

# Decomposition of Homeomorphism on Topological Spaces

Ahmet Z. Ozelik, Serkan Narli

**Abstract**—In this study, two new classes of generalized homeomorphisms are introduced and shown that one of these classes has a group structure. Moreover, some properties of these two homeomorphisms are obtained.

**Keywords**—Generalized closed set, homeomorphism, gsg-homeomorphism, sgs-homeomorphism.

## I. INTRODUCTION

LEVINE [9] has generalized the concept of closed sets to generalized closed sets. Bhattacharyya and Lahiri [2] have generalized the concept of closed sets to semi-generalized closed sets with the help of semi-open sets and obtained various topological properties. Arya and Nour [1] have defined generalized semi-open sets with the help of semi-openness and used them to obtain some characterizations of  $s$ -normal spaces. Devi, Balachandran and Maki [8] defined two classes of maps called semi-generalized homeomorphisms and generalized semi-homeomorphisms and also defined two classes of maps called sgc-homeomorphisms and gsc-homeomorphism. In this paper, we introduce two classes of maps called sgs-homeomorphisms and gsg-homeomorphisms and study their properties.

Throughout the present paper,  $(X, \tau)$  and  $(Y, \delta)$  denote topological spaces on which no separation axioms are assumed unless explicitly stated. Let  $A$  be a subset of  $X$ . We denote the interior of  $A$  (respectively the closure of  $A$ ) with respect to  $\tau$  by  $\text{Int}(A)$  (respectively  $\text{Cl}(A)$ )

## II. PRELIMINARIES

Since we shall use the following definitions and some properties, we recall them in this section.

**a.** A subset  $B$  of a topological space  $(X, \tau)$  is said to be semi-closed if there exists a closed set  $F$  such that  $\text{Int}(F) \subset B \subset F$ . A subset  $B$  of  $(X, \tau)$  is called a semi-open set if its complement  $X \setminus B$  is semi-closed in  $(X, \tau)$ . Every closed (respectively open)

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set is semi-closed (respectively semi-open) [3,5].

**b.** A mapping  $f : (X, \tau) \rightarrow (Y, \delta)$  is said to be semi-closed if the image  $f(F)$  of each closed set  $F$  in  $(X, \tau)$  is semi-closed in  $(Y, \delta)$ . Every closed mapping is semi-closed [10].

**c.** Let  $(X, \tau)$  be a topological space and  $A$  be a subset of  $X$ . Then, the semiinterior and semiclosure of  $A$  are defined by:

$$s\text{Int}(A) = \cup \{G_i : G_i \text{ is a semi-open in } X \text{ and } G_i \subset A\}$$

$$s\text{Cl}(A) = \cap \{K_i : K_i \text{ is a semi-closed in } X \text{ and } A \subset K_i\}$$

**d.** A subset  $B$  of a topological space  $(X, \tau)$  is said to be semi-generalized closed (written in short as sg-closed) if  $s\text{Cl}(B) \subset O$  whenever  $B \subset O$  and  $O$  is semi-open [2]. The complement of a semi-generalized closed set is called a semi-generalized open. Every semi-closed set is sg-closed. The concepts of g-closed sets [7] and sg-closed sets are, in general, independent. The family of all sg-closed sets of any topological space  $(X, \tau)$  is denoted by  $\text{sgc}(X, \tau)$ .

**e.** A subset  $B$  of a topological space  $(X, \tau)$  is said to be generalized semi-open (written in short as gs-open) if  $F \subset s\text{Int}(B)$  whenever  $F \subset B$  and  $F$  is closed.  $B$  is generalized semi-closed (written in short as gs-closed) if and only if  $X \setminus B$  is gs-open. Every closed set (semi-closed set, g-closed set and sg-closed set) is gs-closed. The family of all gs-closed sets of any topological space  $(X, \tau)$  is denoted by  $\text{gsc}(X, \tau)$  [1].

**f.** A map  $f : (X, \tau) \rightarrow (Y, \delta)$  is called a semi-generalized continuous map (written in short as sg-continuous mapping) if  $f^{-1}(V)$  is sg-closed in  $(X, \tau)$  for every closed set  $V$  of  $(Y, \delta)$  [5].

**g.** A map  $f : (X, \tau) \rightarrow (Y, \delta)$  is called a generalized semi-continuous map (written in short as gs-continuous mapping) if  $f^{-1}(V)$  is gs-closed in  $(X, \tau)$  for every closed set  $V$  of  $(Y, \delta)$  [8].

**h.** A map  $f : (X, \tau) \rightarrow (Y, \delta)$  is called a semi-generalized closed map (respectively semi-generalized open map) if  $f(V)$  is semi-generalized closed (respectively semi-generalized open) in  $(Y, \delta)$  for every closed set (respectively open set)  $V$  of  $(X, \tau)$ . Every semi-closed map is a semi-generalized closed map. A semi-generalized closed map (respectively semi-generalized open map) is written shortly as sg-closed map

(respectively sg open map) [7].

**k.** A map  $f : (X, \tau) \rightarrow (Y, \delta)$  is called a generalized semi-closed map (respectively generalized semi-open map) if for each closed set (respectively open set)  $V$  of  $(X, \tau)$ ,  $f(V)$  is gs-closed (respectively gs-open) in  $(Y, \delta)$ . Every semi-closed map, every sg-closed map is a generalized semi-closed map. A generalized semi-closed map (respectively generalized semi-open map) is written shortly as gs-closed map (respectively gs open map) [7].

**l.** A map  $f : (X, \tau) \rightarrow (Y, \delta)$  is said to be a semi-homeomorphism(B) (simply s.h. (B)) if  $f$  is continuous,  $f$  is semi-open (i.e.  $f(U)$  is semi-open for every open set  $U$  of  $(X, \tau)$ ) and  $f$  is bijective [4].

**m.** A map  $f : (X, \tau) \rightarrow (Y, \delta)$  is said to be a semi-homeomorphism (C.H) (simply s.h.(C.H)) if  $f$  is irresolute (i.e.  $f^{-1}(V)$  is semi-open for every semi-open set  $V$  of  $(Y, \delta)$ ),  $f$  is pre-semi-open (i.e.  $f(U)$  is semi-open for every semi-open set  $U$  of  $(X, \tau)$ ) and  $f$  is bijective [6].

**n.** A map  $f : (X, \tau) \rightarrow (Y, \delta)$  is called a sg-irresolute map if  $f^{-1}(V)$  is sg-closed in  $(X, \tau)$  for every sg-closed set  $V$  of  $(Y, \delta)$  [11].

**o.** A map  $f : (X, \tau) \rightarrow (Y, \delta)$  is called a gs-irresolute map if  $f^{-1}(V)$  is gs-closed in  $(X, \tau)$  for every gs-closed set  $V$  of  $(Y, \delta)$  [8].

**p.** A bijection  $f : (X, \tau) \rightarrow (Y, \delta)$  is called a semi-generalized homeomorphism (abbreviated sg-homeomorphism) if  $f$  is both sg-continuous and sg-open [8].

**r.** A bijection  $f : (X, \tau) \rightarrow (Y, \delta)$  is said to be a sgc-homeomorphism if  $f$  is sg-irresolute and its inverse  $f^{-1}$  is also sg-irresolute [8].

**s.** A bijection  $f : (X, \tau) \rightarrow (Y, \delta)$  is called a generalized semi-homeomorphism (abbreviated gs-homeomorphism) if  $f$  is both gs-continuous and gs-open [8].

**t.** A bijection  $f : (X, \tau) \rightarrow (Y, \delta)$  is said to be a gsc-homeomorphism if  $f$  is gs-irresolute and its inverse  $f^{-1}$  is also gs-irresolute [8].

**u.** A space  $(X, \tau)$  is called a  $T_{1/2}$  space if every g-closed set is closed, that is if and only if every gs-closed set is semi-closed [7,9].

**v.** A space  $(X, \tau)$  is called a  $T_b$  space if every gs-closed set is closed [7].

### III. GSG-HOMEOMORPHISM

In this section, the relations between semi-

homeomorphisms (B) and gsc-homeomorphisms are investigated and the diagram of implications is given. Also the gsg-homeomorphism is defined and some of its properties are obtained.

**Remark 3.1.** The following two examples show that the concepts of semi-homeomorphism (B) and gsc-homeomorphisms are independent of each other.

**Example 3.2.**

Let  $X = \{a, b, c\}$ ,  $\tau = \{\emptyset, \{a, b\}, \{b, c\}, \{b, X\}, \delta = \{\emptyset, \{b\}, X\}$ .

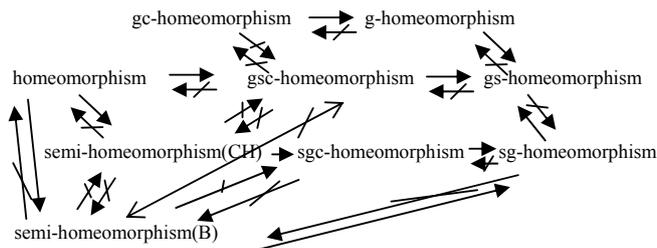
The identity map  $I_x : (X, \tau) \rightarrow (X, \delta)$  is not gsc-homeomorphism. However  $I_x$  is a s.h. (B).

**Example 3.3.**

Let  $X = \{a, b, c\}$ , the topology  $\tau$  on  $X$  be discrete and the topology  $\delta$  on  $X$  be indiscrete.

The identity map  $I_x : (X, \tau) \rightarrow (X, \delta)$  is not sh(B). However  $I_x$  is a gsc-homeomorphism.

**Proposition 3.4.** From remark 3.1 and remark 4.21 of R.Devi, K. Balachandran and H.Maki [8 ], we have the following diagram of implications.



**Definition 3.5.** A map  $f : (X, \tau) \rightarrow (Y, \delta)$  is called a gsg-irresolute map if the set  $f^{-1}(A)$  is sg-closed in  $(X, \tau)$  for every gs-closed set  $A$  of  $(Y, \delta)$ .

**Definition 3.6.** A bijection  $f : (X, \tau) \rightarrow (Y, \delta)$  is called a gsg-homeomorphism if the function  $f$  and the inverse function  $f^{-1}$  are both gsg-irresolute maps. If there exists a gsg-homeomorphism from  $X$  to  $Y$ , then the spaces  $(X, \tau)$  and  $(Y, \delta)$  are said to be gsg-homeomorphic. The family of all gsg-homeomorphism of any topological space  $(X, \tau)$  is denoted by  $gsgh(X, \tau)$ .

**Remark 3.7.** The following two examples show that the concepts of homeomorphism and gsg-homeomorphism are independent of each other.

**Example 3.8.**

Let  $X = \{a, b, c\}$ ,  $\tau = \{\emptyset, \{b\}, X\}$ .

The identity map  $I_x : (X, \tau) \rightarrow (X, \tau)$  is a homeomorphism but is not a gsg-homeomorphism on  $X$ .

**Example 3.9.**

Let  $X$  be any set which contains at least two elements;  $\tau$  and  $\delta$  be discrete and indiscrete topologies on  $X$ , respectively. The identity map  $I_X: (X, \tau) \rightarrow (X, \delta)$  is a gsg-homeomorphism but is not a homeomorphism.

**Remark 3.10.** Every gsg-homeomorphism implies both a gsc-homeomorphism and a sg-homeomorphism.

However the converse is not true as shown by the following example.

**Example 3.11.**

Let  $X = \{a, b, c\}$ ,  $\tau = \{\emptyset, \{b\}, X\}$ . Then

$sgc(X, \tau) = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$  and

$gsc(X, \tau) = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{a, b\}, \{b, c\}, X\}$ .

The identity map  $I_X: (X, \tau) \rightarrow (X, \tau)$  is both gsc-homeomorphism and sgc-homeomorphism. Since the set  $\{b, c\}$  is gs-closed but the set  $I_X^{-1}(\{b, c\}) = \{b, c\}$  is not sg-closed, then the identity map  $I_X$  is not a gsg-homeomorphism on  $X$ .

**Proposition 3.12.** Every gsg-homeomorphism implies both a gs-homeomorphism and a sg-homeomorphism. However its converse is not true.

**Definition 3.13.** Let  $(X, \tau)$  and  $(Y, \delta)$  be any topological spaces. If the following properties are satisfied

a)  $sgc(X, \tau) = gsc(X, \tau)$  and  $sgc(Y, \delta) = gsc(Y, \delta)$

b) there exists a bijective map

$\phi: gsc(X, \tau) \rightarrow gsc(Y, \delta)$  such that

$\forall A \in gsc(X, \tau) \quad \#(\phi(A)) = \#(A)$  ( $\#(A)$  is cardinality of  $A$ ).

then the spaces  $(X, \tau)$  and  $(Y, \delta)$  are called S-related

**Theorem 3.14.** The space  $(X, \tau)$  and  $(Y, \delta)$  are gsg-homeomorphic if and only if these spaces are S-related.

**Proof.** It follows from definition of gsg-homeomorphism and definitions 2.3, 2.4

**Theorem 3.15.**

a) Every gsc(sgc)-homeomorphism from  $T_{1/2}$  space onto itself is a gsg-homeomorphism.

b) Every gs(sg)-homeomorphism from  $T_b$  space onto itself is a gsg-homeomorphism.

**Proof.** Since for any  $T_{1/2}$  space  $(X, \tau)$  the family of sg-closed sets is equal to the family of gs-closed sets, any gsc(sgc)-homeomorphism from  $X$  to  $X$  is a gsg-homeomorphism.

In any  $T_b$  space  $(X, \tau)$  every gs-closed subset is a closed subset so (b) is obvious.

**Result 3.16.** Let  $(X, \tau)$  and  $(Y, \delta)$  be any topological spaces. If there exists any gsg-homeomorphism from  $X$  to  $Y$ , then every gsc(sgc)-homeomorphism from  $X$  to  $Y$  is a gsc(sgc)-

homeomorphism.

**Proof.** It is obtained by theorem 3.14

**Theorem 3.17.** For a topological space  $(X, \tau)$  the following implications hold:

a)  $gsg(X, \tau) \subset gsch(X, \tau) \subset gsh(X, \tau)$  and

$gsg(X, \tau) \subset sgch(X, \tau) \subset sgh(X, \tau)$

b) If  $gsg(X, \tau)$  is nonempty then  $gsg(X, \tau)$  is a group and

$gch(X, \tau) = gsch(X, \tau) = gsg(X, \tau)$

**Proof.** It follows from R. Devi, H. Maki [4], remark 3.10 and result 3.16.

**Theorem 3.18.** If  $f: (X, \tau) \rightarrow (Y, \delta)$  is a gsg-homeomorphism, then it induces an isomorphism from the group  $gsg(X, \tau)$  onto  $gsg(Y, \delta)$ .

**Proof.** The homomorphism  $f_*: gsg(X, \tau) \rightarrow gsg(Y, \delta)$  is induced from  $f$  by  $f_*(h) = f \circ h \circ f^{-1}$  for every  $h \in gsg(X, \tau)$ . Then it easily follows that  $f_*$  is an isomorphism

## IV. SGS-HOMEOMORPHISM

**Definition 4.1.** A map  $f: (X, \tau) \rightarrow (Y, \delta)$  is called a sgs-irresolute map if the set  $f^{-1}(A)$  is gs-closed in  $(X, \tau)$  for every sg-closed set  $A$  of  $(Y, \delta)$ .

**Definition 4.2.** A bijection  $f: (X, \tau) \rightarrow (Y, \delta)$  is called a sgs-homeomorphism if the function  $f$  and its inverse function  $f^{-1}$  are both sgs-irresolute maps. If there exists a sgs-homeomorphism from  $X$  to  $Y$ , then the space  $(X, \tau)$  and  $(Y, \delta)$  are said to be sgs-homeomorphic spaces.

**Remark 4.3.** Every sgc-homeomorphism and gsc-homeomorphism implies a sgs-homeomorphism.

**Example 4.4.**

Let  $X = Y = \{a, b, c\}$  and

$\tau = \{\{a\}, \{b\}, \{a, b\}, \{b, c\}, X, \emptyset\}$ ,  $\delta = \{\emptyset, \{b\}, \{a, b\}, Y\}$ .

Since  $sgc(X, \tau) = gsc(X, \tau) = \mathcal{P}(X) \setminus \{\{b\}, \{a, b\}\}$  ( $\mathcal{P}(X)$  is power set of  $X$ ) and

$sgc(Y, \delta) = \{\{c\}, \{a\}, \{a, c\}, \emptyset, Y\}$ ,  $gsc(Y, \delta) = \mathcal{P}(Y) \setminus \{\{b\}, \{a, b\}\}$ ,

then the identity map  $I_X: (X, \tau) \rightarrow (Y, \delta)$  is a sgs-homeomorphism but is not a gsc-homeomorphism.

**Example 4.5.**

Let  $X = Y = \{a, b, c\}$  and

$\tau = \{\emptyset, \{a\}, X\}$ ,  $\delta = \{\emptyset, \{b\}, \{a, b\}, Y\}$ . Since

$sgc(X, \tau) = \{\{b\}, \{c\}, \{b, c\}, X, \emptyset\}$ ,  $gsc(X, \tau) = \{\{b\}, \{c\},$

$\{b, c\}, \{a, c\}, \{a, b\}, X, \emptyset\}$  and

$sgc(Y, \delta) = \{\{c\}, \{a\}, \{a, c\}, Y, \emptyset\}$ ,  $gsc(Y, \delta) = \{\{a\}, \{c\},$

$\{a, c\}, \{b, c\}, Y, \emptyset\}$  then the mapping

$f: (X, \tau) \rightarrow (Y, \delta)$ , defined by  $f(a) = b$ ,  $f(b) = a$ ,  $f(c) = c$

is a sgs-homeomorphism but is not a gsc-homeomorphism.

**Result 4.6.** Every homeomorphism is a sgs-homeomorphism but the converse is not true.

**Remark 4.7.** Every sgs-homeomorphism is a gs-homeomorphism and the converse is not true as seen from the following example:

**Example 4.8.**

Let  $X = Y = \{a, b, c\}$ ,  $\tau = \{\emptyset, \{a, b\}, X\}$ ,  $\delta = \{\emptyset, \{b\}, \{a, b\}, Y\}$  since  $sgc(X, \tau) = gsc(X, \tau) = \{\{c\}, \{a, c\}, \{b, c\}, X, \emptyset\}$ ,  $sgc(Y, \delta) = \{\{c\}, \{a\}, \{a, c\}, Y, \emptyset\}$  and  $gsc(Y, \delta) = \{\{a\}, \{c\}, \{b, c\}, \{a, c\}, Y, \emptyset\}$

Then, the identity mapping  $I: (X, \tau) \rightarrow (Y, \delta)$  is a gs-homeomorphism but it is not sgs-homeomorphism.

**Example 4.9.**

The map  $I: (X, \tau) \rightarrow (Y, \delta)$  is given by Example 4.8 is a sg-homeomorphism but is not a sgs-homeomorphism.

**Result 4.10.**

- a) From the example 4.9 we can see that any sg-homeomorphism is not a sgs-homeomorphism.
- b) Every gsg-homeomorphism is a sgs-homeomorphism and the converse is not true as seen from the following example.

**Example 4.12.**

Let  $X = Y = \{a, b, c\}$  and  $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$ ,  $\delta = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, Y\}$ . Then the mapping  $f: (X, \tau) \rightarrow (Y, \delta)$  defined by  $f(a) = b$ ,  $f(b) = a$  and  $f(c) = c$  is a sgs-homeomorphism. However  $f$  is not a gsg-homeomorphism.

**Theorem 4.13.**

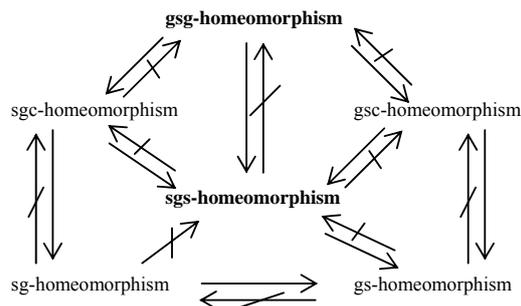
- a) Every sgs-homeomorphism from a  $T_{1/2}$  space onto itself is a gsg-homeomorphism. This implies that sgs-homeomorphism is both a sgc-homeomorphism and gsc-homeomorphism.
- b) Every sgs-homeomorphism from a  $T_b$  space onto itself is a homeomorphism. This implies that sgs-homeomorphism is a gs-homeomorphism, a sg-homeomorphism, a sgc-homeomorphism, a gsc-homeomorphism and a gsg-homeomorphism.
- c) Every sgs-homeomorphism from a  $T_{1/2}$  space onto itself is a sh (CH).

**Proof.**

- a) In a  $T_{1/2}$  space, every gs-closed set is a semi-closed set.
- b) In a  $T_b$  space, every gs-closed set is a closed set.
- c) Follows from the definition of  $T_{1/2}$  space.

V. CONCLUSION

In this paper, we introduce two classes of maps called sgs-homeomorphisms and gsg-homeomorphisms and study their properties. From all of the above statements, we have the following diagram:



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