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Instability of soliton solutions to the Schamel-nonlinear Schrödinger equation

Sarun Phibanchon, and Michael A. Allen

Abstract—A variational method is used to obtain the growth rate of a transverse long-wavelength perturbation applied to the soliton solution of a nonlinear Schrödinger equation with a three-half order potential. We demonstrate numerically that this unstable perturbed soliton will eventually transform into a cylindrical soliton.

Keywords—soliton, instability, variational method, spectral method

I. Introduction

A solitary wave or soliton is an aperiodic wave which can occur in continuous media such as water or plasma. Solitary waves retain their shape as a result of the balance between nonlinearity and dispersion. One of the most important equations which describes such waves is the Korteweg-de Vries (KdV) equation,

$$\phi_t + \phi\phi_x + \phi_{xxx} = 0,$$

in which the subscripts x and t denote differentiation with respect to space and time, respectively (see e.g. [1] for further details). Although originally derived for water, the KdV equation occurs in many other contexts. It governs weakly nonlinear ion-acoustic waves in plasma when the electrons have a Maxwellian distribution [2]. In that case ϕ is the electrostatic potential. Schamel proposed allowing for the trapping of some of the electrons on ion-acoustic waves [3]. The free and trapped electrons have different temperatures, although both still have Maxwellian distributions. This leads a modified equation for ion-acoustic waves,

$$\phi_t + \phi^{1/2}\phi_x + \phi_{xxx} = 0,$$

now known as the Schamel equation.

Solitons also occur in other contexts such as fibre optics [4] and laser physics [5]. The equation governing solitons in these applications is the cubic nonlinear Schrödinger (cNLS) equation,

$$\mathrm{i}\phi_t + \phi_{xx} + |\phi|^2\phi = 0.$$

A connection between the KdV and cNLS equations can be made within the context of Madelung's fluid [6]. If one applies the same approach to the Schamel equation one obtains the Schamel-nonlinear Schrödinger (SNLS) equation,

$$\mathrm{i}\phi_t + \phi_{xx} + |\phi|^{1/2}\phi = 0. \tag{1}$$

Sarun Phibanchon is with Faculty of Science and Art, Burapha University, Chanthaburi, 57 M.1 Kamong, Thamai, Chanthaburi, Thailand 22170 e-mail: sarunp@buu.ac.th

Michael A. Allen is with Physics Department, Mahidol University Rama 6 Road, Bangkok 10400 Thailand e-mail: frmaa@mahidol.ac.th

where the real part of ϕ is the electrostatic potential. The soliton solution of (1) is

$$\phi(x,t) = 400\eta^4 \mathrm{e}^{\mathrm{i}\{\frac{1}{2}Vx + (16\eta^2 - V^2/4)t\}} \operatorname{sech}^4 \eta(x - Vt - x_0) \tag{2}$$

where V is the wave speed and η and x_0 are real constants. As (1) is not integrable and thus does not have analytical N-soliton solutions, an investigation of the collisions of these solitons in one-dimension was performed numerically by the authors [7]. Like the solitons in the integrable cNLS equation, they were found to pass through one another and thus demonstrated their stability with respect to perturbations in the direction of propagation.

Here we consider the two-dimensional form of the SNLS equation,

$$\mathrm{i}\phi_t + \phi_{xx} + \phi_{yy} + |\phi|^{1/2}\phi = 0. \tag{3}$$

In the next section we use a variational method applied to this equation to determine the growth rate of transverse instabilities of the 1-d soliton solution (2). As the SNLS equation is not integrable, there is no possibility of an analytical solution for describing the subsequent behaviour of the unstable perturbed solitons. In section III we therefore use a numerical method to determine the fate of the solitons.

II. Growth rate of instabilities

Large-wavelength perturbations to solitons in plasmas were studied by Rowlands [8]. For small t, the perturbed soliton can be written as

$$\phi(x,y,t) = \phi_0 + \epsilon(u + \mathrm{i}v)\mathrm{e}^{\gamma t + \mathrm{i}ky} \tag{4}$$

in which u, v are functions of x only, ϵ is a small parameter, ϕ_0 is the soliton solution at $t = 0$, k is the wavenumber of the perturbation, and γ is the growth rate of the perturbation. After substituting (4) into (3) and neglecting $O(\epsilon^2)$ terms, we obtain the coupled differential equations,

$$\begin{aligned} u_{xx} + \left(\tfrac{3}{2}\phi_0^{1/2} - 16\eta^2 - k^2\right)u &= \gamma v, \\ v_{xx} + \left(\phi_0^{1/2} - 16\eta^2 - k^2\right)v &= -\gamma u. \end{aligned} \tag{5}$$

Following the application of the variational method given in [9] to the cNLS equation, we start with the integral of the Lagrangian density for (5),

$$\mathcal{S} = \int_{-\infty}^{\infty} \mathcal{L}(u, u_x, v, v_x; x)\,\mathrm{d}x \tag{6}$$

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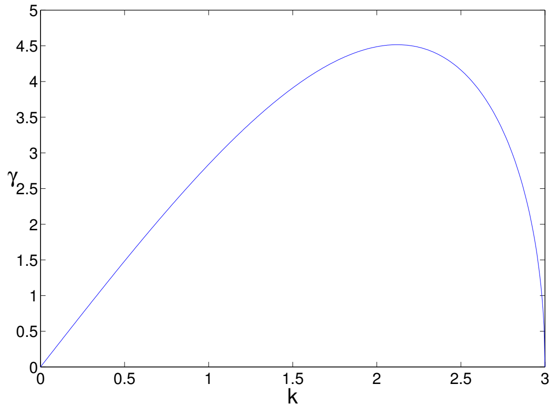

Fig. 1. The growth rate curve for large-wavelength perturbations of solitons as determined by a variational method.

where

$$\mathcal{L} = \frac{u_x^2}{2} - \frac{v_x^2}{2} + \left(16 + k^2 - \frac{3\phi_0^{1/2}}{2}\right)\frac{u^2}{2} + \left(\phi_0^{1/2} - 16\eta^2 - k^2\right)\frac{v^2}{2} + \gamma uv.$$

As trial functions we use the $k = 0$ solutions to (5), namely, $u = \alpha \operatorname{sech}^5 x$ and $v = \beta \operatorname{sech}^4 x$, where α and β are constants. To obtain expressions for α and β, we substitute the trial functions into (6) to obtain

$$\mathcal{S} = \frac{128}{315}\alpha^2\left(k^2 - 9\right) - \frac{16}{35}\beta^2 k^2 + \frac{35\pi}{128}\alpha\beta\gamma.$$

We then use $\delta S = 0$ to obtain

$$\frac{\partial \mathcal{S}}{\partial \alpha} = \frac{\partial \mathcal{S}}{\partial \beta} = 0.$$

These give coupled equations for α and β, and after eliminating these constants we obtain the growth rate

$$\gamma^2 = \frac{134217728}{13505625\pi^2}k^2(9 - k^2).$$

The growth rate is thus real for $0 \leq k \leq 3$. The maximum growth rate of 4.52 occurs at $k = 2.12$ (Fig. 1).

III. Evolution of Perturbed Solitons

To study the evolution of the perturbed soliton, we express the SNLS equation in the form

$$\frac{d\phi}{dt} = -\mathrm{i}\left(F^{-1}\left(\left(\xi^2 + \chi^2\right)F(\phi)\right) - |\phi|^{1/2}\phi\right)$$

where F and F^{-1} denote the Fourier and inverse Fourier transforms, respectively. These are obtained in the numerical scheme using the discrete Fourier transform

$$[F(\phi)]_{p,q} = \sum_{l=0}^{N_x-1}\sum_{m=0}^{N_y-1}\phi_{l,m}e^{i(\xi_p x_l + \chi_q y_m)}$$

where N_x and N_y are the number of mesh points in the x and y directions, $(x_l, y_m) = (lL_x/N_x, mL_y/N_y)$, L_x and L_y are the lengths of the domain in the x and y directions, and $(\xi_p, \chi_q) = 2\pi(p/L_x, q/L_y)$ for $p = 0, \ldots, N_x - 1$ and $q = 0, \ldots, N_y - 1$. The Runge-Kutta method [10] was used for the time derivative.

The initial conditions used were

$$\phi_0(x) = 400\eta^4[1 + \epsilon\cos(2\pi y/L_y)]\mathrm{e}^{\mathrm{i}Vx/2}\operatorname{sech}^4\eta(x - x_0)$$

where x_0 denotes a starting point. In Fig. 2 we show the results for the case $L_x = 120$, $L_y = 25$, $\eta = 0.25$, $\epsilon = 0.01$, $\Delta t = 0.001$, $N_x = 128$, $N_y = 64$, $V = 2$. The perturbed plane soliton initial condition is shown in Fig. 2a. Fig. 2b–f show how the unstable soliton is transformed into a single cylindrical soliton.

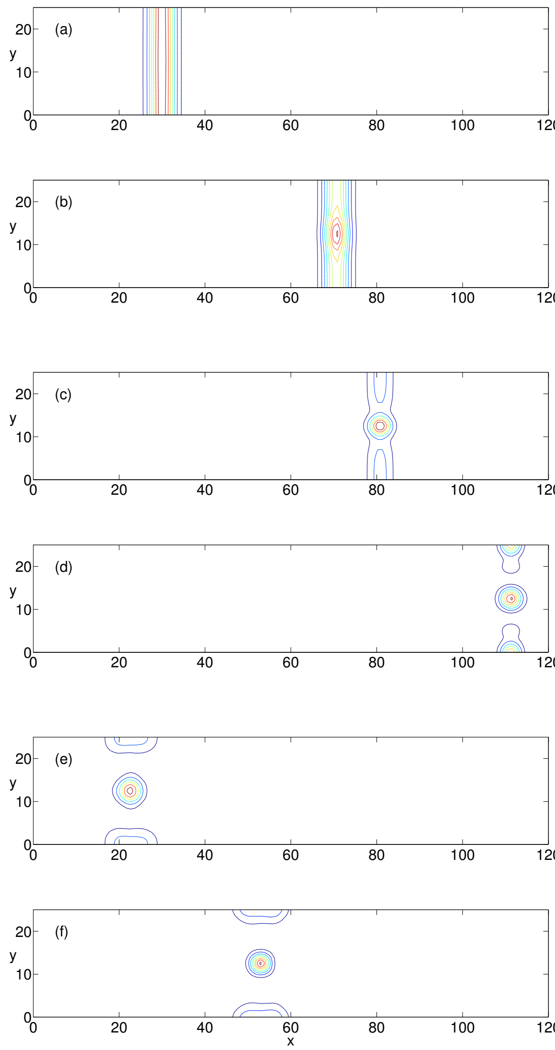

Fig. 2. Contour plot of $|\phi|$ showing the time evolution of a perturbed plane soliton with $V = 2.0$, $\eta = 0.25$: (a) $t = 0.0$, (b) $t = 0.002$, (c) $t = 0.0025$, (d) $t = 0.004$, (e) $t = 0.0175$, (f) $t = 0.019$.

IV. Discussion

The decay of unstable perturbed 1-d solitons into higher-dimensional solitons has been observed for a number of other equations including generalizations of the KdV equation [11], [12]. In future work it would be of interest to further study the nature of the cylindrical solitons seen here, and in particular find out what happens when they collide.

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Sarun Phibanchon is a lecturer at the Faculty of Science and Art, Burapha University, Chanthaburi campus. He received his B.Sc. and Ph.D. in Physics from Mahidol University, Bangkok in 1998 and 2006, respectively. In 2001, he also received his M.Sc. (Physics) from Chulalongkorn University, Bangkok. He mainly studies unstable nonlinear waves in plasmas and related fields.

Michael A. Allen He is an Assistant Professor of Physics at Mahidol University, Thailand. He received his B.A. in Physics from Oxford University in 1990 and his Ph.D. in 1994 from Warwick University. He works on a number of areas related to nonlinear systems.