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Boundary Effect on the Onset of Marangoni Convection with Internal Heat Generation

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***Abstract*—The onset of Marangoni convection in a horizontal fluid layer with internal heat generation overlying a solid layer heated from below is studied. The upper free surface of a fluid is nondeformable and the bottom boundary are rigid and no-slip. The resulting eigenvalue problem is solved exactly. The critical values of the Marangoni numbers for the onset of Marangoni convection are calculated and the latter is found to be critically dependent on the internal heating, depth ratio and conductivity ratio. The effects of the thermal conductivity and the thickness of the solid plate on the onset of convective instability with internal heating are studied in detail.

***Keywords*—Linear stability, Marangoni convection, Internal Heat generation.

I. Introduction

EFFECT of buoyancy or surface tension can become a major mechanism of driving a possible convective instability for a horizontal fluid layer heated from below and cooled from above. The instability of convection driven by buoyancy is referred to as the Rayleigh-Bénard convection and the instability convection driven by surface tension is referred to as the Marangoni convection. The instability of the Bénard-Marangoni convection due to the combined effects of the thermal buoyancy and surface tension. Theoretical analysis of Marangoni convection was started with the linear analysis by Pearson [1] who assumed an infinite fluid layer, nondeformable case and zero gravity. He showed that thermocapillary forces can cause convection when the Marangoni number exceeds a critical value in the absence of buoyancy forces. Sparrow et al. [2] and Roberts [3] analyze the thermal instability in a horizontal fluid layer with the nonlinear temperature distribution which is created by an internal heat generation. The effect of a quadratic basic state temperature profile caused by internal heat generation was first addressed by Char and Chiang [4] for Bénard-Marangoni convection. Later, Wilson [5] investigated the effect of the internal heat generation on the onset of Marangoni convection when the lower boundary is conducting and insulating to temperature perturbations. He found that the effect of increasing the internal heat generation is always to destabilize the layer.

The modern techniques in the recent past has posed challenges in studying convective instability problems in much more complicated two and multilayer fluid dynamical systems. Theoretically and experimentally were studied by considering multilayer of fluid, or a fluid layer separated at the middle or bounded from the above or below by a slab. Even though single layer systems and double layer systems heated from below have received a great deal of attention in the past, there have been very few studies related to the thermal instability and heat transfer phenomena in a system with more than two layers. Yang [6] consider the lower boundary to be a solid plate where it is a perfect insulating boundary condition for thermal disturbances which is difficult compared to conducting boundary condition. It is found that the solid plate with a higher thermal conductivity tends to stabilize the system. The role of the plate thickness is minor in most of the Bénard-Marangoni experiments, while the conductivity of the plate has a significant impact on the stability of the system. Char and Chen [7] focused on Bénard-Marangoni instability with a boundary slab of finite conductivity. They solved the problem numerically and later compared to the asymptotic of the long wavelength. It shown that the critical Rayleigh number increases with thickness of solid layer to the thickness of fluid and thermal conductivity of solid layer to the thermal conductivity of fluid. Recently, Arifin and Pop [8] have studied the onset of Marangoni convection in a fluid-porous-solid layer system and they found that the critical Marangoni number increases with the depth ratio or the thermal conductivity ratio. The onset of Marangoni convection in a horizontal fluid layer with the influences of the variable viscosity and the solid plate have been investigated by Abidin. et al. [9]. They have shown that the viscosity is a destabilizing factor but with thicker solid layer or higher thermal conductivity, the system become more stable.

In this paper, we consider the onset of Marangoni convection in a horizontal fluid layer with the internal heat generation and the solid plate at the bottom surface. The problem has been solved exactly to obtain a detail description of the marginal stability curves for the onset of Marangoni convection.

II. The Mathematical Formulation

Consider a horizontal fluid layer of depth d with a free upper surface overlying a solid layer of thickness d_s. The physical configuration is shown in Fig.1. The lower boundary is subjected to a fixed heat flux, while the upper surface of the fluid is assumed to be non-deformable. We used Cartesian coordinates with two horizontal x- and y-axes located at the lower solid boundary and a positive z-axis is directed towards the free surface. The surface tension, τ is assumed to be a linear function of the temperature

$$\tau = \tau_0 - \gamma (T - T_0), \qquad (1)$$

where τ_0 is the value of τ at temperature T_0 and the constant γ is positive for most fluids. The density of the fluid is given by

$$\rho = \rho_0 \{1 - \alpha(T - T_0)\}, \qquad (2)$$

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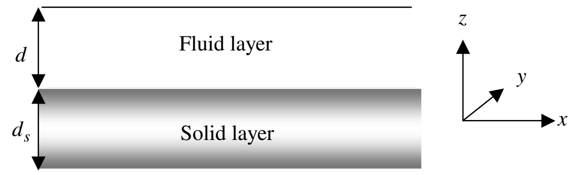

Fig. 1. Geometry of the unperturbed state

where α is the positive coefficient of the thermal liquid expansion and ρ_0 is the value at the reference temperature T_0. Based on the above assumptions together with the Boussinesq approximation, the governing equations for the continuity, momentum and energy in the fluid layer are respectively

$$\nabla \cdot \mathbf{u} = 0, \tag{3}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)\mathbf{u} = -\frac{\nabla p}{\rho} + \nu\Delta\mathbf{u}, \tag{4}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)T = \kappa\nabla^2 T + q \tag{5}$$

and for the solid layer, the energy equation takes the form

$$\frac{\partial T_s}{\partial t} = \kappa_s \nabla^2 T_s, \tag{6}$$

where $\mathbf{u}$, T, p, ρ, ν, κ and q denote the velocity, temperature, pressure, density, kinematic viscosity, thermal diffusivity and uniformly distributed volumetric internal heat generation in the fluid layer, respectively. The subscript s refer to the quantities in the solid layer.

When motion occurs, the upper free surface of the layer will be deformable with its position at $z = d + f(x, y, t)$. At the free surface, we have the usual kinematic condition together with the conditions of continuity for the normal and tangential stresses. The temperature obeys the Newton's law of cooling, $k\partial T/\partial\mathbf{n} = h(T - T_\infty)$, where k and h are the thermal conductivity of the fluid and the heat transfer coefficient between the free surface and the air, respectively, and $\mathbf{n}$ is the outward unit normal to the free surface.

To simplify the analysis, we write the governing equations and the boundary conditions in a dimensionless form. We choose d, κ/d, d^2/κ and ΔT for length, velocity, time and temperature gradient respectively. As a results the following dimensionless group arises, the Marangoni number, $M = \gamma\Delta T d/\rho_0\kappa\nu$, the Biot number, $B_{\mathrm{i}} = hd/k$, the Bond number, $B_{\mathrm{o}} = \rho_0 g d^2/\tau_0$, the Prandtl number, $P_{\mathrm{r}} = \nu/\kappa$, the Crispation number, $C_{\mathrm{r}} = \rho_0\nu\kappa/\tau_0 d$ and the internal heating, $Q = qd^2/2\kappa\Delta T$.

Standard methods of linear stability analysis are used to determine the effect of the controller gain, K, on the critical Marangoni number at the onset of convection with internal heat generation. We start with a linear stability analysis of the basic state in the usual manner by seeking perturbed solutions for any quantity $\Phi(x, y, z, t)$ in terms of normal modes in the form

$$\Phi(x, y, z, t) = \Phi_0(x, y, z) + \phi(z)\exp\left[i(\alpha_x x + \alpha_y y) + st\right], \tag{7}$$

where Φ_0 is the value of Φ in the basic state, $a = (\alpha_x^2 + \alpha_y^2)^{1/2}$ is the total horizontal wave number of the disturbance and s is a complex growth rate with the real part representing the growth rate of the instability and the imaginary part representing its frequency. At marginal stability, the growth rate s of perturbation is zero and the real part of s, Re(s)> 0 represents unstable modes while Re(s)< 0 represents stable modes. Substituting equation (7) into equations (3) – (5), we obtain the corresponding linearized equations involving only the z-dependent parts of the perturbations to the temperature and the z-components of the velocity denoted by T and w respectively, namely

$$(D^2 - a^2)[(D^2 - a^2)w - sP_{\mathrm{r}}^{-1}] = 0, \tag{8}$$

$$(D^2 - a^2 - s)T + [1 - Q(1 - 2z)]w = 0, \tag{9}$$

$$(D^2 - a^2 - s)T_s = 0, \tag{10}$$

subject to

$$w = 0, \tag{11}$$

$$D^2 w + Ma^2 T = 0, \tag{12}$$

$$DT + B_{\mathrm{i}} T = 0, \tag{13}$$

evaluated on the undisturbed position of the upper free surface $z = 1$,

$$w = 0, \tag{14}$$

$$Dw = 0, \tag{15}$$

$$T = T_s, \tag{16}$$

$$DT = \kappa_r DT_s, \tag{17}$$

evaluated on the solid-fluid interface $z = 0$, and

$$DT_s = 0, \tag{18}$$

on $z = -d_r$. The operator D=d/dz denotes differentiation with respect to the vertical coordinate z, $\kappa_r = \kappa/\kappa_s$ is the ratio of the thermal conductivity of the solid plate to that of fluid layer and $d_r = d_s/d$ is the ratio of the solid plate thickness to the fluid layer thickness. Solving the perturbation equation (10) for the solid layer, together with the boundary conditions (16) – (18), the thermal boundary condition at the solid-fluid interface, at $z = 0$ becomes

$$DT = \kappa_r a \tanh(ad_r)T. \tag{19}$$

III. Results and Discussion

By substituting the general solution of equations (8) and (9) into the boundary conditions (11) – (15) and (19), we obtain the closed form analytical expression for M on the marginal stability curves which can be written in the form

$$M = \frac{A_1[A_2\kappa_r\sinh(ad_r) + A_3\cosh(ad_r)]}{\kappa_r[A_4 Q + A_5]\sinh(ad_r) + [A_6 Q + A_7]\cosh(ad_r)}, \tag{20}$$

where

$$
\begin{aligned}
A_1 &= 192a^2(a-SC)(S+C)\\
A_2 &= aC+B_{\rm i}S\\
A_3 &= aS+B_{\rm i}C\\
A_4 &= 3a(2-a)[1+(S+C)^4]+4a^2(a^2+8)\\
&\quad [1+(s+C)^2]+2(8a^3+21a-6)[1-(S+C)^2]+\\
&\quad 6(a-2)(s^2+C^2)\\
A_5 &= -3a[1+(S+C)^4]+12a^4[1+(S+C)^2]-\\
&\quad 6a[1-(S+C)^2]+6a(S^2+C^2)\\
A_6 &= 3(2-a)[1+(S+C)^4]+6a(1-2a^2)\\
&\quad [1+(S+C)^2]+4(3-a^4)[1-(S+C)^2]-\\
&\quad 6(a+2)(s^2+C^2)\\
A_7 &= 3a[1-(S+C)^4]-12a^3[1+(S+C)^2]-\\
&\quad 12a^2(a^2+2)[1-(S+C)^2]+12aSC
\end{aligned}
$$

where we have define $C=\cosh(a)$ and $S=\sinh(a)$. When we set $\kappa_r=0$ or $d_r=0$, the equation (20) reduces to the expression given by Wilson [5].

The marginal curves in the (a,M) plane are obtained by (20) where M is a function of the parameters a, $B_{\rm i}$, Q, d_r and κ_r. For a given set of parameters, the critical Marangoni number for the onset of convection is defined as the minimum of the global minima of marginal curve. We denote this critical value by M_c and the corresponding critical wave number a_c. Numerically calculated values of M and the corresponding values of a are shown in Fig.2 and Fig.3 for a range values of d_r and κ_r respectively with $Q=1$ when the free surface is perfectly insulated ($B_{\rm i}=0$). From Fig.2 and Fig.3, it can be seen that with the larger depth ratio, d_r and thermal conductivity ratio, κ_r, the global minimum occurs at short wavelength ($a\neq 0$) and the critical Marangoni number increases. Numerically calculated values of M and the corresponding values of a are plotted in Fig.4 and Fig.5 for a range values of d_r and κ_r respectively with $Q=1$ when the free surface is conduction ($B_{\rm i}\neq 0$). An inspection of the figures reveals that the critical Marangoni number increases as the values of d_r and κ_r increase.

Fig. 6 – Fig. 9 show the variation of M_c and a_c with Q for a range of values of d_r and κ_r in the case $B_{\rm i}=0$. Fig. 6 and 8 shows that M_c is a monotonically decreasing function of Q, while Fig. 7 and 9 shows that a_c increases monotonically with Q. For $d_r=0$, we reproduce the numerical results obtained by Wilson [5]. From Fig. 7, it is seen that the critical Marangoni number, M_c increases with d_r. The thicker solid layer are clearly a stabilizing factor to make system more stable because its might store more thermal energy. Fig. 8 indicate the study of thermal conductivity influences to stability curves. The results shows that the thermal conductivity ratio always has a stabilizing effect on the Marangoni convection. This is due to the thermal disturbances are easily dissipated deep into the solid layer.

IV. Conclusion

Boundary effect on the onset of Marangoni convection in a horizontal layer of electrically-conducting fluid which is free above and rigid below with internal heat generation is studied. We obtained the closed form analytical solution for the onset of Marangoni convection. The critical values of the Marangoni numbers for the onset of Marangoni convection are calculated and the latter is found to be critically dependent on the internal heating, depth ratio and conductivity ratio. It is found that the critical Marangoni number increases as depth ratio and thermal conductivity ratio increases.

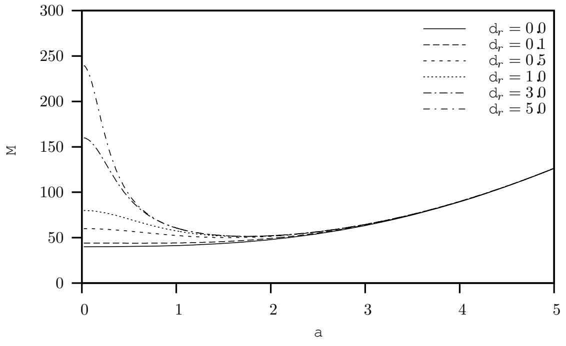

Fig. 2. Numerically-calculated Marangoni number, M as a function of the wave number, a, for various values of d_r, in the case $B_{\rm i}=0$, $\kappa_r=1$ and $Q=1$.

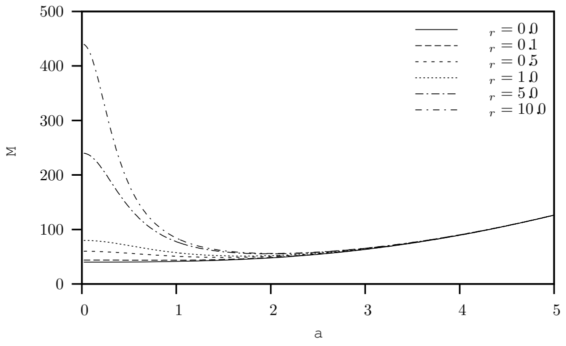

Fig. 3. Numerically-calculated Marangoni number, M as a function of the wave number, a, for various values of κ_r, in the case $B_{\rm i}=0$, $d_r=1$ and $Q=1$.

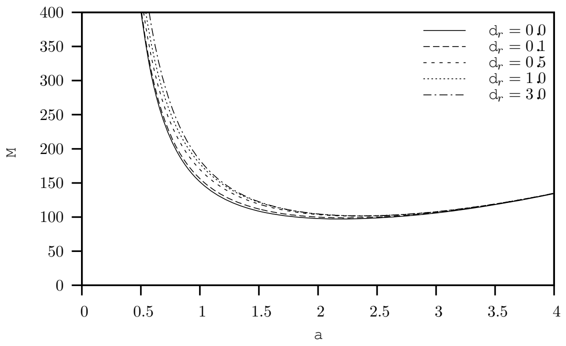

Fig. 4. Numerically-calculated Marangoni number, M as a function of the wave number, a, for various values of d_r, in the case $B_{\rm i}=2$, $\kappa_r=1$ and $Q=1$.

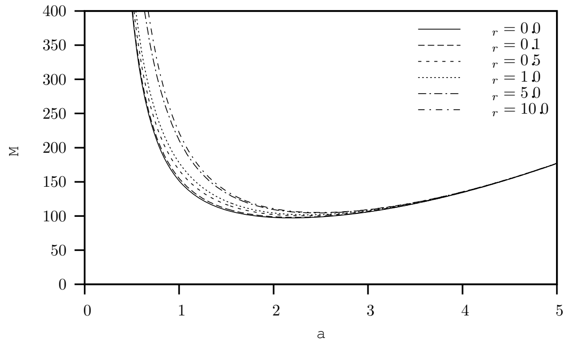

Fig. 5. Numerically-calculated Marangoni number, M as a function of the wave number, a, for various values of κ_r, in the case $B_{\mathrm{i}} = 2$, $d_r = 1$ and $Q = 1$.

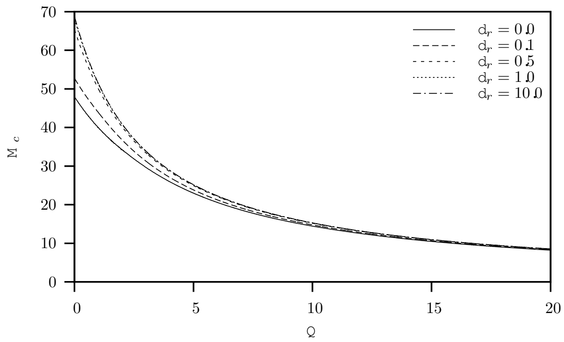

Fig. 6. The critical Marangoni number at the onset of convection as a function of Q, for a range of values of d_r in the case $B_{\mathrm{i}} = 0$, and $\kappa_r = 1$.

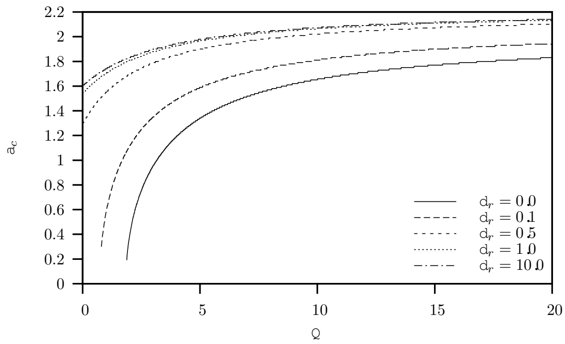

Fig. 7. The critical wave number at the onset of convection as a function of Q, for a range of values of d_r in the case $B_{\mathrm{i}} = 0$ and $\kappa_r = 1$.

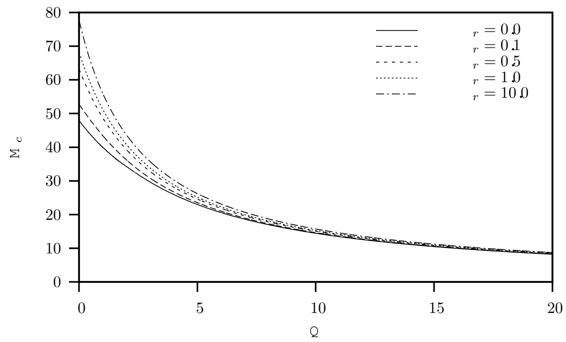

Fig. 8. The critical Marangoni number at the onset of convection as a function of Q, for a range of values of κ_r in the case $B_{\mathrm{i}} = 0$, and $d_r = 1$.

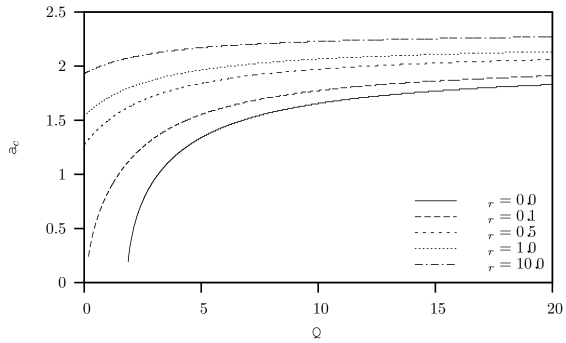

Fig. 9. The critical wave number at the onset of convection as a function of Q, for a range of values of κ_r in the case $B_{\mathrm{i}} = 0$ and $d_r = 1$.

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