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The Game of Maundy Block

Alessandro Cincotti

Abstract—The game of Maundy Block is the three-player variant of Maundy Cake, a classical combinatorial game. Even though to determine the solution of Maundy Cake is trivial, solving Maundy Block is challenging because of the identification of *queer* games, i.e., games where no player has a winning strategy.

Keywords—Combinatorial game, Maundy Cake, Three-player partizan games.

I. Introduction

THE game of Maundy Block is a three-player version of Maundy Cake [1]. Every instance of this game is defined as a set of blocks of integer side-lengths, with edges parallel to the $x-$, $y-$, and $z-$ axes. A legal move for Left is to divide one of the blocks into *any* number of blocks of *equal* integer side-length by means of a certain number of cuts perpendicular to the $x-$ axis; analogously, we define the legal moves for Center and Right. Players take turns making legal moves in cyclic fashion (..., Left, Center, Right, Left, Center, Right, ...). When one of the three players is not more able to move, he/she leaves the game and the remaining players continue in alternation until one of them cannot move. Then that player leaves the game and the remaining player is the winner.

Definition 1: Given a positive integer $n \geq 2$, the prime factorization is written $n = p_1 p_2 \ldots p_k$ where the p_is are the k prime factors. We define $d(n) = k$ and $d(1) = 0$.

We recall that in the game of Maundy Cake the outcome for a l by r rectangle depends on the dimension of l and r as shown in Table I.

TABLE I

	Left starts	Right starts
$d(l) > d(r)$	Left wins	Left wins
$d(l) < d(r)$	Right wins	Right wins
$d(l) = d(r)$	Right wins	Left wins

II. Three-Player Partizan Games

For the sake of self-containment we recall the basic definitions and main results concerning a mathematical theory to classify three-player partizan games [2]. Such a theory is an extension of Conway's theory of partizan games [3] and, as a consequence, it is both a theory of games and a theory of numbers.

Definition 2: If L, C, R are any three sets of numbers previously defined and

1) no element of L is $\geq_L$ any element of $C \cup R$, and
2) no element of C is $\geq_C$ any element of $L \cup R$, and
3) no element of R is $\geq_R$ any element of $L \cup C$,

then $\{L|C|R\}$ is a number. All numbers are constructed in this way.

This definition for numbers is based on the definition and comparison operators for games given in the following two definitions.

Definition 3: If L, C, R are any three sets of games previously defined then $\{L|C|R\}$ is a game. All games are constructed in this way.

Definition 4: We say that

- $x \geq_L y$ iff ($y \geq_L$ no x^C, $y \geq_L$ no x^R and no $y^L \geq_L x$),
- $x \leq_L y$ iff $y \geq_L x$,
- $x \geq_C y$ iff ($y \geq_C$ no x^L, $y \geq_C$ no x^R and no $y^C \geq_C x$),
- $x \leq_C y$ iff $y \geq_C x$,
- $x \geq_R y$ iff ($y \geq_R$ no x^L, $y \geq_R$ no x^C and no $y^R \geq_R x$),
- $x \leq_R y$ iff $y \geq_R x$.

where x^L, x^C, x^R are respectively the typical elements of L, C, and R.

We write

- $x \ngeq_L y$ to mean that $x \geq_L y$ does not hold,
- $x \ngeq_C y$ to mean that $x \geq_C y$ does not hold,
- $x \ngeq_R y$ to mean that $x \geq_R y$ does not hold.

Definition 5: We say that

- $x =_L y$ iff ($x \geq_L y$ and $y \geq_L x$),
- $x >_L y$ iff ($x \geq_L y$ and $y \ngeq_L x$),
- $x <_L y$ iff $y >_L x$,
- $x =_C y$ iff ($x \geq_C y$ and $y \geq_C x$),
- $x >_C y$ iff ($x \geq_C y$ and $y \ngeq_C x$),
- $x <_C y$ iff $y >_C x$,
- $x =_R y$ iff ($x \geq_R y$ and $y \geq_R x$),
- $x >_R y$ iff ($x \geq_R y$ and $y \ngeq_R x$),
- $x <_R y$ iff $y >_R x$,
- $x = y$ if and only if ($x =_L y$, $x =_C y$, and $x =_R y$).

All the given definition are inductive, so that to decide whether $x \geq_L y$ we check the pairs (x^C, y), (x^R, y), and (x, y^L).

Theorem 1: For any number x

- $x^L <_L x, x <_L x^C, x <_L x^R$,
- $x^C <_C x, x <_C x^L, x <_C x^R$,
- $x^R <_R x, x <_R x^L, x <_R x^C$

and, for any two numbers x and y

- either $x \geq_L y$ or $y \geq_L x$,
- either $x \geq_C y$ or $y \geq_C x$,
- either $x \geq_R y$ or $y \geq_R x$.

Numbers are totally ordered with respect to $\geq_L$, $\geq_C$, and $\geq_R$ but games are partially-ordered, i.e., there exist games x and y for which we have neither $x \geq_L y$ nor $y \geq_L x$.

Definition 6: We define the sum of two numbers as follows

$$x + y = \{x^L + y, x + y^L | x^C + y, x + y^C | x^R + y, x + y^R\}.$$

Alessandro Cincotti is with the School of Information Science, Japan Advanced Institute of Science and Technology, 1-1 Asahidai, Nomi, Ishikawa 923-1292 Japan, (phone: +81-761-51-1293; fax: +81-761-51-1149; email: cincotti@jaist.ac.jp).

TABLE II

Class	Left starts	Center starts	Right starts
$=$	Right wins	Left wins	Center wins
$>_L$	Left wins	Left wins	Left wins
$>_C$	Center wins	Center wins	Center wins
$>_R$	Right wins	Right wins	Right wins
$=_{LC}$	Center wins	Left wins	Center wins
$=_{LR}$	Right wins	Left wins	Left wins
$=_{CR}$	Right wins	Right wins	Center wins
$<_{CR}$	?	Left wins	Left wins
$<_{LR}$	Center wins	?	Center wins
$<_{LC}$	Right wins	Right wins	?
$<$	?	?	?

All numbers can be classified in 11 outcome classes as shown in Table II. For further details, please refer to [2].

III. Classifying the Instances of Maundy Block

Theorem 2: Let

$$G = [l_1, c_1, r_1] + \ldots + [l_i, c_i, r_i] + \ldots + [l_n, c_n, r_n]$$

be a general instance of Maundy Block. Then, G is a number.

Proof: Let $G = \{G^L | G^C | G^R\}$ be a general instance of Maundy Block. By induction hypothesis, G^L, G^C, and G^R are numbers; moreover, for every couple of options G^L and G^C, we can distinguish two different subcases:

1) if G^L and G^C concern the same block then

$$G^L <_L G^{LC} <_L \ldots <_L G^{LC\ldots C} \equiv G^{CL\ldots L} <_L \ldots <_L G^{CL} <_L G^C$$

It follows $G^L <_L G^C$ where the number of center options following G^L is equal to the number of blocks created by G^L and the number of left options following G^C is equal to the number of blocks created by G^C.

2) if G^L and G^C concern two different blocks then

$$G^L <_L G^{LC} \equiv G^{CL} <_L G^C \Rightarrow G^L <_L G^C$$

In the same way, we prove that $G^L <_L G^R$, $G^C <_C G^L$, $G^C <_C G^R$, $G^R <_R G^L$, and $G^R <_C G^C$. ∎

Example 1: Let $G = [3, 2, 4]$ be a block of Maundy Block. We observe that

$$\begin{aligned} G^L &= [1,2,4] + [1,2,4] + [1,2,4] \\ &<_L [1,1,4] + [1,1,4] + [1,2,4] + [1,2,4] \\ &<_L [1,1,4] + [1,1,4] + [1,1,4] + [1,1,4] + [1,2,4] \\ &<_L [1,1,4] + [1,1,4] + [1,1,4] + [1,1,4] + [1,1,4] + \\ &\quad [1,1,4] \\ &<_L [1,1,4] + [1,1,4] + [1,1,4] + [3,1,4] \\ &<_L [3,1,4] + [3,1,4] \\ &= G^C \end{aligned}$$

Theorem 3: In the game of Maundy Block

1) $G = [1, 1, 1] = 0$
2) $G = [l, 1, 1] >_L 0, l > 1$
3) $G = [1, c, 1] >_C 0, c > 1$
4) $G = [1, 1, r] >_R 0, r > 1$

Proof:

1) Trivial.
2) By induction hypothesis $G^L \geq_L 0$ and $G >_L 0$.
3) Analogous to (2).
4) Analogous to (2).

∎

Theorem 4: In the game of Maundy Block

1) if $d(l) = d(c)$ then $G = [l, c, 1] =_{LC} 0$,
2) if $d(l) > d(c)$ then $G = [l, c, 1] >_L 0$,
3) if $d(l) < d(c)$ then $G = [l, c, 1] >_C 0$,
4) if $d(l) = d(r)$ then $G = [l, 1, r] =_{LR} 0$,
5) if $d(l) > d(r)$ then $G = [l, 1, r] >_L 0$,
6) if $d(l) < d(r)$ then $G = [l, 1, r] >_R 0$,
7) if $d(c) = d(r)$ then $G = [1, c, r] =_{CR} 0$,
8) if $d(c) > d(r)$ then $G = [1, c, r] >_C 0$,
9) if $d(c) < d(r)$ then $G = [1, c, r] >_R 0$,

where $l, c, r > 1$.

Proof:

1) A generic left option G^L is represented by

$$[l_1, c, 1] + [l_2, c, 1] + \ldots + [l_k, c, 1]$$

where $l_1 = l_2 = \ldots = l_k$ and $d(l_i) < d(c)$ for all $1 \leq i \leq k$. By induction hypothesis, $[l_i, c, 1] >_C 0$ for all $1 \leq i \leq k$ therefore $G^L >_C 0$.
By similar reasoning, we can prove that $G^C >_L 0$ therefore $G =_{LC} 0$.

2) We observe that there exists at least a left option

$$G^L = [l_1, c, 1] + \ldots + [l_k, c, 1]$$

where $d(l_i) \geq d(c)$ therefore, by induction hypothesis, either $G^L >_L 0$ or $G^L =_{LC} 0$. In both cases we have $G >_L 0$.

3) Analogous to (2).

The other 6 cases can be proved analogously. ∎

Theorem 5: Let $G = [l, c, r]$ be a block of Maundy Block where $l, c, r > 1$. If

- $d(l) < d(c) + d(r)$
- $d(c) < d(l) + d(r)$
- $d(r) < d(l) + d(c)$

then $G < 0$ else one of the following 6 cases occurs

1) if $d(l) > d(c) + d(r)$ then $G >_L 0$,
2) if $d(l) = d(c) + d(r)$ then $G <_{CR} 0$,
3) if $d(c) > d(l) + d(r)$ then $G >_C 0$,
4) if $d(c) = d(l) + d(r)$ then $G <_{LR} 0$,
5) if $d(r) > d(l) + d(c)$ then $G >_R 0$,
6) if $d(r) = d(l) + d(c)$ then $G <_{LC} 0$.

Proof: Let's assume that $d(l) < d(c) + d(r)$, $d(c) < d(l) + d(r)$, and $d(r) < d(l) + d(c)$. We have two subcases:

- $d(l) = 1$. In this case, $d(c) = d(r)$ therefore

$$G^L = [1, c, r] + \ldots + [1, c, r] =_{CR} 0$$

as shown in the previous theorem.

- $d(l) > 1$. In this case, there exist at least a left option

$$G^L = [l_1, c, r] + \ldots + [l_k, c, r]$$

where $d(l_i) = d(l) - 1$ for all $1 \leq i \leq k$. By induction hypothesis, $[l_i, c, r]$ is

- $<_{LR} 0$ if $d(c) = d(l_1) + d(r)$,
- $<_{LC} 0$ if $d(r) = d(l_1) + d(c)$,
- < 0 otherwise.

Therefore G^L is $<_{LR} 0$, $<_{LC} 0$, or < 0.

It follows that for each of the 2 aforementioned cases there exists at least a left option $G^L \leq_C 0$ and $G^L \leq_R 0$ therefore $G <_C 0$ and $G <_R 0$. Analogously, we can prove that $G <_L 0$ ($G <_R 0$) considering

$$G^C = [l, c_1, r] + \ldots + [l, c_k, r]$$

where $d(c_i) = d(c) - 1$ for all $1 \leq i \leq k$ therefore $G < 0$.
Now, let's suppose that the hypothesis $d(l) < d(c) + d(r)$, $d(c) < d(l) + d(r)$, and $d(r) < d(l) + d(c)$ is false. In this case, only one of the 6 cases mentioned previously can be true.

1) In this case, there exists at least a left option

$$G^L = [l_1, c, r] + \ldots + [l_k, c, r]$$

where $d(l_i) = d(l) - 1$ such that by induction hypothesis either $G^L >_L 0$ or $G^L <_{CR} 0$. In both cases we have $G >_L 0$.

2) We observe that, for any center option

$$G^C = [l, c_1, r] + \ldots + [l, c_k, r]$$

$d(c_i) < d(c) \Rightarrow d(l) > d(c_i) + d(r)$ for all $1 \leq i \leq k$ therefore, by induction hypothesis, $G^C >_L 0$. In the same way, we prove that $G^R >_L 0$. Let's consider a generic left option

$$G^L = [l_1, c, r] + \ldots + [l_k, c, r]$$

where $d(l_i) < d(l)$ for all $1 \leq i \leq k$. It follows that $d(l_i) \ngeq d(c) + d(r)$ therefore $[l_i, c, r]$ can only be $>_C 0$, $>_R 0$, $=_{CR} 0$, $<_{LR} 0$, $<_{LC} 0$, or < 0. In any case, $G^L <_L 0$ and therefore $G <_{CR} 0$.

We can prove the other 4 cases analogously. ∎

Theorem 6: Let

$$G = [l_1, c_1, r_1] + \ldots + [l_i, c_i, r_i] + \ldots + [l_n, c_n, r_n]$$

be a general instance of Maundy Block. If $d(l_i) \leq d(c_i)$ for all $1 \leq i \leq n$ and Left has to play then Left has not a winning strategy.

Proof: Let's suppose that Left plays in the i-th block

$$[l_i, c_i, r_i] \rightarrow [l_{i_1}, c_i, r_i] + \ldots + [l_{i_k}, c_i, r_i].$$

In every of these blocks $d(l_{i_j}) < d(c_i)$ for all $1 \leq j \leq k$ and Center can play in any of these blocks.
Successively, Right has to play but we observe that his/her move cannot affect the relation between Left and Center inside a block. When Left will move again, in every block $[l, c, r]$, we have $d(l) \leq d(c)$ therefore, by induction hypothesis, Left has not a winning strategy. ∎

The following theorem can be proven in the same way.

Theorem 7: Let

$$G = [l_1, c_1, r_1] + \ldots + [l_i, c_i, r_i] + \ldots + [l_n, c_n, r_n]$$

be a general instance of Maundy Block. If $d(l_i) \leq d(r_i)$ for all $1 \leq i \leq n$ and Left has to play then Left has not a winning strategy.
Analogously, we can get the same results for Center and Right.
The previous theorems give us some further information about the outcome of $G = [l, c, r] <_{CR} 0$, $G = [l, c, r] <_{LR} 0$, $G = [l, c, r] <_{LC} 0$, and $G = [l, c, r] < 0$ as shown in Table III and IV.

TABLE III

	Left starts	Center starts	Right starts
$G <_{CR} 0$	Left wins/q	Left wins	Left wins
$G <_{LR} 0$	Center wins	Center wins/q	Center wins
$G <_{LC} 0$	Right wins	Right wins	Right wins/q

q = queer.

TABLE IV

$G < 0$	Left starts	Center starts	Right starts
$L > C$, $L > R$	Left wins/q	Left wins/q	Left wins/q
$C > L$, $C > R$	Center wins/q	Center wins/q	Center wins/q
$R > L$, $R > C$	Right wins/q	Right wins/q	Right wins/q
$L = C$, $L > R$	Center wins/q	Left wins/q	Center wins/q
$L = R$, $L > C$	Right wins/q	Left wins/q	Left wins/q
$C = R$, $C > L$	Right wins/q	Right wins/q	Center wins/q
$L = C$, $L = R$	Right wins/q	Left wins/q	Center wins/q

$L = d(l)$,C = d(c), $R = d(r)$, q = queer.

We briefly recall the definition of queer game introduced by Propp [4]:

Definition 7: A position in a three-player combinatorial game is called queer if no player can force a win.

IV. $[25, 2, 2]$ IS A QUEER GAME

Let's consider the game $G = [25, 2, 2]$. We observe that $d(25) = d(2) + d(2)$ therefore $G <_{CR} 0$. When Center or Right makes the first move Left has always a winning strategy. When Left makes the first move we know, by previous theorems, that neither Center nor Right has a winning strategy; therefore, we have two possible cases: either Left has a winning strategy or G is a queer game. We show that Left has not a winning strategy.
In the beginning, Left has only one plausible move:

$$[25, 2, 2] \rightarrow [5, 2, 2] + [5, 2, 2] + [5, 2, 2] + [5, 2, 2] + [5, 2, 2].$$

Successively, Center moves

$$[5, 2, 2] \rightarrow [5, 1, 2] + [5, 1, 2]$$

and Right moves

$$[5, 2, 2] \rightarrow [5, 2, 1] + [5, 2, 1]$$

obtaining the instance

$$[5, 1, 2] + [5, 1, 2] + [5, 2, 2] + [5, 2, 2] + [5, 2, 2] + [5, 2, 1] + [5, 2, 1].$$

Now, Left has 3 possible moves:

- If Left moves in $[5, 1, 2]$ we have

$$[5, 1, 2] \rightarrow [1, 1, 2] + [1, 1, 2] + [1, 1, 2] + [1, 1, 2] + [1, 1, 2].$$

In this case, Center moves

$$[5, 2, 2] \to [5, 1, 2] + [5, 1, 2]$$

and Right moves

$$[1, 1, 2] \to [1, 1, 1] + [1, 1, 1].$$

Now, Left has to move and you can check easily that he/she has not a winning strategy.

- If Left moves in $[5, 2, 2]$ we have

$$[5, 2, 2] \to [1, 2, 2]+[1, 2, 2]+[1, 2, 2]+[1, 2, 2]+[1, 2, 2].$$

In these 5 blocks, Center and Right can make 7 moves each one and Left can make only 6 moves in the other blocks therefore he/she has not a winning strategy.

- If Left moves in $[5, 2, 1]$ we have

$$[5, 2, 1] \to [1, 2, 1]+[1, 2, 1]+[1, 2, 1]+[1, 2, 1]+[1, 2, 1].$$

Analogous to the first case.

It is amazing to observe that both $[25, 2, 2]$ and $[4, 2, 2]$ are $<_{CR} 0$ but in $[4, 2, 2]$ Left has still a winning strategy when he/she makes the first move.

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