{-4})-(a_1a_0-a_2a_{-1})(a_0a_{-3}-a_1a_{-4})]\}\}$$
$$\div\{a_0^3(a_0^2-a_1a_{-1})^2[(a_0^2-a_1a_{-1})(a_0^2-a_2a_{-2})-(a_1a_0-a_2a_{-1})(a_0a_{-1}-a_1a_{-2})]\}\}$$

Corollary: The inverses of T_2, T_3, T_4, T_5 are listed as follows:

$$T_2^{-1} = \text{Transpose of } \frac{(-1)^{i+j}\det((\tilde{T}_2)_{ij})}{a_0^2 - a_1a_{-1}};$$

$$T_3^{-1} = \text{Transpose of } \frac{(-1)^{i+j}\det((\tilde{T}_3)_{ij})}{\dfrac{(a_0^2-a_1a_{-1})(a_0^2-a_2a_{-2})-(a_1a_0-a_2a_{-1})(a_0a_{-1}-a_1a_{-2})}{a_0}};$$

$$T_4^{-1} = \text{Transpose of } \frac{(-1)^{i+j}\det((\tilde{T}_4)_{ij})}{\det(T_4)}; \text{ where}$$

$$\det T_4 = \{[(a_0^2-a_1a_{-1})(a_0^2-a_2a_{-2})-(a_1a_0-a_2a_{-1})(a_0a_{-1}-a_1a_{-2})]$$
$$\times[(a_0^2-a_1a_{-1})(a_0^2-a_3a_{-3})-(a_2a_0-a_3a_{-1})(a_0a_{-2}-a_1a_{-3})]$$
$$-[(a_0^2-a_1a_{-1})(a_1a_0-a_3a_{-2})-(a_2a_0-a_3a_{-1})(a_0a_{-1}-a_1a_{-2})]$$
$$\times[(a_0^2-a_1a_{-1})(a_0a_{-1}-a_2a_{-3})-(a_1a_0-a_2a_{-1})(a_0a_{-2}-a_1a_{-3})]\}$$
$$\div\{a_0^2(a_0^2-a_1a_{-1})\};$$

$$T_5^{-1} = \text{Transpose of } \frac{(-1)^{i+j}\det((\tilde{T}_5)_{ij})}{\det(T_5)}; \text{ where}$$

$$\det T_5 = \{\{[(a_0^2-a_1a_{-1})(a_0^2-a_2a_{-2})-(a_1a_0-a_2a_{-1})(a_0a_{-1}-a_1a_{-2})]$$
$$\times[(a_0^2-a_1a_{-1})(a_0^2-a_3a_{-3})-(a_2a_0-a_3a_{-1})(a_0a_{-2}-a_1a_{-3})]$$
$$-[(a_0^2-a_1a_{-1})(a_1a_0-a_3a_{-2})-(a_2a_0-a_3a_{-1})(a_0a_{-1}-a_1a_{-2})]$$
$$\times[(a_0^2-a_1a_{-1})(a_0a_{-1}-a_2a_{-3})-(a_1a_0-a_2a_{-1})(a_0a_{-2}-a_1a_{-3})]\}$$
$$\times\{[(a_0^2-a_1a_{-1})(a_0^2-a_2a_{-2})-(a_1a_0-a_2a_{-1})(a_0a_{-1}-a_1a_{-2})]$$
$$\times[(a_0^2-a_1a_{-1})(a_0^2-a_4a_{-4})-(a_3a_0-a_4a_{-1})(a_0a_{-3}-a_1a_{-4})]$$
$$-[(a_0^2-a_1a_{-1})(a_2a_0-a_4a_{-2})-(a_3a_0-a_4a_{-1})(a_0a_{-1}-a_1a_{-2})]$$
$$\times[(a_0^2-a_1a_{-1})(a_0a_{-2}-a_2a_{-4})-(a_1a_0-a_2a_{-1})(a_0a_{-3}-a_1a_{-4})]\}$$
$$-\{[(a_0^2-a_1a_{-1})(a_0^2-a_2a_{-2})-(a_1a_0-a_2a_{-1})(a_0a_{-1}-a_1a_{-2})]$$
$$\times[(a_0^2-a_1a_{-1})(a_1a_0-a_4a_{-3})-(a_3a_0-a_4a_{-1})(a_0a_{-2}-a_1a_{-3})]$$
$$-[(a_0^2-a_1a_{-1})(a_2a_0-a_4a_{-2})-(a_3a_0-a_4a_{-1})(a_0a_{-1}-a_1a_{-2})]$$
$$\times[(a_0^2-a_1a_{-1})(a_0a_{-1}-a_2a_{-3})-(a_1a_0-a_2a_{-1})(a_0a_{-2}-a_1a_{-3})]\}$$
$$\times\{[(a_0^2-a_1a_{-1})(a_0^2-a_2a_{-2})-(a_1a_0-a_2a_{-1})(a_0a_{-1}-a_1a_{-2})]$$
$$\times[(a_0^2-a_1a_{-1})(a_0a_{-1}-a_3a_{-4})-(a_2a_0-a_3a_{-1})(a_0a_{-3}-a_1a_{-4})]$$
$$-[(a_0^2-a_1a_{-1})(a_1a_0-a_3a_{-2})-(a_2a_0-a_3a_{-1})(a_0a_{-1}-a_1a_{-2})]$$

$$\times[(a_0^2-a_1a_{-1})(a_0a_{-2}-a_2a_{-4})-(a_1a_0-a_2a_{-1})(a_0a_{-3}-a_1a_{-4})]\}\}$$
$$\div\{a_0^3(a_0^2-a_1a_{-1})^2[(a_0^2-a_1a_{-1})(a_0^2-a_2a_{-2})-(a_1a_0-a_2a_{-1})(a_0a_{-1}-a_1a_{-2})]\}\}$$

Example: Let $a_4 = 2, a_3 = 7, a_2 = 4, a_1 = 15, a_0 = 6, a_{-1} = 21, a_{-2} = 8, a_{-3} = 28, a_{-4} = 10,$ then $T_5 = L_5U_5$, where

$$T_5 = \begin{bmatrix} 6 & 21 & 8 & 28 & 10 \\ 14 & 6 & 21 & 8 & 28 \\ 4 & 14 & 6 & 21 & 8 \\ 7 & 4 & 14 & 6 & 21 \\ 2 & 7 & 4 & 14 & 6 \end{bmatrix};$$

$$L_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{7}{3} & 1 & 0 & 0 & 0 \\ \frac{2}{3} & 0 & 1 & 0 & 0 \\ \frac{7}{6} & \frac{41}{86} & \frac{917}{258} & 1 & 0 \\ \frac{1}{3} & 0 & 2 & 0 & 1 \end{bmatrix};$$

$$U_5 = \begin{bmatrix} 6 & 21 & 8 & 28 & 10 \\ 0 & -43 & \frac{7}{3} & -\frac{172}{3} & \frac{14}{3} \\ 0 & 0 & \frac{2}{3} & \frac{7}{3} & \frac{4}{3} \\ 0 & 0 & 0 & -\frac{5903}{774} & \frac{917}{387} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix};$$

and

$$\det T_5 = \prod_{i=1}^{5} U_5(i,i)$$
$$= 6 \times (-43) \times \frac{2}{3} \times \left(-\frac{5903}{774}\right) \times 0$$
$$= 0.$$

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