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Synchronization of Non-Identical Chaotic Systems with Different Orders Based on Vector Norms Approach

Rihab Gam, Anis Sakly, and Faouzi M'sahli

Abstract**—**A new strategy of control is formulated for chaos synchronization of non-identical chaotic systems with different orders using the Borne and Gentina practical criterion associated with the Benrejeb canonical arrow form matrix, to drift the stability property of dynamic complex systems. The designed controller ensures that the state variables of controlled chaotic slave systems globally synchronize with the state variables of the master systems, respectively. Numerical simulations are performed to illustrate the efficiency of the proposed method.

Keywords**—**Synchronization, Non-identical chaotic systems, Different orders, Arrow form matrix.

I. Introduction

OVER the last two decades, since the pioneering work of Pecora and Carroll in 1990 [1], synchronization of chaotic systems has attracted increasing attention in different fields of physics and engineering systems, such as in power converters, chemical reactions, biological systems, information processing, and especially for secure communication [18]. Basically, the chaos synchronization problem means making two systems oscillate in a synchronized manner. Given a chaotic system, which is considered as the master system, and another one, which is considered as the slave system, the dynamical behaviours of these two systems may be identical after a transient when the slave system is driven by a control input [12, 13]. To date, different techniques and methods have been proposed to achieve chaos synchronization such as impulsive control [19], adaptive control [20], sliding mode control, fuzzy control [21], optimal control [22], digital redesign control [23], active control law [2,9,15], and so on [7]. Throughout the present paper, the Benrejeb arrow form matrix [3, 6] is applied to synchronize two non-identical chaotic systems with different orders namely the Lorenz system and the Chen-Lee system.

It is proved that by applying the proposed control scheme, the variance of the synchronization error can converge to any arbitrarily small bound around zero.

Rihab Gam is with the National School of Engineers of Monastir ENIM, University of Monastir-Tunisia (e-mail: rihabgam@hotmail.com).

Anis Sakly is with the National School of Engineers of Monastir ENIM, University of Monastir-Tunisia (e-mail: Sakly_anis@yahoo.fr).

Faouzi M'sahli is with the National School of Engineers of Monastir ENIM, University of Monastir-Tunisia (e-mail: faouzi.msahli@enim.rnu.tn).

This paper is organized as follows. Section II deals with non-identical synchronization between Lorenz and Chen-Lee systems with different orders. In Section III, some concluding remarks are given.

Simulation results show that the proposed method can be successfully used in synchronization of chaotic systems.

II. Synchronization of Non-Identical Chaotic Systems with Different Orders

In this subsection, we study the problem of synchronization process of two different chaotic systems having different orders, namely the Chen-Lee system and the Lorenz system.

A. The Chen-Lee System

The Chen-Lee system is described by the following four couple first-order autonomous ordinary differential equations and is given by [4, 10]:

$$\begin{cases} \dot{x}_{1m}(t) = -x_{2m}(t)x_{3m}(t) + a_1 x_{1m}(t) \\ \dot{x}_{2m}(t) = x_{1m}(t)x_{3m}(t) + b_1 x_{2m}(t) \\ \dot{x}_{3m}(t) = \frac{1}{3}x_{1m}(t)x_{2m}(t) + c_1 x_{3m}(t) + \frac{1}{5}x_{4m}(t) \\ \dot{x}_{4m}(t) = d_1 x_{1m}(t) + \frac{1}{2}x_{2m}(t)x_{3m}(t) + \frac{1}{20}x_{4m}(t) \end{cases} \quad (1)$$

Where x_{1m}, x_{2m}, x_{3m} and x_{4m} are state variables, a_1, b_1, c_1 are three system parameters and d_1 is a constant defining the dynamic behaviours of the system.

Chaos of the Chen-Lee system appears, when initial conditions $(x_{1m0}, x_{2m0}, x_{3m0}, x_{4m0}) = (-0.1, 0.2, 0.3, -0.2)$ and parameters $(a_1, b_1, c_1, d_1) = (5, -10, -3.8, 1.3)$.

The time histories of the four Chen-Lee state variables are drawn in Fig. 1.

Its phase projection portraits with hyper-chaotic behaviour are shown in Fig. 2.

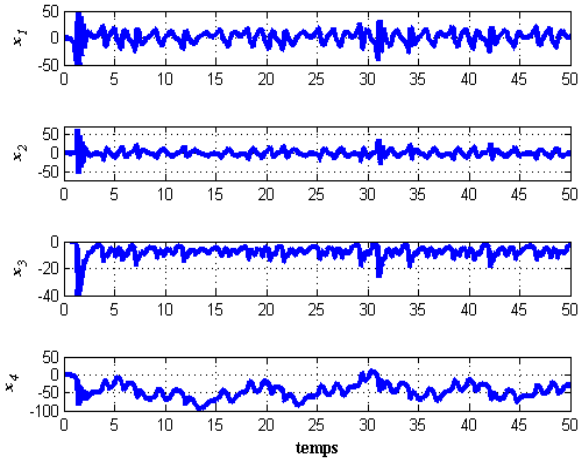

Fig. 1 Time histories of states of the Chen-Lee chaotic system

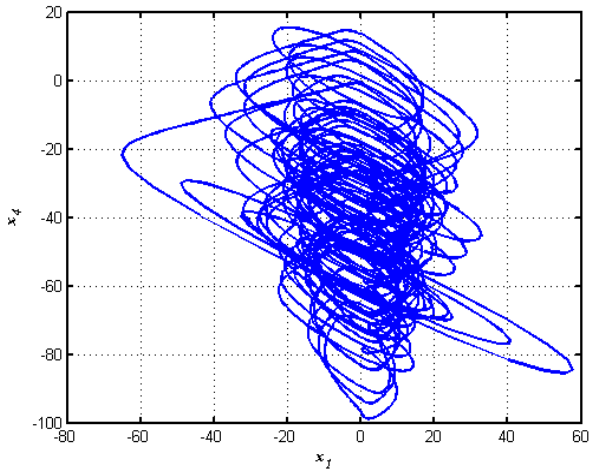

Fig. 2 Projections of phase portrait of chaotic Chen-Lee system with $a_1 = 5, b_1 = -10, c_1 = -3.8, d_1 = 1.3$

B. The Lorenz System

Here, we consider the following three coupled nonlinear autonomous first order differential equations, characterizing the Lorenz system [4, 11]

$$\begin{cases} \dot{x}_{1s}(t) = a\left(x_{2s}(t) - x_{1s}(t)\right) \\ \dot{x}_{2s}(t) = cx_{1s}(t) - x_{1s}(t)x_{3s}(t) - x_{2s}(t) \\ \dot{x}_{3s}(t) = x_{1s}(t)x_{2s}(t) - bx_{3s}(t) \end{cases} \quad (2)$$

Where x_{1s}, x_{2s}, x_{3s} are state variables, a, b, c are three system parameters. Chaos of the Lorenz system appears, when initial conditions $(x_{1s0}, x_{2s0}, x_{3s0}) = (-0.1, 0.2, 0.3)$ and parameters $a = 10, b = \frac{8}{3}, c = 28$.

The time histories of states of the Lorenz chaotic system are shown in Fig. 3 and its chaotic behaviours are drawn in Fig. 4.

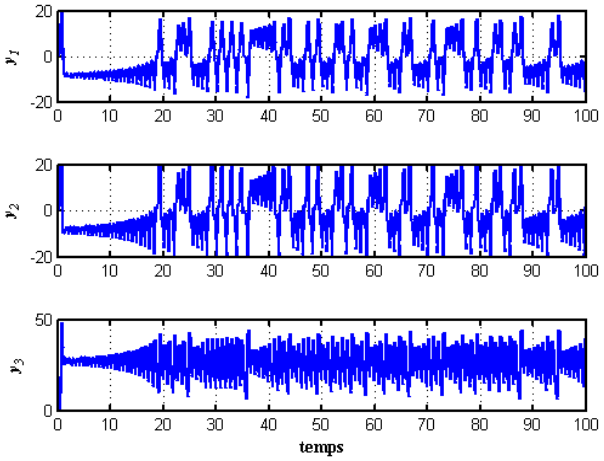

Fig. 3 Time histories of states of the Lorenz chaotic system

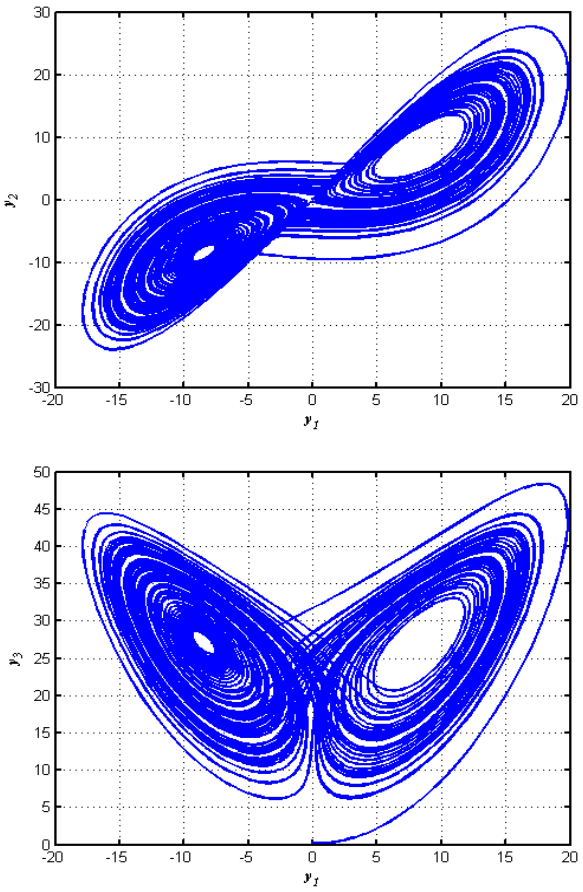

Fig. 4 Projections of phase portrait of chaotic Lorenz system with $a = 10, b = \frac{8}{3}, c = 28$

C. Problem Statement

We choose the Chen-Lee system as the drive system and the Lorenz system as the response system.

This implies that when the drive–response system is synchronized, the Lorenz system will follow the dynamics of the Chen-Lee system.

Let us consider the four order master Chen-Lee system given as

$$\begin{cases} \dot{x}_{1m}(t) = -x_{2m}(t)x_{3m}(t) + a_1 x_{1m}(t) \\ \dot{x}_{2m}(t) = x_{1m}(t)x_{3m}(t) + b_1 x_{2m}(t) \\ \dot{x}_{3m}(t) = \frac{1}{3}x_{1m}(t)x_{2m}(t) + c_1 x_{3m}(t) + \frac{1}{5}x_{4m}(t) \\ \dot{x}_{4m}(t) = d_1 x_{1m}(t) + \frac{1}{2}x_{2m}(t)x_{3m}(t) + \frac{1}{20}x_{4m}(t) \end{cases} \quad (3)$$

Which drives a third order Lorenz system given as?

$$\begin{cases} \dot{x}_{1s}(t) = a\left(x_{2s}(t) - x_{1s}(t)\right) + u_1(t) \\ \dot{x}_{2s}(t) = c x_{1s}(t) - x_{1s}(t)x_{3s}(t) - x_{2s}(t) + u_2(t) \\ \dot{x}_{3s}(t) = x_{1s}(t)x_{2s}(t) - b x_{3s}(t) + u_3(t) \end{cases} \quad (4)$$

Where $U(t) = \left[u_1(t), u_2(t), u_3(t)\right]^T$ is the active control function? Here, our objective is to determine the controller U which is required for system (4) to synchronize with system (3).

For this purpose, we define the error dynamics between (4) and (3) as

$$[e] = \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix} = \begin{bmatrix} x_{1s}(t) - x_{1m}(t) \\ x_{2s}(t) - x_{3m}(t) \\ x_{3s}(t) - x_{4m}(t) \end{bmatrix} \quad (5)$$

From Eq. (5), we have the following error dynamics:

$$\begin{cases} \dot{e}_1(t) = a\left(x_{2s}(t) - x_{1s}(t)\right) + x_{2m}(t)x_{3m}(t) - a_1 x_{1m}(t) + u_1(t) \\ \dot{e}_2(t) = c x_{1s}(t) - x_{1s}(t)x_{3s}(t) - x_{2s}(t) - \frac{1}{3}x_{1m}(t)x_{2m}(t) \\ -c_1 x_{3m}(t) - \frac{1}{5}x_{4m}(t) + u_2(t) \\ \dot{e}_3(t) = (x_{1s}(t)x_{2s}(t) - b x_{3s}(t) - d_1 x_{1m}(t) - \frac{1}{2}x_{2m}(t)x_{3m}(t) \\ -\frac{1}{20}x_{4m}(t) + u_3(t) \end{cases} \quad (6)$$

Which can be written under the following matrix description [4]?

$$\dot{e}(t) = A(.)e(t) + BU(t) + f_i(.) \quad (7)$$

With

$$e(t) = \left[e_1(t)\ e_2(t)\ e_3(t)\right]^T \quad (8)$$

$$A(.) = \begin{pmatrix} a_1 - a & a & 0 \\ c & c_1 - 1 & \frac{1}{5} \\ d_1 & x_{1s} & \frac{1}{20} - b \end{pmatrix} \quad (9)$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (10)$$

And

$$\begin{cases} f_1(.) = a_1 x_{1s}(t) - a x_{1m}(t) + x_{3m}(t)\left(a + x_{2m}(t)\right) \\ f_2(.) = c x_{1m}(t) - c_1 x_{2s}(t) - x_{3m}(t) \\ + x_{3s}(t)\left(-\frac{1}{5} - x_{1s}(t)\right) \\ f_3(.) = x_{1s}(t)\left(x_{3m}(t) - d_1\right) + b x_{3s}(t) + \frac{1}{20}x_{4m}(t) \\ -\frac{1}{2}x_{2m}(t)x_{3m}(t) \end{cases} \quad (11)$$

Our purpose is to design the most sufficient structure of controllers $u_i(t), i = 1,..,3$, which will make the response system fulfils synchronism with the drive system. We try, in the next part of the paper, to design a state feedback control law assuring the stabilization of the error dynamic system.

D. New Control Law Synchronizing Lorenz and Chen-Lee Systems [3]

Now, a control law must be designed to follow asymptotically the trajectories of the master attractor into those of the slave one.

By putting in prominent position the practical criterion of Borne and Gentina [5, 17], associated to the Benrejeb arrow form matrix [6, 16], we redefine the active control functions $u_i(t), i = 1,..,3$, as follows:

$$u_i(t) = -f_i(.) - k_{ij}(.)e_i \forall i, j = 1,..,3 \quad (12)$$

In such a way that the closed loop system

$$\dot{e}(t) = A(.)e(t) + Bu(t) + f_i(.)$$

$$A_c(.) = A(.) - BK(.)$$

$$= \begin{pmatrix} a_1 - a - k_{11}(.) & a - k_{12}(.) & -k_{13}(.) \\ c - k_{21}(.) & c_1 - 1 - k_{22}(.) & \frac{1}{5} - k_{23}(.) \\ d_1 - k_{31}(.) & x_{1s} - k_{32}(.) & \frac{1}{20} - b - k_{33}(.) \end{pmatrix} \quad (13)$$

Being described by a particular canonical matrix form, namely the Benrejeb arrow form matrix. To satisfy this aim,

the parameters of correction $k_{ij}(.), \forall i, j = 1,2,3; i \neq j$, can be chosen as follows:

$$\begin{cases} a - k_{12}(.) = 0 \\ k_{13}(.) = 0 \\ \frac{1}{5} - k_{23}(.) = 0 \\ c - k_{21}(.) = 0 \\ d_1 - k_{31}(.) = 0 \\ x_{1s} - k_{32}(.) = 0 \end{cases} \Rightarrow \begin{cases} k_{12}(.) = a \\ k_{13}(.) = 0 \\ k_{23}(.) = \frac{1}{5} \\ k_{21}(.) = c \\ k_{31}(.) = d_1 \\ k_{32}(.) = x_{1s}(t) \end{cases} \quad (14)$$

When the considered system (7) is stabilized by the feedback U, the error will converge to zero as $t \to +\infty$, which implies that the systems (3) and (4) are globally synchronized. To achieve this objective, U should be chosen such that the instantaneous gain matrix $K(.)$ defined by (12), is a 3×3 matrix.

The following theorem is due to the application of the classical Borne and Gentina stability criterion [6], associated to the particular canonical Benrejeb arrow form matrix.

Theorem 1. [3] The process, described by (13) is stabilized by the control law defined by (12), if the matrix $A_c(.)$, defined by (14), is in the arrow form and such that:

(i) the nonlinear elements are isolated in either one row or one column of the matrix $A_c(.)$

(ii) the diagonal elements, $a_{c_{ii}}(.)$, of the matrix $A_c(.)$ are such that:

$$a_{c_{ii}}(.) \langle 0 \, \forall i = 1,2,..,n-1 \quad (15)$$

(iii) there exist $\varepsilon \rangle 0$ such that:

$$a_{c_{nn}}(.) - \sum_{i=1}^{n-1} \left(\left| a_{c_{ni}}(.) a_{c_{in}}(.) \right| \right) a_{c_{ii}}^{-1}(.) \leq -\varepsilon \quad (16)$$

Corollary 1. [3] The process, described by (7) is stabilized by the control law defined by (12) if the characteristic matrix $A_c(.)$, defined by (14), is under the arrow form and such that:

(i) all the nonlinearities are located in either one row or one column of $A_c(.)$

(ii) the diagonal elements $a_{c_{ii}}(.), \forall i = 1,2,...,n-1$, of the matrix $A_c(.)$ are strictly negative

(iii) the products of the off-diagonal elements $a_{c_{ni}}(.) a_{c_{in}}(.), \forall i = 1,...,n-1$, of the matrix $A_c(.)$ are non-negative

(iv) the characteristic instantaneous polynomial $P_{A_c}(\lambda,.)$, defined by

$$P_{A_c}(\lambda,.) = \det\left(\lambda - A_c(.)\right) \quad (17)$$

is strictly positive for $\lambda = 0$

The parameters of correction $K_{ij}(.), \forall i, j = 1,..,3$, which ensure the stability of Eq. (13), can be chosen so that the following constraints are accomplished:

(i) all the nonlinearities are located in either one row or one column of the matrix $A_c(.)$

(ii) the diagonal elements of the matrix $A_c(.)$ are such that

$$\begin{cases} a_1 - a - k_{11}(.) \langle 0 \\ c_1 - 1 - k_{22}(.) \langle 0 \\ \frac{1}{20} - b - k_{33}(.) \langle 0 \end{cases} \Rightarrow \begin{cases} a_1 - a \langle k_{11}(.) \\ c_1 - 1 \langle k_{22}(.) \\ \frac{1}{20} - b \langle k_{33}(.) \end{cases} \quad (18)$$

(iii) there exist $\varepsilon \rangle 0$, such that

$$\left(\frac{1}{20} - b - k_{33}(.) \right) - \begin{pmatrix} (d_1 - k_{31}(.))(-k_{13}(.))(a_1 - a - k_{11}(.))^{-1} \\ (x_{1s}(t) - k_{32}(.)) \left(\frac{1}{5} - k_{23}(.) \right) (c_1 - 1 - k_{22}(.))^{-1} \end{pmatrix} \leq -\varepsilon \quad (19)$$

With both Eqs. (11) and (12), the error system (6) is reduced, in this particular case, to a nonlinear system with control inputs: $-k_{ij}(.) e_i, \forall i, j = 1,..,3$ as functions of the error states e_1, e_2, e_3.

We make the most appropriate choice for the instantaneous gain matrix $k(.) = \{k_{ij}(.)\}, \forall i, j = 1,..,3$ so that the synchronization, between the two non-identical systems (3) and (4), is fulfilled.

Among various choices of the gain matrix $K(.)$, one possible solution is the following:

$$k(.) = \begin{pmatrix} (a_1 - a) + 1 & a & 0 \\ c & c_1 & \frac{1}{5} \\ d_1 & x_{1s} & \left(\frac{1}{20} - b \right) + 1 \end{pmatrix} \quad (20)$$

The error defined by Eq. (5) will converge to zero as $t \to +\infty$ implying that system (4) will globally synchronize with system (3). This can be achieved by choosing the previous instantaneous gain matrix $K(.)$ such that

$$\begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix} = -\begin{pmatrix} (a_1 - a)+1 & a & 0 \\ c & c_1 & \frac{1}{5} \\ d_1 & x_{1s}(t) & \left(\frac{1}{20} - b\right)+1 \end{pmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix} - \begin{bmatrix} f_1(.) \\ f_2(.) \\ f_3(.) \end{bmatrix} \quad (21)$$

From equations (11) and (21), it has been proved that the state variables of the Lorenz system (4) should follow the state variables of the Chen-Lee system (3).

The error dynamics between the Chen Lee system and the Lorenz system when controller is deactivated is drawn in Fig. 5. Hence, synchronization has been provided thanks to the designed controller.

The error dynamics of the coupled Chen Lee and Lorenz systems when controller is switched on is shown in Fig. 6.

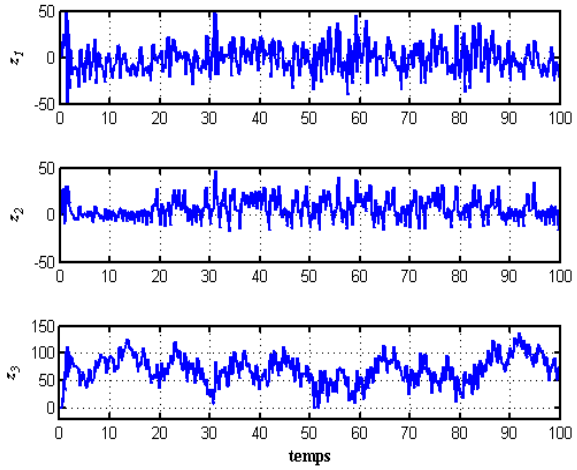

Fig. 5 Error dynamics between the Chen Lee system and the Lorenz system when controller is deactivated

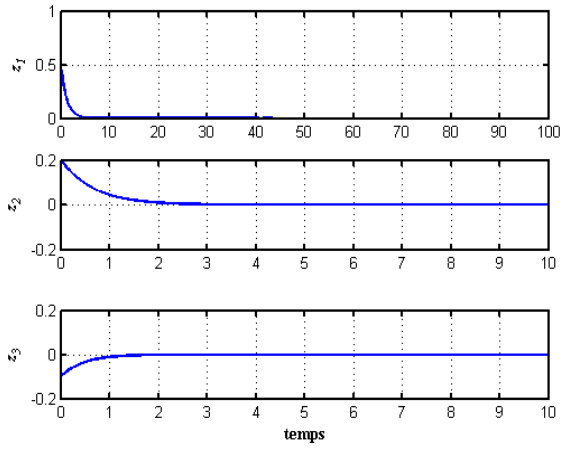

Fig. 6 Error dynamics of the coupled Chen Lee and Lorenz systems when controller is switched on

III. Conclusion

This paper deals with the synchronization of non-identical chaotic systems with different orders: The Chen-Lee system, as drive, and the Lorenz system, as response, using the practical stability criterion of Borne and Gentina, associated to the Benrejeb particular arrow form matrix description.

The different active controllers designed, with the control law described in this paper, ensure synchronization between the states of both slave system and the master system. Numerical simulations are also given to validate the synchronization approach and to prove its efficiency for non-identical systems having different orders.

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