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A Laplace Transform Dual-Reciprocity Boundary Element Method for Axisymmetric Elastodynamic Problems

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Abstract—A dual-reciprocity boundary element method is presented for the numerical solution of a class of axisymmetric elastodynamic problems. The domain integrals that arise in the integro-differential formulation are converted to line integrals by using the dual-reciprocity method together suitably constructed interpolating functions. The second order time derivatives of the displacement in the governing partial differential equations are suppressed by using Laplace transformation. In the Laplace transform domain, the problem under consideration is eventually reduced to solving a system of linear algebraic equations. Once the linear algebraic equations are solved, the displacement and stress fields in the physical domain can be recovered by using a numerical technique for inverting Laplace transforms.

Keywords—Axisymmetric elasticity, boundary element method, dual-reciprocity method, Laplace transform.

I. Introduction

THE axisymmetric boundary element method (BEM) has been studied by many researchers. Some of the earliest works on the axisymmetric BEM are Cruse, Snow, and Wilson [1] (elasticity) and Wrobel and Brebbia [2] (transient heat conductions). The boundary integral equations in [1] and [2] are obtained by axially integrating the corresponding three-dimensional boundary integral equations. The fundamental solutions in the axisymmetric boundary integral equations involve complete elliptic integrals of the first and second kind.

The axisymmetric BEM approach can be applied to axisymmetric elastodynamic problems. Depending on the fundamental solutions used, the axisymmetric integral formulations may contain domain integrals in addition to the usual boundary integrals. For axisymmetric elastic problems, domain integrals may be due to body force terms, nonhomogeneous elastic properties and time dependent terms. One way to deal with the domain integrals is to discretize the domain into elements, but the advantages of the BEM will be lost. There are different approaches to treat the domain integrals without discretizing the solution domain [3]. Some of the methods are particular integral method [4], Galerkin vector approach [1], dual-reciprocity method [5] and multiple reciprocity method [6].

Here, a dual reciprocity boundary element method is used to solve the homogeneous and isotropic elastodynamic problems in Laplace transform domain. The domain integrals, which contain body forces and the second order time derivatives of the displacements, can be transformed to line integrals by dual reciprocity method together with suitable interpolating functions. In Agnantiaris, Polyzos and Beskos [7], the interpolating functions obtained by axially integrating the corresponding three dimensional interpolating functions are complicated to evaluate. The axisymmetric interpolating functions used here are in relatively simple forms and are easy to compute.

The boundary of the solution domain is discretized into elements and a system of linear algebraic equations is formed in the Laplace transform domain. Once the algebraic equations are solved together with the initial-boundary conditions, the displacement and stress fields in physical domain can be obtained and recovered by a numerical method for inverting Laplace transforms. A specific problem is given to check the validity and accuracy of the numerical approach proposed here.

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II. Basic Equations of Axisymmetric Elasticity

The solid is isotropic and homogeneous. Therefore, the shear modulus μ, the Poisson's ratio ν and the mass density ρ are constant. The elastic field in the solid is transient and varies with the spatial coordinates r and z. In the cylindrical coordinates (r, θ, z), the only non-zero displacement components are u_r, u_z and non-zero stress components are σ_{rr}, σ_{rz}, σ_{zz} and $\sigma_{\theta\theta}$.

The equilibrium equations of the axisymmetric elastodynamics are given by

$$\begin{aligned} \frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + F_r &= \rho \frac{\partial^2 u_r}{\partial t^2}, \\ \frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} + F_z &= \rho \frac{\partial^2 u_z}{\partial t^2}, \end{aligned} \quad (1)$$

where F_r and F_z are the body force in r and z direction respectively.

The stress components are given in terms of displacements u_r and u_z by

$$\begin{aligned} \sigma_{rr} &= 2\mu(\frac{\partial u_r}{\partial r} + \frac{\nu}{1-2\nu}[\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}]), \\ \sigma_{zz} &= 2\mu(\frac{\partial u_z}{\partial z} + \frac{\nu}{1-2\nu}[\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}]), \\ \sigma_{\theta\theta} &= 2\mu(\frac{u_r}{r} + \frac{\nu}{1-2\nu}[\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}]), \\ \sigma_{rz} &= \mu(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}). \end{aligned} \quad (2)$$

From (1) and (2), the governing equations are obtained and written as

$$\begin{aligned}
&\nabla^2_{\text{axis}} u_r - \frac{u_r}{r^2} + \frac{1}{1-2\nu}\frac{\partial}{\partial r}(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}) \\
&= \frac{1}{\mu}(-F_r + \rho\frac{\partial^2 u_r}{\partial t^2}), \\
&\nabla^2_{\text{axis}} u_z + \frac{1}{1-2\nu}\frac{\partial}{\partial z}(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}) \\
&= \frac{1}{\mu}(-F_z + \rho\frac{\partial^2 u_z}{\partial t^2}), \quad (3)
\end{aligned}$$

where $\nabla^2_{\text{axis}} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$.

III. Integro-Differential Formulation

Following closely the analysis in Bakr [8] and Gao and Davies [9], the partial differential equations in (3) can be recast to integro-differential form

$$\begin{aligned}
&\gamma(\underline{\mathbf{x}}_0) u_K(\underline{\mathbf{x}}_0, t) \\
&= \int_\Gamma (\Phi_{JK}(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) p_J(\underline{\mathbf{x}}, t; \underline{\mathbf{n}}(\underline{\mathbf{x}})) \\
&- \Psi_{JK}(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}})) u_J(\underline{\mathbf{x}}, t)) r ds(\underline{\mathbf{x}}) \\
&+ \iint_\Omega \frac{\Phi_{JK}(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)}{\mu}(F_J(\underline{\mathbf{x}}, t) - \rho\frac{\partial^2 u_J(\underline{\mathbf{x}}, t)}{\partial t^2}) r dr dz \\
&\text{for } \underline{\mathbf{x}}_0 \in \Omega \cup \Gamma, \quad (4)
\end{aligned}$$

where Ω is the solution domain in the Orz (axisymmetric coordinate) plane, Γ is the line boundary of Ω, $\underline{\mathbf{x}} = (r, z)$, $\underline{\mathbf{x}}_0 = (r_0, z_0)$, the uppercase Latin subscripts such as J and K are given the values r and z, $x_r = r$, $x_z = z$, $\underline{\mathbf{n}}(\underline{\mathbf{x}})$ is the unit normal vector to Γ pointing away from Ω, $p_J(\underline{\mathbf{x}}; \underline{\mathbf{n}}(\underline{\mathbf{x}})) = \sigma_{IJ} n_I/\mu$ and $\Phi_{JK}(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)$ and $\Psi_{JK}(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}}))$ are axisymmetric fundamental solution for homogeneous isotropic elasticity as given in the Appendix. Note that the Einsteinian convention of summing over a repeated uppercase Latin subscript is adopted here.

Applying Laplace transformation (with respect to time) on (4) yields

$$\begin{aligned}
&\gamma(\underline{\mathbf{x}}_0) \widetilde{u}_K(\underline{\mathbf{x}}_0, p) \\
&= \int_\Gamma (\Phi_{JK}(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) \widetilde{p}_J(\underline{\mathbf{x}}, p; \underline{\mathbf{n}}(\underline{\mathbf{x}})) \\
&- \Psi_{JK}(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}})) \widetilde{u}_J(\underline{\mathbf{x}}, p)) r ds(\underline{\mathbf{x}}) \\
&+ \iint_\Omega \frac{\Phi_{JK}(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)}{\mu}(\widetilde{F}_J(\underline{\mathbf{x}}, p) \\
&- \rho[p^2 \widetilde{u}_J(\underline{\mathbf{x}}, p) - p f_J(\underline{\mathbf{x}}) - g_J(\underline{\mathbf{x}})]) r dr dz \\
&\text{for } \underline{\mathbf{x}}_0 \in \Omega \cup \Gamma \text{ and } p > 0, \quad (5)
\end{aligned}$$

where p is the Laplace transform parameter, $f_J(\underline{\mathbf{x}})$ and $g_J(\underline{\mathbf{x}})$ are the known initial conditions given by $f_J(\underline{\mathbf{x}}) = u_J(\underline{\mathbf{x}}, 0)$ and $g_J(\underline{\mathbf{x}}) = \frac{\partial}{\partial t}[u_J(\underline{\mathbf{x}}, t)]\big|_{t=0}$, and $\widetilde{u}_J(\underline{\mathbf{x}}, p)$, $\widetilde{p}_J(\underline{\mathbf{x}}, p; \underline{\mathbf{n}}(\underline{\mathbf{x}}))$ and $\widetilde{F}_J(\underline{\mathbf{x}}, p)$ are the Laplace transform of $u_J(\underline{\mathbf{x}}, t)$, $p_J(\underline{\mathbf{x}}, t; \underline{\mathbf{n}}(\underline{\mathbf{x}}))$ and $F_J(\underline{\mathbf{x}}, t)$ respectively which are given by

$$\begin{aligned}
\widetilde{u}_J(\underline{\mathbf{x}}, p) &= \int_0^\infty u_J(\underline{\mathbf{x}}, t)\exp(-pt) dt, \\
\widetilde{p}_J(\underline{\mathbf{x}}, p; \underline{\mathbf{n}}(\underline{\mathbf{x}})) &= \int_0^\infty p_J(\underline{\mathbf{x}}, t; \underline{\mathbf{n}}(\underline{\mathbf{x}}))\exp(-pt) dt, \\
\widetilde{F}_J(\underline{\mathbf{x}}, p) &= \int_0^\infty F_J(\underline{\mathbf{x}}, t)\exp(-pt) dt. \quad (6)
\end{aligned}$$

IV. Dual-Reciprocity Method

By using the dual-reciprocity method, the double integral in (5) can be approximated into line integral. Select M well spaced out collocations points as $\underline{\mathbf{y}}^{(1)}, \underline{\mathbf{y}}^{(2)}, \cdots, \underline{\mathbf{y}}^{(M-1)}$ and $\underline{\mathbf{y}}^{(M)}$ inside the solution domain $\Omega \cup \Gamma$, and the functions which appear in the domain integral can be approximated as

$$\begin{aligned}
&\frac{1}{\mu}(-\widetilde{F}_J(\underline{\mathbf{x}}, p) + \rho[p^2 \widetilde{u}_J(\underline{\mathbf{x}}, p) - p f_J(\underline{\mathbf{x}}) - g_J(\underline{\mathbf{x}})]) \\
&\simeq \sum_{n=1}^{M} \phi_{JN}(\underline{\mathbf{x}}; \underline{\mathbf{y}}^{(n)}) \alpha_N^{(n)}(p) \text{ for } \underline{\mathbf{x}} \in \Omega \cup \Gamma, \quad (7)
\end{aligned}$$

where $\alpha_N^{(n)}(p)$ are constant coefficients and $\phi_{JN}(\underline{\mathbf{x}}; \underline{\mathbf{y}})$ are interpolating functions given by

$$\begin{aligned}
\phi_{rJ}(\underline{\mathbf{x}}; \underline{\mathbf{y}}) &= \nabla^2_{\text{axis}} \chi_{rJ}(\underline{\mathbf{x}}; \underline{\mathbf{y}}) - \frac{\chi_{rJ}(\underline{\mathbf{x}}; \underline{\mathbf{y}})}{r^2} \\
&+ \frac{1}{1-2\nu}\frac{\partial}{\partial r}(\frac{\partial \chi_{rJ}(\underline{\mathbf{x}}; \underline{\mathbf{y}})}{\partial r} \\
&+ \frac{\chi_{rJ}(\underline{\mathbf{x}}; \underline{\mathbf{y}})}{r} + \frac{\partial \chi_{zJ}(\underline{\mathbf{x}}; \underline{\mathbf{y}})}{\partial z}), \\
\phi_{zJ}(\underline{\mathbf{x}}; \underline{\mathbf{y}}) &= \nabla^2_{\text{axis}} \chi_{zJ}(\underline{\mathbf{x}}; \underline{\mathbf{y}}) \\
&+ \frac{1}{1-2\nu}\frac{\partial}{\partial z}(\frac{\partial \chi_{rJ}(\underline{\mathbf{x}}; \underline{\mathbf{y}})}{\partial r} \\
&+ \frac{\chi_{rJ}(\underline{\mathbf{x}}; \underline{\mathbf{y}})}{r} + \frac{\partial \chi_{zJ}(\underline{\mathbf{x}}; \underline{\mathbf{y}})}{\partial z}), \quad (8)
\end{aligned}$$

where $\chi_{KJ}(\underline{\mathbf{x}}; \underline{\mathbf{y}})$ are

$$\begin{aligned}
\chi_{rr}(\underline{\mathbf{x}}; \underline{\mathbf{y}}) &= \frac{1}{9}\{[\sigma(\underline{\mathbf{x}}; \underline{\mathbf{y}})]^3 + [\sigma(\underline{\mathbf{x}}; -\beta, \zeta)]^3\} \\
&- \frac{2}{9}[\sigma(0, z; \underline{\mathbf{y}})]^3, \\
\chi_{zr}(\underline{\mathbf{x}}; \underline{\mathbf{y}}) &= \chi_{rz}(\underline{\mathbf{x}}; \underline{\mathbf{y}}) = 0, \\
\chi_{zz}(\underline{\mathbf{x}}; \underline{\mathbf{y}}) &= \frac{1}{9}\{[\sigma(\underline{\mathbf{x}}; \underline{\mathbf{y}})]^3 + [\sigma(\underline{\mathbf{x}}; -\beta, \zeta)]^3\}, \quad (9)
\end{aligned}$$

$\underline{\mathbf{y}} = (\beta, \zeta)$ and $\sigma(\underline{\mathbf{x}}; \underline{\mathbf{y}}) = \sqrt{(r-\beta)^2 + (z-\zeta)^2}$.

Note that the choice of $\chi_{rJ}(\underline{\mathbf{x}}; \underline{\mathbf{y}})$ and $\chi_{zJ}(\underline{\mathbf{x}}; \underline{\mathbf{y}})$ in (9) ensures that the interpolating functions $\phi_{JN}(\underline{\mathbf{x}}; \underline{\mathbf{y}})$ are bounded for $r > 0$.

From (7) and (8), the double integral in (5) can be written as

$$\iint\limits_{\Omega} \frac{\Phi_{JK}(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)}{\mu}(\widetilde{F}_J(\underline{\mathbf{x}},p) - \rho[p^2\widetilde{u}_J(\underline{\mathbf{x}},p) - pf_J(\underline{\mathbf{x}}) - g_J(\underline{\mathbf{x}})])rdrdz \simeq \sum_{n=1}^{M} \alpha_N^{(n)}(p) W_{KN}^{(n)}(\underline{\mathbf{x}}_0), \quad (10)$$

where

$$W_{KN}^{(n)}(\underline{\mathbf{x}}_0) = -\gamma(\underline{\mathbf{x}}_0)\chi_{KN}(\underline{\mathbf{x}}_0;\underline{\mathbf{y}}^{(n)}) + \int\limits_{\Gamma}(\Phi_{JK}(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)\tau_{JN}(\underline{\mathbf{x}};\underline{\mathbf{y}}^{(n)};\underline{\mathbf{n}}(\underline{\mathbf{x}})) - \Psi_{JK}(\underline{\mathbf{x}};\underline{\mathbf{x}}_0;\underline{\mathbf{n}}(\underline{\mathbf{x}}))\chi_{JN}(\underline{\mathbf{x}};\underline{\mathbf{y}}^{(n)}))rds(\underline{\mathbf{x}}), \quad (11)$$

the parameter $\gamma(\underline{\mathbf{x}}_0)$ has the value 1 when $\underline{\mathbf{x}}_0$ lies in the interior of Ω, $\gamma(\underline{\mathbf{x}}_0)$ has the value $1/2$ when $\underline{\mathbf{x}}_0$ lies on a smooth part of Γ,

$$\begin{aligned}
&\tau_{rN}(\underline{\mathbf{x}};\underline{\mathbf{y}};\underline{\mathbf{n}}(\underline{\mathbf{x}})) \\
&= 2n_r(\underline{\mathbf{x}})\{\frac{\partial}{\partial r}[\chi_{rN}(\underline{\mathbf{x}};\underline{\mathbf{y}})] + \frac{\nu}{1-2\nu}(\frac{\partial}{\partial r}[\chi_{rN}(\underline{\mathbf{x}};\underline{\mathbf{y}})] \\
&+\frac{\chi_{rN}(\underline{\mathbf{x}};\underline{\mathbf{y}})}{r} + \frac{\partial}{\partial z}[\chi_{zN}(\underline{\mathbf{x}};\underline{\mathbf{y}})])\} \\
&+n_z(\underline{\mathbf{x}})\{\frac{\partial}{\partial z}[\chi_{rN}(\underline{\mathbf{x}};\underline{\mathbf{y}})] + \frac{\partial}{\partial r}[\chi_{zN}(\underline{\mathbf{x}};\underline{\mathbf{y}})]\}, \\
&\tau_{zN}(\underline{\mathbf{x}};\underline{\mathbf{y}};\underline{\mathbf{n}}(\underline{\mathbf{x}})) \\
&= n_r(\underline{\mathbf{x}})\{\frac{\partial}{\partial z}[\chi_{rN}(\underline{\mathbf{x}};\underline{\mathbf{y}})] + \frac{\partial}{\partial r}[\chi_{zN}(\underline{\mathbf{x}};\underline{\mathbf{y}})]\} \\
&+2n_z(\underline{\mathbf{x}})\{\frac{\partial}{\partial z}[\chi_{zN}(\underline{\mathbf{x}};\underline{\mathbf{y}})] + \frac{\nu}{1-2\nu}(\frac{\partial}{\partial r}[\chi_{rN}(\underline{\mathbf{x}};\underline{\mathbf{y}})] \\
&+\frac{\chi_{rN}(\underline{\mathbf{x}};\underline{\mathbf{y}})}{r} + \frac{\partial}{\partial z}[\chi_{zN}(\underline{\mathbf{x}};\underline{\mathbf{y}})])\}. \quad (12)
\end{aligned}$$

Taking $\underline{\mathbf{x}} = \underline{\mathbf{y}}^{(k)}$ $(k = 1, 2, \cdots, M)$, (7) becomes

$$\sum_{n=1}^{M} \phi_{JN}(\underline{\mathbf{y}}^{(k)};\underline{\mathbf{y}}^{(n)})\alpha_N^{(n)}(p) \simeq \frac{1}{\mu}(-\widetilde{F}_J(\underline{\mathbf{y}}^{(k)},p) + \rho[p^2\widetilde{u}_J(\underline{\mathbf{y}}^{(k)},p) - pf_J(\underline{\mathbf{y}}^{(k)}) - g_J(\underline{\mathbf{y}}^{(k)})]) \quad \text{for } k = 1, 2, \cdots, M. \quad (13)$$

By inverting (13), the constant coefficients $\alpha_N^{(n)}(p)$ can be obtained.

The interpolating functions $\phi_{JN}(\underline{\mathbf{x}};\underline{\mathbf{y}})$ which can be used to convert (10) into line integrals are not unique. In [7], the interpolating functions are obtained by integrating axially the corresponding three-dimensional interpolating functions and are more complicated to evaluate. The interpolating functions here as given by (8), (9) and (12) are in simple elementary forms. Similar interpolating functions are used in Yun and Ang [10] for axisymmetric heat conduction problems.

V. Boundary Element Solution

To solve the integral equation in (5), the boundary Γ is discretized into N elements, which are $\Gamma^{(1)}, \Gamma^{(2)}, \cdots, \Gamma^{(N-1)}$ and $\Gamma^{(N)}$. Along the curve Γ, the collocation points denoted as $\underline{\mathbf{y}}^{(1)}, \underline{\mathbf{y}}^{(2)}, \cdots, \underline{\mathbf{y}}^{(N-1)}$ and $\underline{\mathbf{y}}^{(N)}$ which are the mid point of $\Gamma^{(1)}, \Gamma^{(2)}, \cdots, \Gamma^{(N-1)}$ and $\Gamma^{(N)}$ respectively. In the interior region Ω, L well space out collocation points are denoted as $\underline{\mathbf{y}}^{(N+1)}, \underline{\mathbf{y}}^{(N+2)}, \cdots, \underline{\mathbf{y}}^{(N+L-1)}$ and $\underline{\mathbf{y}}^{(N+L)}$. Over the boundary $\Gamma^{(k)}$, the functions $\widetilde{u}_J(\underline{\mathbf{y}}^{(k)},p)$, $\widetilde{p}_J(\underline{\mathbf{y}}^{(k)},p)$ are approximated as constants given by $u_K^{(k)}(p)$ and $p_J^{(k)}(p)$ respectively. Therefore, the integral equations in (5) together with (10) now become

$$\begin{aligned}
&\gamma(\underline{\mathbf{y}}^{(m)})u_K^{(m)}(p) \\
&= \sum_{n=1}^{N+L} \alpha_N^{(n)}(p) W_{KN}^{(n)}(\underline{\mathbf{y}}^{(m)}) \\
&+ \sum_{k=1}^{N} p_J^{(k)}(p) \int\limits_{\Gamma^{(k)}} \Phi_{JK}(\underline{\mathbf{x}};\underline{\mathbf{y}}^{(m)})rds(\underline{\mathbf{x}}) \\
&- \sum_{k=1}^{N} u_J^{(k)}(p) \int\limits_{\Gamma^{(k)}} \Psi_{JK}(\underline{\mathbf{x}};\underline{\mathbf{y}}^{(m)};\underline{\mathbf{n}}(\underline{\mathbf{x}}))rds(\underline{\mathbf{x}}) \\
&\text{for } m = 1, 2, \cdots, N+L, \quad (14)
\end{aligned}$$

and

$$\sum_{n=1}^{N+L} \phi_{JN}(\underline{\mathbf{y}}^{(k)};\underline{\mathbf{y}}^{(n)})\alpha_N^{(n)}(p) \simeq \frac{1}{\mu}(-F_J^{(k)}(p) + \rho[p^2u_J^{(k)}(p) - pf_J^{(k)} - g_J^{(k)}]), \quad (15)$$

where $F_J^{(k)}(p) = \widetilde{F}_J(\underline{\mathbf{y}}^{(k)},p)$, $f_J^{(k)} = f_J(\underline{\mathbf{y}}^{(k)})$ and $g_J^{(k)} = g_J(\underline{\mathbf{y}}^{(k)})$.

In any suitably prescribed boundary value problem, two of the four components $\widetilde{u}_r$, $\widetilde{u}_z$, $\widetilde{p}_r$ and $\widetilde{p}_z$ are known. Thus, equation (14) can be solved together with (15). There are total of $2(N+L)$ algebraic equations and $2(N+L)$ unknowns in (14) after the constant $\alpha_N^{(n)}(p)$ are calculated from (15). The unknowns are two of the four components $u_r^{(k)}(p)$, $u_z^{(k)}(p)$, $p_J^{(k)}(p)$ and $p_z^{(k)}(p)$ on the boundary element $\Gamma^{(k)}$ for $k = 1, 2, \cdots, N$, and $u_r^{(N+n)}(p)$, $u_z^{(N+n)}(p)$ at the interior collocation points for $n = 1, 2, \cdots, L$.

VI. Inversion of Laplace Transform

The solution $\widetilde{u}_J(\underline{\mathbf{x}},p)$ in Laplace transform can be inverted to $u_J(\underline{\mathbf{x}},t)$ using numerical technique in Stehfest [11], that is,

$$u_J(\underline{\mathbf{x}},t) \approx \frac{\ln(2)}{t}\sum_{n=1}^{2M} V_n\widetilde{u}_J(\underline{\mathbf{x}}, \frac{n\ln(2)}{t}), \quad (16)$$

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where M is the positive integer and

$$V_n = (-1)^{n+M} \times \sum_{m=[(n+1)/2]}^{\min(n,M)} m^M (2m)!/[(M-m)!m! \times (m-1)!(n-m)!(2m-n)!], \qquad (17)$$

where $[a]$ denotes the integer part of the real number a.

The Stehfest's formula can also be used to invert $\widetilde{p}_J(\underline{\mathbf{x}}, p; \underline{\mathbf{n}}(\underline{\mathbf{x}}))$ to obtain $p_J(\underline{\mathbf{x}}, t; \underline{\mathbf{n}}(\underline{\mathbf{x}}))$, hence giving the tractions $t_J(\underline{\mathbf{x}}, t; \underline{\mathbf{n}}(\underline{\mathbf{x}}))$ $(t_J(\underline{\mathbf{x}}, t; \underline{\mathbf{n}}(\underline{\mathbf{x}})) = \mu p_J(\underline{\mathbf{x}}, t; \underline{\mathbf{n}}(\underline{\mathbf{x}})))$.

VII. Stress Fields

In order to obtain the stresses $\sigma_{rr}, \sigma_{rz}, \sigma_{zz}$ and $\sigma_{\theta\theta}$, the partial derivatives of displacements are required. Once the $u_J(\underline{\mathbf{x}}, t)$ are obtained from the BEM procedure, the partial derivatives of $u_J(\underline{\mathbf{x}}, t)$ can be approximated as functions of $u_J(\underline{\mathbf{x}}, t)$. Therefore, the stresses can be calculated numerically.

Let

$$\begin{aligned} u_r(\underline{\mathbf{x}}, t) &\simeq \sum_{m=1}^{N+L} v_r^{(m)}(t) \overline{\chi}(\underline{\mathbf{x}}; \underline{\mathbf{y}}^{(m)}), \\ u_z(\underline{\mathbf{x}}, t) &\simeq \sum_{m=1}^{N+L} v_z^{(m)}(t) \chi(\underline{\mathbf{x}}; \underline{\mathbf{y}}^{(m)}), \end{aligned} \qquad (18)$$

where $\chi(\underline{\mathbf{x}}; \underline{\mathbf{y}}^{(m)})$ is defined the same as χ_{zz} in (9), and

$$\overline{\chi}(\underline{\mathbf{x}}; \underline{\mathbf{y}}^{(m)}) = \frac{1}{9}\{[\sigma(\underline{\mathbf{x}}; \underline{\mathbf{y}})]^3 - [\sigma(\underline{\mathbf{x}}; -\beta, \zeta)]^3\}. \qquad (19)$$

By letting $\underline{\mathbf{x}} = \underline{\mathbf{y}}^{(k)}$ for $k = 1, 2, \cdots, N+L$ and invert (18) to get the constant variables $v_r^{(m)}(t)$ and $v_z^{(m)}(t)$, we obtain the partial derivatives of $u_J(\underline{\mathbf{x}}, t)$ in terms of the known displacements

$$\begin{aligned} \frac{\partial}{\partial x_J}[u_r(\underline{\mathbf{x}}, t)] &= \sum_{q=1}^{N+L} u_r(\underline{\mathbf{x}}^{(q)}, t) \overline{\varphi}_J^{(q)}(\underline{\mathbf{x}}), \\ \frac{\partial}{\partial x_J}[u_z(\underline{\mathbf{x}}, t)] &= \sum_{q=1}^{N+L} u_z(\underline{\mathbf{x}}^{(q)}, t) \varphi_J^{(q)}(\underline{\mathbf{x}}), \end{aligned} \qquad (20)$$

where

$$\begin{aligned} \varphi_J^{(q)}(\underline{\mathbf{x}}) &= \sum_{m=1}^{N+L} \omega^{(mq)} \frac{\partial}{\partial x_J}[\chi(\underline{\mathbf{x}}; \underline{\mathbf{y}}^{(m)})], \\ \overline{\varphi}_J^{(q)}(\underline{\mathbf{x}}) &= \sum_{m=1}^{N+L} \overline{\omega}^{(mq)} \frac{\partial}{\partial x_J}[\overline{\chi}(\underline{\mathbf{x}}; \underline{\mathbf{y}}^{(m)})], \end{aligned} \qquad (21)$$

$$\begin{aligned} \sum_{m=1}^{N+L} \chi(\underline{\mathbf{x}}^{(k)}; \underline{\mathbf{y}}^{(m)}) \omega^{(mq)} &= \begin{cases} 1 \text{ if } k = q, \\ 0 \text{ if } k \neq q, \end{cases} \\ \sum_{m=1}^{N+L} \overline{\chi}(\underline{\mathbf{x}}^{(k)}; \underline{\mathbf{y}}^{(m)}) \overline{\omega}^{(mq)} &= \begin{cases} 1 \text{ if } k = q, \\ 0 \text{ if } k \neq q. \end{cases} \end{aligned} \qquad (22)$$

TABLE I
The displacement u_r at $t = 1$ at selected interior points

	u_r		
(r, z)	Set A	Set B	Exact
$(0.25, 0.25)$	0.097356	0.097569	0.097718
$(0.50, 0.25)$	0.195586	0.195389	0.195436
$(0.75, 0.25)$	0.294116	0.293337	0.293154
$(0.25, 0.50)$	0.114536	0.114675	0.114962
$(0.50, 0.50)$	0.229946	0.229641	0.229925
$(0.75, 0.50)$	0.345698	0.344876	0.344887
$(0.25, 0.75)$	0.143857	0.143474	0.143703
$(0.50, 0.75)$	0.287798	0.287107	0.287406
$(0.75, 0.75)$	0.431538	0.430973	0.431109

VIII. Specific Problem

For a test problem, consider the solution domain $0 < r < 1$, $0 < z < 1$. The shear modulus, Poisson's ratio and the density are taken to be $\mu = 0.5$, $\nu = 0.3$ and $\rho = 2$ respectively.

The body force terms are taken to be

$$\begin{aligned} F_r(\underline{\mathbf{x}}, t) &= (r + 2rz^2)\exp(-t), \\ F_z(\underline{\mathbf{x}}, t) &= -5z\exp(-t) \\ &\quad + (-7.5 + r^2 + 0.5z^2)\exp(-0.5t). \end{aligned}$$

The initial conditions are

$$\left.\begin{aligned} &u_r(\underline{\mathbf{x}}, 0) = r + rz^2, \\ &u_z(\underline{\mathbf{x}}, 0) = 2r^2 + z^2, \\ &\frac{\partial}{\partial t}[u_r(\underline{\mathbf{x}}, t)]\Big|_{t=0} = -(r + rz^2), \\ &\frac{\partial}{\partial t}[u_z(\underline{\mathbf{x}}, t)]\Big|_{t=0} = -(r^2 + 0.5z^2), \end{aligned}\right\} \text{for} \quad \begin{aligned} 0 < r < 1, \\ 0 < z < 1, \end{aligned}$$

and the boundary conditions

$$\left.\begin{aligned} t_r(r, 1, t) &= r\exp(-t) + 2r\exp(-0.5t)), \\ t_z(r, 1, t) &= 3\exp(-t) + 3.5\exp(-0.5t), \end{aligned}\right\} \text{for } 0 < r < 1, \ t > 0,$$

$$\left.\begin{aligned} u_r(1, z, t) &= (1 + z^2)\exp(-t), \\ u_z(1, z, t) &= (2 + z^2)\exp(-0.5t), \end{aligned}\right\} \text{for } 0 < z < 1, \ t > 0,$$

$$\left.\begin{aligned} u_r(r, 0, t) &= r\exp(-t), \\ u_z(r, 0, t) &= 2r^2\exp(-0.5t), \end{aligned}\right\} \text{for } 0 < r < 1, \ t > 0.$$

In order to obtain the numerical results, the boundary curve Γ is discretized into N equal length elements and L well distributed interior collocation points are chosen. The numerical results are obtained using $(N, L) = (30, 49)$ (Set A) and $(N, L) = (120, 225)$ (Set B) at $t = 1$. The numerical results of displacement u_r and u_z are compared with the exact solutions,

$$\begin{aligned} u_r(\underline{\mathbf{x}}, t) &= (r + rz^2)\exp(-t), \\ u_z(\underline{\mathbf{x}}, t) &= (2r^2 + z^2)\exp(-0.5t), \end{aligned}$$

and are shown in Table I and II. To invert the solution in Laplace transform in (16), we choose $M = 5$.

From Table I and II, two sets of numerical results are compared with the exact solutions. The values in Set B, which are obtained using more elements and collocation points in the numerical procedure, are more accurate.

TABLE II
THE DISPLACEMENT u_z AT $t = 1$ AT SELECTED INTERIOR POINTS

(r, z)	u_z Set A	Set B	Exact
(0.25, 0.25)	0.111441	0.112953	0.113724
(0.50, 0.25)	0.338387	0.340272	0.341173
(0.75, 0.25)	0.718264	0.719634	0.720255
(0.25, 0.50)	0.224238	0.226426	0.227449
(0.50, 0.50)	0.449987	0.453387	0.454898
(0.75, 0.50)	0.830329	0.832832	0.833980
(0.25, 0.75)	0.413560	0.416107	0.416990
(0.50, 0.75)	0.638269	0.642619	0.644439
(0.75, 0.75)	1.018955	1.021915	1.023520

In Fig. 1, the numerical displacements u_r and u_z are plotted against time t at the point $(r, z) = (0.5, 0.5)$ using $(N, L) = (120, 225)$ and are compared with the exact solution.

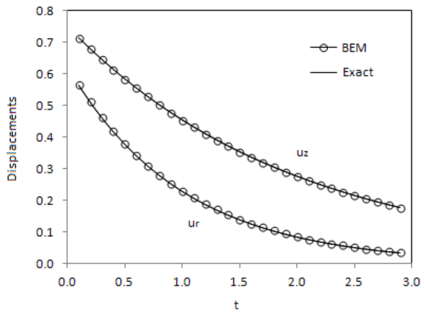

Fig. 1. The displacements u_r and u_z are plotted against time t at point $(r, z) = (0.5, 0.5)$.

From Fig. 1, it is clear that the numerical values of u_r and u_z agree well with those calculated from the exact solution.

The stresses in the solution domain can be obtained from (2) together with (20), (21) and (22). In Fig. 2 and Fig. 3, the stresses are calculated numerically using $(N, L) = (120, 225)$. In Fig. 2, the stresses (except $\sigma_{\theta\theta}$, because it has the same values as σ_{rr} in this problem) are plotted at $r = 0.5$, $0 < z < 1$ at $t = 1$ and compared with the exact solutions.

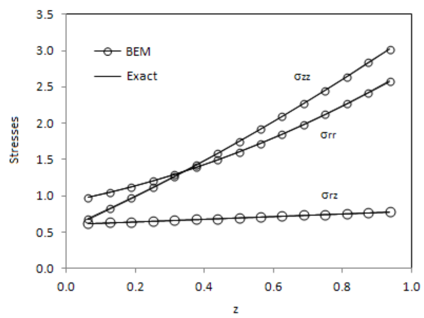

Fig. 2. The stresses $\sigma_{rr}, \sigma_{rz}, \sigma_{zz}$ at $r = 0.5$ are plotted against z at $t = 1$.

The stress σ_{rr} at different values of t ($t = 0.5$, $t = 1.0$, and $t = 1.5$) at $r = 0.5$, $0 < z < 1$ are given in Fig. 3.

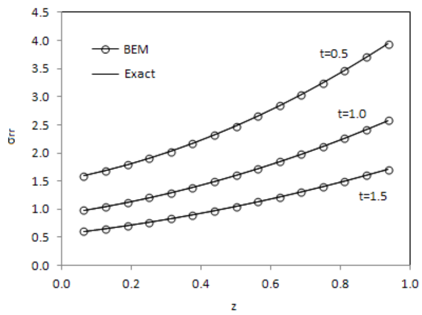

Fig. 3. The stress σ_{rr} at $r = 0.5$ is plotted against z at time $t = 0.5$, $t = 1.0$, and $t = 1.5$.

From both Fig. 2 and Fig. 3, the numerical stresses are in good agreement with the exact solutions. Furthermore, the stress in Fig. 3 is decreasing exponentially with time t as expected.

IX. SUMMARY

A dual-reciprocity boundary element method is proposed for solving numerically a class of axisymmetric elastodynamic problems. Relative simple interpolating functions are used in the dual-reciprocity method for transforming domain integrals to boundary integrals. The problem under consideration is first solved in the Laplace transform domain. The displacement and the stress fields in the Laplace transform domain can be recovered to physical time domain by using a numerical algorithm for inverting Laplace transforms.

From the specific test problem, the numerical solutions obtained shows that the dual-reciprocity boundary element approach is reliable and accurate to solve the axisymmetric elastodynamic problem in homogeneous solids. For further potential applications, this BEM procedure with the relative simple interpolating functions can be applied to axisymmetric thermoelastic problems in nonhomogeneous material with appropriate modifications.

APPENDIX

The functions $\Phi_{IJ}(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)$ and $\Psi_{IJ}(\underline{\mathbf{x}};\underline{\mathbf{x}}_0;\underline{\mathbf{n}}(\underline{\mathbf{x}}))$ in (4) are given by

$$\begin{aligned}\Phi_{rr}(\underline{\mathbf{x}};\underline{\mathbf{x}}_0) &= \frac{1}{8\pi(1-\nu)r_0C(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)}\{((3-4\nu)(r_0^2+r^2) \\ &\quad +4(1-\nu)(z_0-z)^2)K(m(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)) \\ &\quad +(-[C(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)]^2(3-4\nu) \\ &\quad -\frac{(z_0-z)^2}{D(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)}A(\underline{\mathbf{x}};\underline{\mathbf{x}}_0))E(m(\underline{\mathbf{x}};\underline{\mathbf{x}}_0))\},\end{aligned}$$

$$\begin{aligned}\Phi_{rz}(\underline{\mathbf{x}};\underline{\mathbf{x}}_0) &= \frac{(z_0-z)}{8\pi(1-\nu)C(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)}\{-K(m(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)) \\ &\quad +\frac{B(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)}{D(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)}E(m(\underline{\mathbf{x}};\underline{\mathbf{x}}_0))\},\end{aligned}$$

$$\Phi_{zr}(\underline{\mathbf{x}};\underline{\mathbf{x}}_0) = \frac{r(z_0-z)}{8\pi(1-\nu)r_0C(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)}\{K(m(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)) - \frac{A(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)-2r_0^2}{D(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)}E(m(\underline{\mathbf{x}};\underline{\mathbf{x}}_0))\},$$

$$\Phi_{zz}(\underline{\mathbf{x}};\underline{\mathbf{x}}_0) = \frac{r}{4\pi(1-\nu)C(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)}\{(3-4\nu)K(m(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)) + \frac{(z_0-z)^2}{D(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)}E(m(\underline{\mathbf{x}};\underline{\mathbf{x}}_0))\},$$

$$\Psi_{rr}(\underline{\mathbf{x}};\underline{\mathbf{x}}_0;\underline{\mathbf{n}}(\underline{\mathbf{x}})) = -\frac{r}{2\pi(1-\nu)}(\Lambda_1(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)n_r(\underline{\mathbf{x}}) + \Lambda_2(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)n_z(\underline{\mathbf{x}})),$$

$$\Psi_{rz}(\underline{\mathbf{x}};\underline{\mathbf{x}}_0;\underline{\mathbf{n}}(\underline{\mathbf{x}})) = -\frac{r}{2\pi(1-\nu)}(\Lambda_3(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)n_r(\underline{\mathbf{x}}) + \Lambda_4(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)n_z(\underline{\mathbf{x}})),$$

$$\Psi_{zr}(\underline{\mathbf{x}};\underline{\mathbf{x}}_0;\underline{\mathbf{n}}(\underline{\mathbf{x}})) = -\frac{r}{2\pi(1-\nu)}(\Lambda_5(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)n_r(\underline{\mathbf{x}}) + \Lambda_6(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)n_z(\underline{\mathbf{x}})),$$

$$\Psi_{zz}(\underline{\mathbf{x}};\underline{\mathbf{x}}_0;\underline{\mathbf{n}}(\underline{\mathbf{x}})) = -\frac{r}{2\pi(1-\nu)}(\Lambda_7(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)n_r(\underline{\mathbf{x}}) + \Lambda_8(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)n_z(\underline{\mathbf{x}})),$$

where $K(m)$ and $E(m)$ being the complete elliptic integrals of the first and second kind respectively and

$$m(\underline{\mathbf{x}};\underline{\mathbf{x}}_0) = \frac{2b(r;r_0)}{a(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)+b(r;r_0)},$$

and

$$\begin{aligned}\Lambda_1(\underline{\mathbf{x}};\underline{\mathbf{x}}_0) &= \frac{1}{2r_0r^2C(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)}\{(1-2\nu)(A(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)+H(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)) \\ &-\frac{1}{[C(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)]^2D(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)}(-2(z_0-z)^6 \\ &+(-5r_0^2-4r^2)(z_0-z)^4+(5r_0^2r^2-4r_0^4-r^4)(z_0-z)^2 \\ &+(r^2-r_0^2)^3)\}K(m(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)) \\ &+\frac{1}{2r_0r^2C(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)D(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)}\{-(1-2\nu)(2A(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)B(\underline{\mathbf{x}};\underline{\mathbf{x}}_0) \\ &+3r^2(A(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)-2r_0^2)) \\ &+\frac{1}{[C(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)]^2D(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)}(-2(z_0-z)^8 \\ &+(-6r^2-7r_0^2)(z_0-z)^6 \\ &+(-9r_0^4+2r_0^2r^2-5r^4)(z_0-z)^4 \\ &+(-5r_0^6+10r_0^4r^2-5r_0^2r^4)(z_0-z)^2 \\ &+(-r_0^8+2r_0^6r^2-2r_0^2r^4+r^8))\}E(m(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)),\end{aligned}$$

$$\begin{aligned}\Lambda_2(\underline{\mathbf{x}};\underline{\mathbf{x}}_0) &= \Lambda_5(\underline{\mathbf{x}};\underline{\mathbf{x}}_0) \\ &= \frac{z_0-z}{2r_0rC(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)}\{(1-2\nu) \\ &+\frac{1}{[C(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)]^2D(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)}((z_0-z)^2(3A(\underline{\mathbf{x}};\underline{\mathbf{x}}_0) \\ &-2(z_0-z)^2)+2(r_0^2-r^2)^2)\}K(m(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)) \\ &+\frac{z_0-z}{2r_0rC(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)D(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)}\{-(1-2\nu)A(\underline{\mathbf{x}};\underline{\mathbf{x}}_0) \\ &-\frac{1}{[C(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)]^2D(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)}((z_0-z)^4(4A(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)-3(z_0-z)^2) \\ &+(r_0^2-r^2)^2(2A(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)+3(z_0-z)^2))\}E(m(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)),\end{aligned}$$

$$\begin{aligned}\Lambda_3(\underline{\mathbf{x}};\underline{\mathbf{x}}_0) &= -\frac{z_0-z}{2r^2[C(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)]^3D(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)}(2r^2(r^2-r_0^2+2(z_0-z)^2) \\ &+A(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)B(\underline{\mathbf{x}};\underline{\mathbf{x}}_0))K(m(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)) \\ &+\frac{z_0-z}{C(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)D(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)}\{(1-2\nu)-\frac{1}{2r^2[C(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)]^2D(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)} \\ &\times(-[H(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)]^3+r^2(z_0-z)^2(2r_0^2+r^2-5(z_0-z)^2) \\ &+r^2(7r_0^4-11r_0^2r^2+5r^4))\}E(m(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)),\end{aligned}$$

$$\begin{aligned}\Lambda_4(\underline{\mathbf{x}};\underline{\mathbf{x}}_0) &= \frac{1}{2rC(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)}\{(1-2\nu) \\ &+\frac{(z_0-z)^2}{[C(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)]^2D(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)}B(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)\}K(m(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)) \\ &+\frac{1}{2rC(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)D(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)}\{-(1-2\nu)B(\underline{\mathbf{x}};\underline{\mathbf{x}}_0) \\ &+\frac{(z_0-z)^2}{[C(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)]^2D(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)}(-A(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)B(\underline{\mathbf{x}};\underline{\mathbf{x}}_0) \\ &+6r^2(A(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)-2r_0^2))\}E(m(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)),\end{aligned}$$

$$\begin{aligned}\Lambda_6(\underline{\mathbf{x}};\underline{\mathbf{x}}_0) &= \Lambda_7(\underline{\mathbf{x}};\underline{\mathbf{x}}_0) \\ &= \frac{1}{2r_0C(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)}\{(1-2\nu) \\ &-\frac{(z_0-z)^2}{[C(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)]^2D(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)}(A(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)-2r_0^2)\}K(m(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)) \\ &+\frac{1}{2r_0C(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)D(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)}\{-(1-2\nu)(A(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)-2r_0^2) \\ &+\frac{(z_0-z)^2}{[C(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)]^2D(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)}(A(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)(A(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)-2r_0^2) \\ &-6r_0^2B(\underline{\mathbf{x}};\underline{\mathbf{x}}_0))\}E(m(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)),\end{aligned}$$

$$\begin{aligned}\Lambda_8(\underline{\mathbf{x}};\underline{\mathbf{x}}_0) &= \frac{(z_0-z)^3}{[C(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)]^3D(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)}K(m(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)) + \frac{(z_0-z)}{C(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)D(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)} \\ &\times\{-(1-2\nu)-\frac{4(z_0-z)^2}{[C(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)]^2D(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)}A(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)\}E(m(\underline{\mathbf{x}};\underline{\mathbf{x}}_0)),\end{aligned}$$

$$\begin{aligned}A(\underline{\mathbf{x}};\underline{\mathbf{x}}_0) &= r_0^2+r^2+(z_0-z)^2, \\ B(\underline{\mathbf{x}};\underline{\mathbf{x}}_0) &= r_0^2-r^2+(z_0-z)^2, \\ C(\underline{\mathbf{x}};\underline{\mathbf{x}}_0) &= \sqrt{(r_0+r)^2+(z_0-z)^2}, \\ D(\underline{\mathbf{x}};\underline{\mathbf{x}}_0) &= (r_0-r)^2+(z_0-z)^2, \\ H(\underline{\mathbf{x}};\underline{\mathbf{x}}_0) &= r_0^2+(z_0-z)^2.\end{aligned}$$

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