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An Efficient Hamiltonian for Discrete Fractional Fourier Transform

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***Abstract*—**Fractional Fourier Transform, which is a generalization of the classical Fourier Transform, is a powerful tool for the analysis of transient signals. The discrete Fractional Fourier Transform Hamiltonians have been proposed in the past with varying degrees of correlation between their eigenvectors and Hermite Gaussian functions. In this paper, we propose a new Hamiltonian for the discrete Fractional Fourier Transform and show that the eigenvectors of the proposed matrix has a higher degree of correlation with the Hermite Gaussian functions. Also, the proposed matrix is shown to give better Fractional Fourier responses with various transform orders for different signals.

***Keywords*—** Fractional Fourier Transform, Hamiltonian, Eigen Vectors, Discrete Hermite Gaussians.

I. Introduction

FRACTIONAL Fourier transform is a generalization of classical Fourier Transform. The traditional Fourier transform decomposes the signal in terms of sinusoids, which are perfectly localized in frequency, but are not at all localized in time [1]. FrFT expresses the signal in terms of an orthonormal basis formed by linear chirps, whose instantaneous frequency varies linearly with time.

The Kernal for continuous Fractional Fourier Transform is given by [2]

$$K_\alpha(t,u)=\sqrt{\frac{1-j\cot\alpha}{2\pi}}e^{j\frac{t^2+u^2}{2}\cot\alpha-jut\operatorname{cosec}\alpha}$$

Using this kernel of FrFT, the FRFT of signal *x(t)* with transform order (α) is computed as

$$X_\alpha(u)=\int_{-\infty}^{\infty}x(t)K_\alpha(t,u)dt$$

And x(t) can be recovered from the following equation,

$$x(t)=\int_{-\infty}^{\infty}X_\alpha(u)K_{-\alpha}(u,t)du$$

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II. Discrete Fractional Fourier Transform

It is well known that Hermite Gaussians are the eigen functions of the DFT matrix, forming a complete and orthonormal set in L_2. The k^{th} order Hermite Gaussian can be given as follows:

$$\varphi_k(t)=\frac{2^{1/4}}{\sqrt{2^k k!}}H_k(\sqrt{2\pi}t)e^{-\pi t^2}$$

The Hermite polynomials for the first six orders can be given as follows:

$$H_0(x)=1$$

$$H_1(x)=x$$

$$H_2(x)=x^2-1$$

$$H_3(x)=x^3-3x$$

$$H_4(x)=x^4-6x^2+3$$

$$H_5(x)=x^5-10x^3+15x$$

$$H_6(x)=x^6-15x^4+45x^2-15$$

Defining the differential equations for the Hermite Gaussians, we arrive at the following expression,

$$\frac{d^2f(t)}{dt^2}-4\pi^2t^2f(t)=\lambda f(t)$$

which can be equivalently written as,

$$(D^2+FD^2F^{-1})f(t)=\lambda f(t)$$

(D = d/dt is the differentiation and F is the classical Fourier transformation).

or,

$$Sf(t)=\lambda f(t)$$

The operator ***S*** is the Hamiltonian operator associated with the harmonic oscillator.
The development of the Discrete Fractional Fourier Transform starts with the decomposition of the DFT matrix, and in order for a generalized expansion from DFT to DFrFT, it is required that S and F are commutative[3]. The decomposition for odd and even N can be respectively given as,

$$X_\alpha[n] = \Re_\alpha x[n]$$

Where,
$\Re_\alpha =$

$$\begin{cases} \sum_{k=0}^{N-1} e^{-jk\alpha} v_k v_k^* \\ \sum_{k=0}^{N-2} e^{-jk\alpha} v_k v_k^* + e^{-jN\alpha} v_{N-1} v_{N-1}^* \end{cases}$$

with v_k being the eigen vector obtained from matrix ***S***.

The decomposition becomes the same as classical Fourier transform for α = 1.

The common eigen function set of the commuting operators S and F also resolves the ambiguity of the eigen value degeneracy for the DFT matrix. The common commuting operators are best chosen to be the Hermite Gaussian functions. As can be clearly observed, there can be many commuting matrices to the DFT matrix. However, we have the following two constraints in choosing the commuting matrix S.

The commuting matrix should have the eigen functions that are nearly equal to the discrete counterparts of the Hermits Gaussian functions.

The commuting matrix should be expressed in a generalized form for the generalized computation of the DFrFT.

The second constraint more or less dominates in the development of the Hamiltonian for DFrFT while not deviating significantly from the first constraint. As a result, the matrices proposed in the past and the one proposed in this paper are both not perfectly commuting matrices with the DFT matrix, but are nearly commuting. The measure of the near commutation is the degree of correlation of the eigen functions of the matrix with the discrete counterparts of the Hermite Gaussian functions.

III. DEVELOPMENT OF THE EFFICIENT HAMILTONIAN

We provide an overview of the method used for developing a Hamiltonian which gives more efficient Fractional Fourier responses as compared to the previously proposed ones.

The entire procedure is described as follows:

We take the Hermite Gaussian functions and form matrices with three different orders of 4x4, 8x8, 16x16 taking them as the eigenvectors. We find that the matrices found cannot be combined in a generalized form with a significant degree of accuracy.

We add some random small error in the Hermite Gaussian functions and then find the desired matrices with the three different orders.

We try to find the generalized expression for the matrices by Method I described below.

If found, we end the procedure, else we proceed with the second step with a more degree of randomness.

The Method for finding a generalized form of matrices with the three varying orders is described as follows:

A. Method I:

Take the generalized expressions of the matrices described by Steiglits[4], Candan[3]. These matrices serve as the reference matrices.

Take the Eucledian distance measure between the corresponding values of the matrix formed by the considered approximation to the Hermite Gaussians with the references, varying the values of S(1,1) and the constant in the cosine term in the matrices. The tolerance limit is set to be les than or equal to 0.001.

If the tolerance limit is satisfied by the Euclidean distance measure for a certain set of constants, then check for the orders varying from 1x1 to 200x200 with a programming loop.

If the expression is found to be generalized for the orders from 1x1 to 200x200 with any set of constant values, the matrix is found else go to Step 2.

The entire procedure is based on the theory of training sets, wherein we form the reference set of matrices, and compare our found matrices while varying two constants and adding the degree of randomness in the approximation to the Hermite Gaussian functions at each subsequent level.

The proposed matrix is found to be as follows:

$$S = \begin{matrix} C_0 & 1 & 0 & \cdots & 1 \\ 1 & C_1 & 1 & \cdots & 0 \\ 0 & 1 & C_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & C_{N-1} \end{matrix}$$

Where,

$C_0 = 7$; $C_n = 2(\cos(\frac{2\pi}{N}n) - fac), n = 1.....N-1$, $fac = -3$

IV. PERFORMANCE OF THE PROPOSED HAMILTONIAN

The proposed matrix finds a greater degree of correlation with the discrete counterparts of the Hermite Gaussian functions as can be seen from Fig. 1. The 0^{th} , 2^{nd}, 4^{th} , 7^{th} order Hermite Gaussian functions are compared with the corresponding eigen functions of the proposed matrix. The figure shows the eigen functions of the matrices proposed in [3] which is already known to be better than that proposed in [4]. As a result, the eigen functions of the matrix in [4] are not plotted in Fig.1. In all the cases, diamonds on the dotted line denote the eigen functions plot of the proposed matrix, while stars on a dashed line denote the eigen functions plot of the matrix proposed by Candan in [4]. The solid continuous curve with no marks for the points is the discrete Hermite Gaussian function of the respective order.

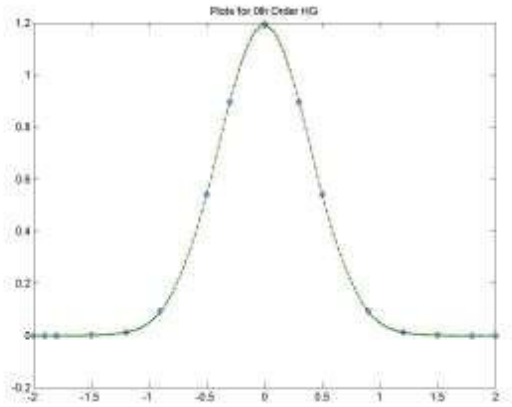

Fig.1 (a) 0^{th} order Hermite Gaussian functions (red) compared with the corresponding eigen functions of the proposed matrix (green) and the one by Candan (blue).

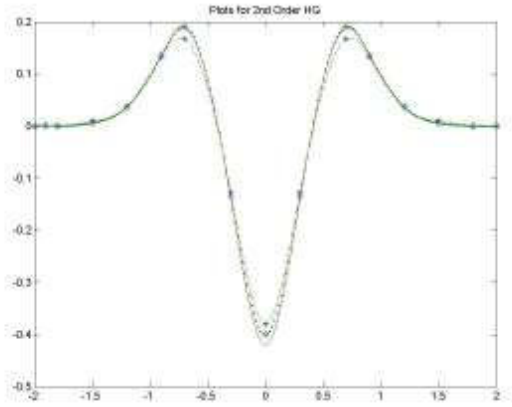

Fig.1 (b) 2th order Hermite Gaussian functions (red) compared with the corresponding eigen functions of the proposed matrix (green) and the one by Candan (blue).

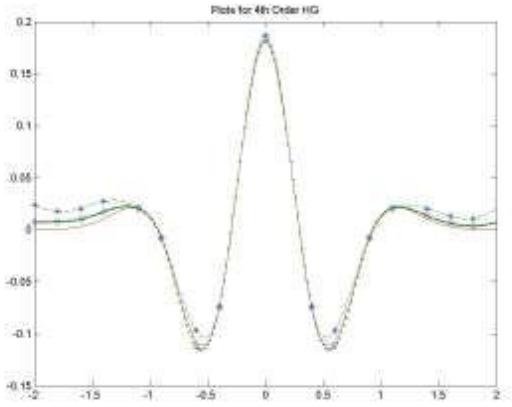

Fig.1 (c) 4^{th} order Hermite Gaussian functions (red) compared with the corresponding eigen functions of the proposed matrix (green) and the one by Candan (blue).

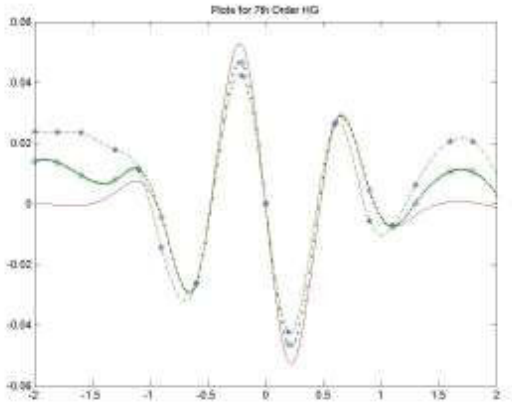

Fig.1 (d) 7^{th} order Hermite Gaussian functions (red) compared with the corresponding eigen functions of the proposed matrix (green) and the one by Candan (blue).

We now show the Fractional Fourier responses with the proposed matrix and compare them to the responses with that proposed in [2] and [4]. In each of the cases, the magnitude of the amplitude response is plotted.

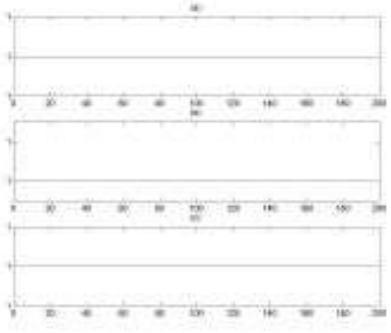

Fig.2 Signal Taken is a constant at 1. Transform Order = 0 (a) Response with the proposed matrix (b) Response with the matrix considered by Pei (c) Response with the matrix proposed by Candan.

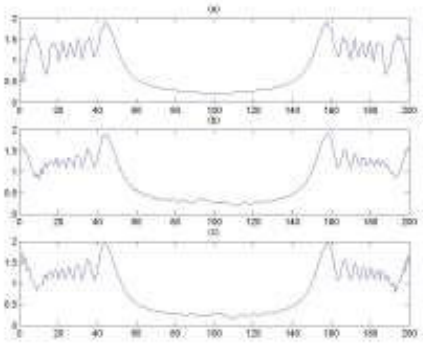

Fig.3 Signal Taken is a constant at 1. Transform Order = π/4 (a) Response with the proposed matrix (b) Response with the matrix considered by Pei (c) Response with the matrix proposed by Candan.

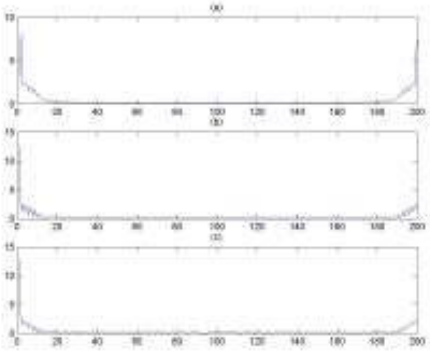

Fig.4 Signal Taken is a constant at 1. Transform Order = π/2 (a) Response with the proposed matrix (b) Response with the matrix considered by Pei (c) Response with the matrix proposed by Candan.

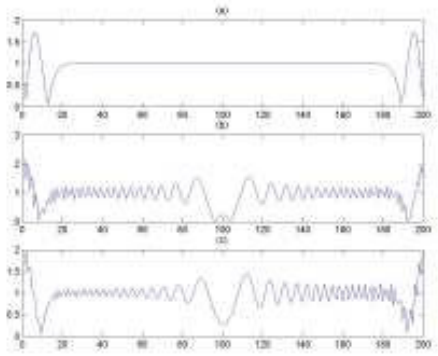

Fig.5 Signal Taken is a constant at 1. Transform Order = π (a) Response with the proposed matrix (b) Response with the matrix considered by Pei (c) Response with the matrix proposed by Candan.

It can be easily observed that the proposed matrix response gives a far lesser number of ripples and has a much better response at the transform orders of π. Also, it is optimal for transform orders ranging from 0 to π/2.

We now present the Fractional Fourier responses for a quadratic chirp, sinusoid and a rectangular function for various transform orders. The response with the proposed matrix is found to be better for orders in between 0 and π/2, and significantly better for π. Though, the transform order of π is only the inversion of the original time signal, and for obtaining the same, one does not need to use the Fractional Fourier kernel, a better response at π provides a critical analysis of the proposed DFrFT kernel. A better resemblance of the eigen functions of the DFrFT kernel with discrete Hermite Gaussian functions assures better response at π.

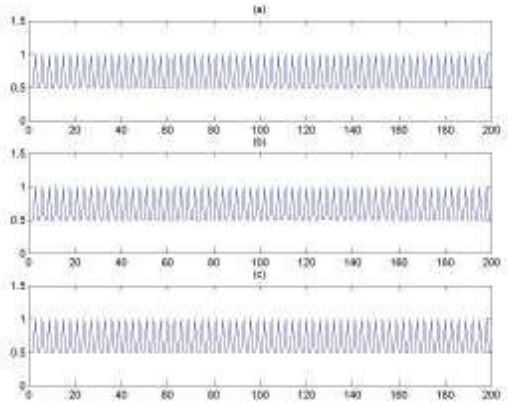

Fig.6 Signal Taken is a quadratic chirp. Transform Order = 0 (a) Response with the proposed matrix (b) Response with the matrix considered by Pei (c) Response with the matrix proposed by Candan.

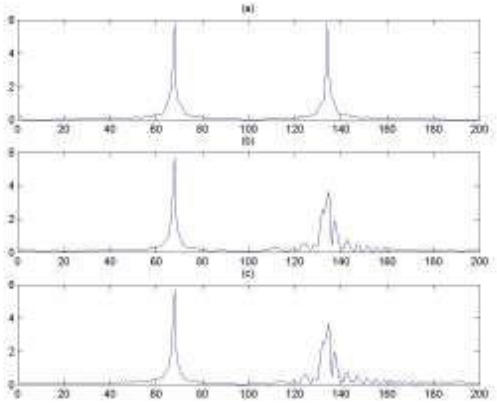

Fig.7 Signal Taken is a quadratic chirp. Transform Order = π/4 (a) Response with the proposed matrix (b) Response with the matrix considered by Pei (c) Response with the matrix proposed by Candan.

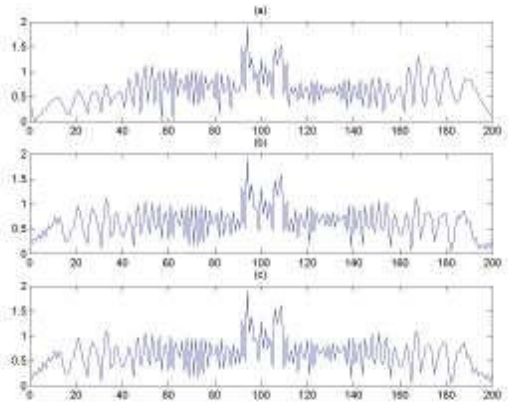

Fig.8 Signal Taken is a quadratic chirp. Transform Order = π/2 (a) Response with the proposed matrix (b) Response with the matrix considered by Pei (c) Response with the matrix proposed by Candan.

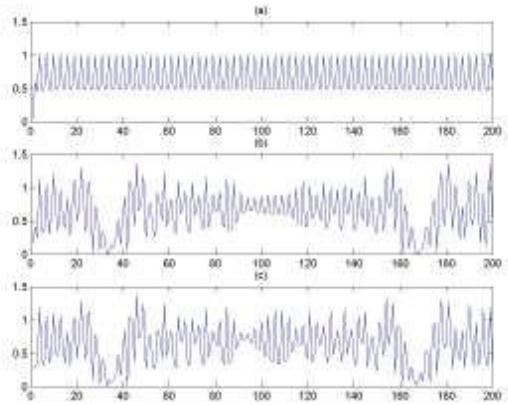

Fig.9 Signal Taken is a quadratic chirp. Transform Order = π (a) Response with the proposed matrix (b) Response with the matrix considered by Pei (c) Response with the matrix proposed by Candan.

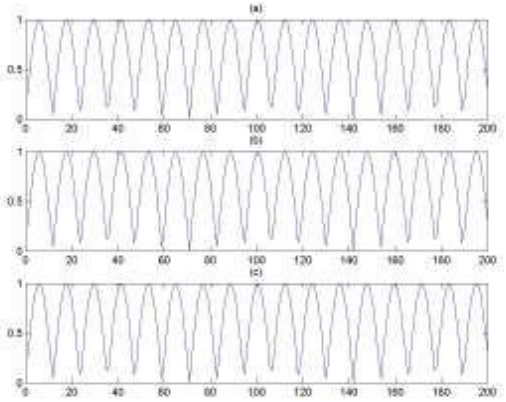

Fig.10 Signal Taken is a sinusoid. Transform Order = 0 (a) Response with the proposed matrix (b) Response with the matrix considered by Pei (c) Response with the matrix proposed by Candan.

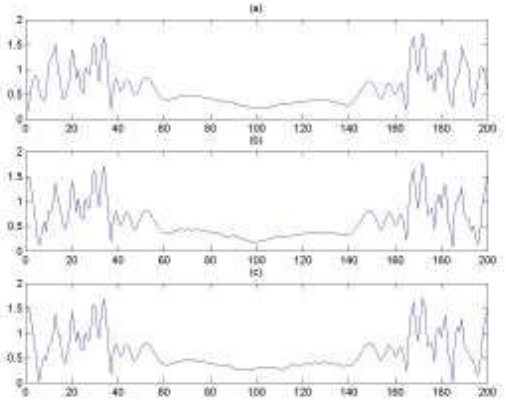

Fig.11 Signal Taken is a sinusoid. Transform Order = π/4 (a) Response with the proposed matrix (b) Response with the matrix considered by Pei (c) Response with the matrix proposed by Candan.

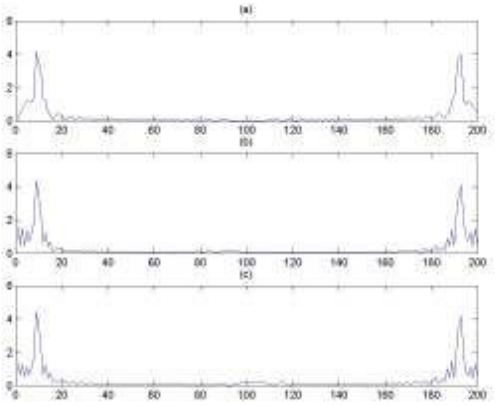

Fig.12 Signal Taken is a sinusoid. Transform Order = π/2 (a) Response with the proposed matrix (b) Response with the matrix considered by Pei (c) Response with the matrix proposed by Candan.

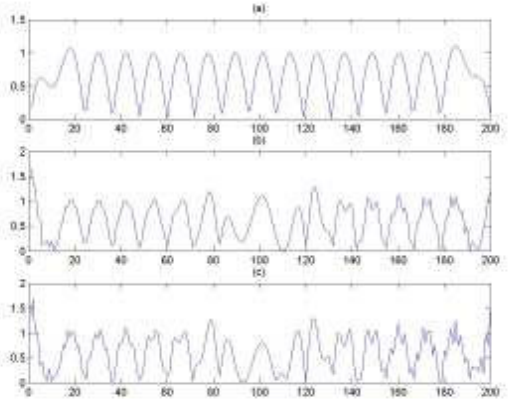

Fig.13 Signal Taken is a sinusoid. Transform Order = π (a) Response with the proposed matrix (b) Response with the matrix considered by Pei (c) Response with the matrix proposed by Candan.

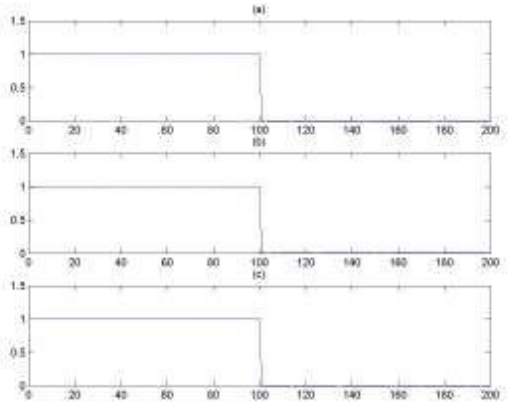

Fig.14 Signal Taken is a rectangular function. Transform Order = 0 (a) Response with the proposed matrix (b) Response with the matrix considered by Pei (c) Response with the matrix proposed by Candan.

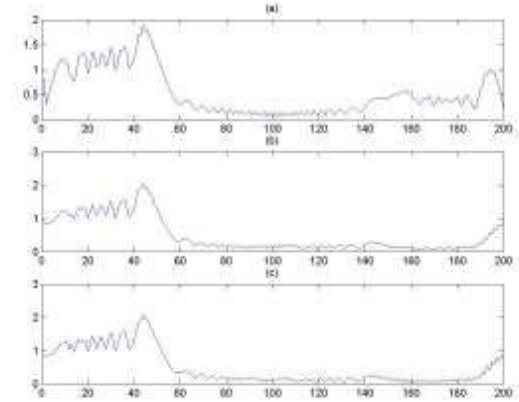

Fig.15 Signal Taken is a rectangular function. Transform Order = π/4 (a) Response with the proposed matrix (b) Response with the matrix considered by Pei (c) Response with the matrix proposed by Candan.

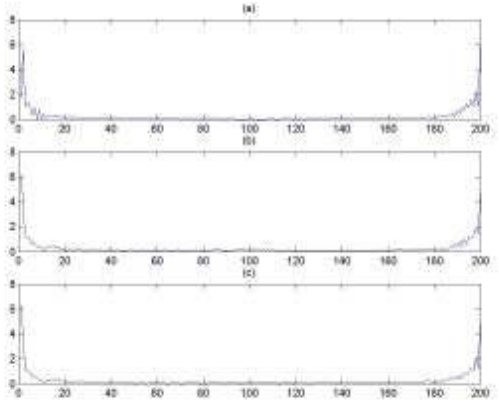

Fig.16 Signal Taken is a rectangular function. Transform Order = π/2 (a) Response with the proposed matrix (b) Response with the matrix considered by Pei (c) Response with the matrix proposed by Candan.

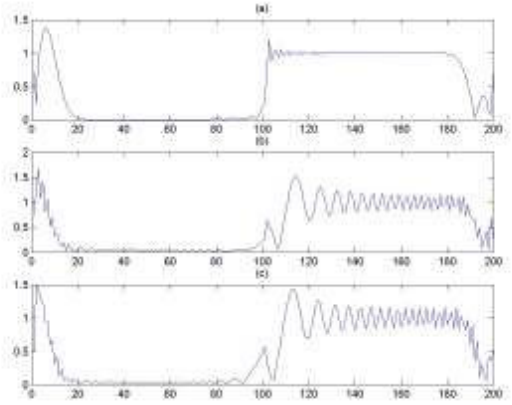

Fig.17 Signal Taken is a rectangular function. Transform Order = π (a) Response with the proposed matrix (b) Response with the matrix considered by Pei (c) Response with the matrix proposed by Candan.

IV. Conclusion

We have proposed a more efficient Hamiltonian for describing the kernel of the discrete Fractional Fourier Transform by matching with the training set formed by the variation of the two constants in the already proposed matrices. The eigen functions of the proposed matrix provide a better approximation to the discrete Hermite Gaussian functions and a better Fractional Fourier response.

References

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